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A note on some conditions on integrality of the intersection of two simple ring extensions

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Abstract

Let R be a Noetherian domain and α, β algebraic elements over the quotient field of R. In this note, we give some conditions that $R[\alpha] \cap R[\beta]$ is integral over R.

Let R be an integral domain with quotient field K and $R[X]$ a polynomial ring over R in an indeterminate X. Let α be an element of an algebraic field extension of K and $\pi : R[X] \longrightarrow R[\alpha]$ the R-algebra homomorphism defined by $\pi(X) = \alpha$. Let $\varphi_\alpha(X)$ be the monic minimal polynomial of α over K with $\deg \varphi_\alpha = d$, and write

$$\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d, \ (\eta_1, \ldots, \eta_d \in K).$$

We define $I_{[\alpha]} = \bigcap_{i=1}^{d}(R :_R \eta_i)$ where $(R :_R \eta_i) = \{c \in R; c\eta_i \in R\}$. We set $\mathrm{Dp}_1(R) = \{p \in \mathrm{Spec}R; \mathrm{depth}R_p = 1\}$.

Our general reference for unexplained terms is [2].

Lemma 1.(cf. [3, Theorem 2.2.]) *Let R be an integral domain and α an algebraic element over the quotient field of R. If $I_{[\alpha]} = R$, then $R[\alpha]$ is integral over R.*

Proof. Since 1 is in $I_{[\alpha]} = \bigcap_{i=1}^{d}(R :_R \eta_i)$, we see that $\eta_1, \ldots, \eta_d$ are in R. Hence $\varphi_\alpha(X)$ is in $R[X]$, and α is integral over R. Therefore $R[\alpha]$ is integral over R. Q.E.D.

Lemma 2. ([1, Theorem 11.11.] and [3, Proposition 2.1.]) *Let R be an integral domain and A an integral domain containing R.*

(1) *If A_p is integral over R_p for all $p \in \mathrm{Spec}R$, then A is integral over R.*

(2) *If A_p is integral over R_p for all $p \in \mathrm{Dp}_1(R)$, then A is integral over R.*

Proposition 3. *Let R be a Noetherian domain and α, β algebraic elements over the quotient field of R. If $I_{[\alpha]} + I_{[\beta]} = R$, then $R[\alpha] \cap R[\beta]$ is integral over R.*

Proof. Let p be an arbitrary element of Spec$R$. Then $p \not\supset I_{[\alpha]}$ or $p \not\supset I_{[\beta]}$. If $p \not\supset I_{[\alpha]}$, then $R_p[\alpha]$ is integral over R_p by Lemma 1. Since $R_p \subset R_p[\alpha] \cap R_p[\beta] \subset R_p[\alpha]$, we know that $R_p[\alpha] \cap R_p[\beta]$ is integral over R_p. If $p \not\supset I_{[\beta]}$, then, by the same argument as above, $R_p[\alpha] \cap R_p[\beta]$ is integral over R_p. Then Lemma 2 (1) implies that $R[\alpha] \cap R[\beta]$ is integral over R. Q.E.D

We define grade$(R) = \infty$.

Theorem 4. *Let R be a Noetherian domain and α, β algebraic elements over the quotient field of R. If grade$(I_{[\alpha]} + I_{[\beta]}) > 1$, then $R[\alpha] \cap R[\beta]$ is integral over R.*

Proof. Let p be an arbitrary element of $\mathrm{Dp}_1(R)$. Then by the assumption, we have $p \not\supset I_{[\alpha]} + I_{[\beta]}$. Then Proposition 3 assers that $R_p[\alpha] \cap R_p[\beta]$ is integral over R_p. Hence $R[\alpha] \cap R[\beta]$ is integral over R by Lemma 2 (2). Q.E.D.

References

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