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Abelian solution in Poincaré gauge theory and extended phase transformations

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We treat a model of Poincaré gauge theory (PGT) which can be compatible with general relativity. The equations obtained are rewritten in complex Einstein-Yang-Mills forms. We pick up its special case, "Abelian solution" where the equations are essentially reducible to complex Einstein-Maxwell equations. On the other hand, we define certain "complex" phase transformations by extending ordinary phase transformations. Then we find that the same complex Maxwell equations as those from PGT can be obtained from an gauge theory based on the "complex" phase transformati ons. On the basis of this fact we conclude that the sources for the Lorentz gauge field may be a composed of at least two Dirac fields, if the sources are Dirac particles.

1 Introduction

Poincaré gauge theory (PGT) was founded by Utiyama[1] and Kibble[2], and later developed to a more general model with nine independent parameters by Hayashi[3] and Hehl and his collaborators[4]. Since then, some special solutions have been solved in several models[5], and also variant conditions have been imposed on the parameters from various requirements[6].

Generally speaking, PGT is a theory with two gauge fields, namely a translational and a Lorentz gauge field, and it is geometrically interpreted as a Riemann-Cartan theory with a curvature and a torsion. This is done by identifying the Lorentz gauge field with an affine connection through a requirement: Kibble's covariant derivative of a vierbein field should vanish. However, we adopt here a model which can be interpreted as an Einstein theory associated with the Lorentz gauge field. This selection is based on both the simplicity and experimental or observational results which appear to justify the Einstein theory at least in macroscale.

It is well-known[7] that any Yang-Mills theories based on arbitrary internal gauge symmetries are reducible to the Maxwell theory in a special case. And, of course, the Maxwell theory can be derived from gauging any phase transformations. In the same way, PGT is reducible to complex Einstein-Maxwell theory through complex Einstein-Yang-Mills theory[8].

In this paper it is shown that the complex Maxwell theory can be derived from a gauge theory based on gauged and extended phase transformations.

And also we consider the gauge theory in such a case that the complex Maxwell field is interacting with Dirac fields. Then we conclude that the complex Maxwell field must be generated by a pair of Dirac fields. From this conclusion we infer for the source of the Lorentz gauge field to be a combination of at least two Dirac fields, if the sources are composed of Dirac particles.

In the next section we review how a compatible PGT model with Einstein theory is reducible to complex Einstein-Maxwell theory. And then we consider the transformation property of the Lorentz gauge field as a complex Maxwell field under the Poicaré gauge transformations. As a result, we find that the complex Maxwell field is transformed like the ordinary Maxwell field except that it is a complex quantity. That is, the Poicaré gauge transformations behave like general coordinate transformations plus gauged *complex* phase transformations for the complex Maxwell field.

In Sec.3 we extend ordinary phase transformations and research a gauge theory based on the extended phase transformations. And such a case is considered that the source is composed of Dirac particles. Finally, the gauge theory is compared with the theory reduced from PGT. The final section is devoted to conclusions.

2 Compatible model with general relativity

2.1 Preliminaries

Let us consider a set of matter fields $q = \{q^A/A = 1, 2 \cdots N\}$ which transforms as $\delta q = \frac{i}{2}\omega_{km}S^{km}q$ under any Poincaré transformations: $\delta x^k = \omega^k{}_m x^m + \epsilon^k$. Here S^{km} are generators for Lorentz transformations and, in particular, $S^{km} = -\frac{1}{2}\sigma^{km} = -\frac{i}{4}[\gamma^k, \gamma^m]$ for q =Dirac fields. Now, we assume that the behavior of q is controled by an action $\int d^4x L_M(q, q_{,k})$, which is invariant under the Poincaré transformations. Here $q_{,k}$ means the derivative $\frac{\partial q}{\partial x^k}$, and $L_M(q, q_{,k})$ is Lagragian density for q.

It is well-known[2] that when the Poincaré transformations are gauged, namely

$$\begin{aligned} &\delta x^k = \omega^k{}_m(x)x^m + \epsilon^k(x) \to \delta x^\mu = \xi^\mu(x) \\ &\delta q = \tfrac{i}{2}\omega_{km}(x)S^{km}q \end{aligned} \tag{2.1}$$

then the ordinary derivative $q_{,k}$ must be replaced by the covariant one $D_k q$ for the Lagrangian to keep its invariance. Here $D_k q$ is defined by two gauge fields, a translational gauge field $c_k{}^\mu$ and a Lorentz gauge field $A_{km\mu}$, as follow:

$$D_k q = b_k{}^\mu D_\mu q = b_k{}^\mu \{q_{,\mu} + \frac{i}{2}A_{mn\mu}S^{mn}q\} \tag{2.2}$$

with

$$b_k{}^\mu = \delta_k{}^\mu + c_k{}^\mu.$$

Incidentally, $b_k{}^\mu$ is called vier-bein or tetrad field. The field strengths for these gauge fields are defined as

$$F_{km\mu\nu} = A_{km\nu,\mu} - A_{km\mu,\nu} + A_{kr\mu}A^r{}_{m\nu} - A_{kr\nu}A^r{}_{m\mu}$$

and

$$C_{kmn} = c_{kmn} + 2A_{k[mn]},$$

where we put $A_{kmn} = A_{km\mu}b_n{}^\mu$ and $A^r{}_{m\mu} = \eta^{rk}A_{km\mu}$ with Minkowski metric $\eta^{km}(= \eta_{km}) = diag.(+1, -1, -1, -1)$. And also $c_{kmn} = b_{k\mu,\nu}(b_m{}^\mu b_n{}^\nu - b_n{}^\mu b_m{}^\nu)$ with $b_{k\mu}$ defined by $b_{k\mu}b_m{}^\mu = \eta_{km}$. Hereafter we shall use the Minkowski metrics η^{km}, η_{km} to raise or lower the latin indices and the metric $g^{\mu\nu} = b_k{}^\mu b^{k\nu}$, $g_{\mu\nu} = b^k{}_\mu b_{k\nu}$ to raise or lower the greek indices.

2.2 The action

In our purpose we adopt the following invariant action

$$\begin{aligned} I &= \int d^4x b(L_M(q, D_k q) \\ &- \frac{2}{3}a^T C_{kmn}{}^T C^{kmn} + \frac{2}{3}a^V C_k{}^V C^k - \frac{3}{2}a^V C_k{}^V C^k \\ &+ a_1 F_{km\mu\nu}F^{km\mu\nu} + aF). \end{aligned} \tag{2.3}$$

Here a and a_1 are coupling constants and ${}^T C_{kmn}, \cdots$ are the irreducible components of a field strength C_{kmn}. And F is defined as

$$F = b^{k\mu}b^{m\nu}F_{km\mu\nu}.$$

And also $b = -det(b_{k\mu})$ is needed to make the action invariant.

The Action (2.3) can be also written as

$$I = \int d^4x b(L_M(q, D_k q) + aR + a_1 F_{km\mu\nu}F^{km\mu\nu}), \tag{2.4}$$

omitting a divergence term and using the identity

$$\begin{aligned} F = \; & R + \frac{2}{3}{}^T C_{kmn}{}^T C^{kmn} - \frac{2}{3}{}^V C_k{}^V C^k \\ + \; & \frac{3}{2}{}^A C_k{}^A C^k + (2bb^{m\mu V}C_m)_{,\mu}/b. \end{aligned}$$

Here R is a scalar curvature which is made from Ricci tensor $R_{\mu\nu}$ through Riemann tensor $R^\lambda{}_{\kappa\mu\nu}$ defined in terms of the metric $g_{\mu\nu}$.

By the variational principle we can then derive the following equations for the Lorentz gauge field and for the translational gauge field, respectively:

$$\begin{aligned} F_{kmnp,}{}^p \; & - \; A^r{}_{kp}F_{rmn}{}^p - A^r{}_{mp}F_{krn}{}^p \\ & - \; \Delta^r{}_{pr}F_{kmn}{}^p - \Delta_{nrp}F_{km}{}^{rp} \\ & = \; \frac{1}{4a_1}S_{Mkmn}. \end{aligned} \tag{2.5}$$

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = \frac{1}{2a}(T_M{}^{\mu\nu} + T_L{}^{\mu\nu}) \tag{2.6}$$

Here we put $F_{kmnp} = b_n{}^\mu b_p{}^\nu F_{km\mu\nu}$ and $F_{kmnp,}{}^p = b^{p\mu}F_{kmnp,\mu}$. On the other hand, Δ_{kmn} are Ricci's rotation coefficients defined in terms of b's and its first derivatives as

$$\Delta_{kmn} = \Delta_{km\mu}b_n{}^\mu = \frac{1}{2}(c_{kmn} + c_{mnk} + c_{nmk}).$$

S_M and T_M are the spin-angular momentum and the energy-momentum tensors of the matter field q, respectively. And $T_L{}^{\mu\nu}$ is the energy-momentum tensor of the Lorentz gauge field:

$$T_L{}^{\mu\nu} = -4a_1(g^{\mu\kappa}F_{km\lambda\kappa}F^{km\lambda\nu} - \frac{1}{4}g^{\mu\nu}F_{km\lambda\kappa}F^{km\lambda\kappa}). \tag{2.7}$$

2.3 Covariant derivatives

The original Kibble's covariant derivative[2] has been defined by

$$D_\mu \alpha = \alpha_{,\mu} + \frac{i}{2} A_{km\mu} S^{km} \alpha + \Gamma^\lambda{}_{\nu\mu} \Sigma_\lambda{}^\nu \alpha$$

for a generic field α transforming according to

$$\delta\alpha = \frac{i}{2}\omega_{km} S^{km} \alpha + \xi^\lambda{}_{,\mu} \Sigma_\lambda{}^\mu \alpha.$$

The affine connection $\Gamma^\lambda{}_{\mu\nu}$ is then given by

$$\Gamma^\lambda{}_{\mu\nu} = b_i{}^\lambda (b^i{}_{\mu,\nu} + A^i{}_{j\nu} b^j{}_\mu).$$

This connection has been derived from the requirement that the covariant derivatives of the vier-bein components should vanish,

$$D_\mu b_k{}^\nu = 0, \ etc. \tag{2.8}$$

However, we adopt here an extended Einstein's covariant derivative[1] which is defined as

$$\nabla_\mu \alpha = \alpha_{,\mu} - \frac{i}{2}\Delta_{km\mu} S^{km} \alpha + \left\{ \begin{array}{c} \lambda \\ \mu\nu \end{array} \right\} \Sigma_\lambda{}^\nu \alpha \tag{2.9}$$

for the same α as above, where $\left\{ \begin{array}{c} \lambda \\ \mu\nu \end{array} \right\}$ is a Christoffel symbol. The definition is coming from the original $D_\mu \alpha$. In fact, for the *generic* field α, the covariant derivatives $D_\mu \alpha$ and $\nabla_\mu \alpha$ are related by

$$D_\mu \alpha = \nabla_\mu \alpha + \frac{i}{2} K_{km\mu} S^{km} \alpha + K_{km\mu} b^{k\lambda} b^m{}_\nu \Sigma_\lambda{}^\nu \alpha.$$

This is easily known by noting the relations $A_{km\mu} = K_{km\mu} - \Delta_{km\mu}$ and $\Delta_{kmn} = -\left\{ \begin{array}{c} \lambda \\ \mu\nu \end{array} \right\} b_{k\lambda} b_m{}^\mu b_n{}^\nu + b_m{}^\mu b_n{}^\nu b_{k\mu,\nu}$, where $K_{kmn} = b_n{}^\mu K_{km\mu}$ is called a contortion and given by $K_{kmn} = \frac{1}{2}(\mathcal{C}_{kmn} + \mathcal{C}_{mnk} + \mathcal{C}_{nmk})$.

The covariant derivative ∇_μ satisfies the condition

$$\nabla_\mu b_k{}^\nu = 0, \ etc., \tag{2.10}$$

in place of the condition (2.8) and therefore the metric condition $\nabla_\mu g_{\lambda\kappa} = 0$ is automatically satisfied.

We shall also use another covariant derivative $(\alpha_{;\mu})$, which is equivalent to $\nabla_\mu \alpha$ but it is restricted to operate on *world* tensors only. Accordingly, the covariant derivative is just equal to the Einstein's covariant derivative. It should be remarked that it also satisfies the metric condition

$$g_{\mu\nu;\lambda} = \nabla_\lambda g_{\mu\nu} = 0,$$

but

$$b_k{}^\mu{}_{;\nu} \neq 0.$$

2.4 Field equations

Using the covariant derivative ∇_μ, the equation (2.5) can be written as

$$\begin{aligned} b^{p\mu} \nabla_\mu F_{kmnp} &- \Delta^r{}_{k\mu} F_{rmn}{}^\mu - \Delta^r{}_{m\mu} F_{krn}{}^\mu \\ &- A^r{}_{k\mu} F_{rmn}{}^\mu - A^r{}_{m\mu} F_{krn}{}^\mu \\ &= \frac{1}{4a_1} S_{Mkmn}. \end{aligned}$$

Furthermore, when making use of the condition (2.10) and the covariant derivative (;), then the equation can be reduced to

$$F^{km\mu\nu}{}_{;\nu} + A^k{}_{r\nu} F^{rm\mu\nu} + A^m{}_{r\nu} F^{kr\mu\nu} = \frac{1}{4a_1} S_M{}^{km\mu}. \tag{2.11}$$

We can also have the following equation, which can be driven from the Bianchi identity in PGT in exactly the same way as above:

$$F^{\dagger km\mu\nu}{}_{;\nu} + A^k{}_{r\nu} F^{\dagger rm\mu\nu} + A^m{}_{r\nu} F^{\dagger kr\mu\nu} = 0, \tag{2.12}$$

where $F^{\dagger km\mu\nu}$ is the dual of $F^{km\mu\nu}$, namely $F^{\dagger km\mu\nu} = \frac{1}{2} F^{km}{}_{\alpha\beta} \epsilon^{\alpha\beta\mu\nu} = \frac{1}{2} F^{kmrs} \epsilon_{rs}{}^{np} b_n{}^\mu b_p{}^\nu$ with the Levi-Civita symbol ϵ^{kmnp} ($\epsilon^{0123} = 1$).

2.5 Abelian solutions

We already know[8] that above equations (2.11) and (2.12) can be written in the *complex* Einstein-Yang-Mills form

$$\vec{\mathcal{F}}^{\mu\nu}{}_{;\nu} - i\vec{\mathcal{A}}_\nu \times \vec{\mathcal{F}}^{\mu\nu} = \frac{1}{4a_1} \vec{\mathcal{S}}_M^\mu, \tag{2.13}$$

$$\vec{\mathcal{F}}^{\dagger\mu\nu}{}_{;\nu} - i\vec{\mathcal{A}}_\nu \times \vec{\mathcal{F}}^{\dagger\mu\nu} = 0, \tag{2.14}$$

if we define

$$\begin{aligned} (\vec{\mathcal{A}}_\mu)_a &= A_{0a\mu} + \frac{i}{2}\epsilon_{abc} A_{bc\mu}, \\ (\vec{\mathcal{F}}_{\mu\nu})_a &= F_{0a\mu\nu} + \frac{i}{2}\epsilon_{abc} F_{bc\mu\nu}, \\ (\vec{\mathcal{S}}_{M\mu})_a &= S_{M0a\mu} + \frac{i}{2}\epsilon_{abc} S_{Mbc\mu}. \end{aligned}$$

And also the field strength for the complex Lorentz gauge field $\vec{\mathcal{A}}_\mu$ can be written as

$$\vec{\mathcal{F}}_{\mu\nu} = \vec{\mathcal{A}}_{\nu,\mu} - \vec{\mathcal{A}}_{\mu,\nu} - i\vec{\mathcal{A}}_\mu \times \vec{\mathcal{A}}_\nu. \tag{2.15}$$

Let us now consider so-called Abelian solutions whose existence is well-known in the ordinary Yang-Mills

[1]For simplicity we name the relativistic covariant derivatives with Christoffel connections Einstein's covariant derivatives.

theory. There the Yang-Mills equations can be reduced to the Maxwell equations. Here we get, however, the *complex* Maxwell equations. In fact, if we put

$$\vec{\mathcal{A}}_\mu = \vec{\beta}\mathcal{A}_\mu, \tag{2.16}$$

then we get at once from the equations (2.13) and (2.14)

$$\mathcal{F}^{\mu\nu}{}_{;\nu} = \frac{1}{4a_1}\vec{\alpha}\cdot\vec{\mathcal{S}}^\mu_M \tag{2.17}$$

$$\mathcal{F}^{\dagger\mu\nu}{}_{;\nu} = 0, \tag{2.18}$$

where $\vec{\beta}$ is assumed to be some constant complex vector field and $\vec{\alpha}$ is its inverse, i.e., $\vec{\alpha} = \frac{\vec{\beta}}{\vec{\beta}^2}$. And $\mathcal{F}^{\mu\nu}$ is a strength of the complex Maxwell field $\mathcal{A}_\mu$:

$$\mathcal{F}_{\mu\nu} = \mathcal{A}_{\nu,\mu} - \mathcal{A}_{\mu,\nu}.$$

Additionally, we note also that the field equation (2.17) can be written as

$$F^{1\mu\nu}{}_{;\nu} = \frac{1}{8a_1}\left(\vec{\alpha}\cdot\vec{\mathcal{S}}^\mu_M + (\vec{\alpha}\cdot\vec{\mathcal{S}}^\mu_M)^*\right), \tag{2.19}$$

$$F^{2\mu\nu}{}_{;\nu} = \frac{1}{8ia_1}\left(\vec{\alpha}\cdot\vec{\mathcal{S}}^\mu_M - (\vec{\alpha}\cdot\vec{\mathcal{S}}^\mu_M)^*\right) \tag{2.20}$$

where the real fields A_μ and B_μ are defined by $\mathcal{A}_\mu = A_\mu + iB_\mu$, and $F^1_{\mu\nu}$ and $F^2_{\mu\nu}$ are the corresponding real field strengths, respectively.

On the other hand, the energy-momentum tensor (2.7) for the Lorentz gauge field is then given by

$$\begin{aligned} T_{L\mu\nu} = {} & 8a_1\beta_1\left\{F^1_{\mu\lambda}F^{1\lambda}_\nu - \frac{g_{\mu\nu}}{4}F^1_{\lambda\kappa}F^{1\lambda\kappa} - F^2_{\mu\lambda}F^{2\lambda}_\nu\right. \\ & \left.+\frac{g_{\mu\nu}}{4}F^2_{\lambda\kappa}F^{2\lambda\kappa}\right\} - 8a_1\beta_2\left\{F^1_{\mu\lambda}F^{2\lambda}_\nu\right. \\ & \left.+F^2_{\mu\lambda}F^{1\lambda}_\nu - \frac{g_{\mu\nu}}{2}F^1_{\lambda\kappa}F^{2\lambda\kappa}\right\} \end{aligned} \tag{2.21}$$

where we put $\vec{\beta}\cdot\vec{\beta} \equiv \beta_1 + i\beta_2$. We shall see later that β_1 and β_2 are invariant under any Poincaré gauge transformations.

2.6 Gauge transformations

In this subsection we consider the transformation properties of the constant fields $\vec{\beta}$ and the complex Maxwell field $\mathcal{A}^\mu$ under the Poincar'e gauge transformations (2.1).

We can easily see from the transformation property of $F_{km\mu\nu}$ that the complex field strength $\vec{\mathcal{F}}^{\mu\nu}$ transforms as

$$\delta\vec{\mathcal{F}}^{\mu\nu} = -i\vec{\mathcal{W}}\times\vec{\mathcal{F}}^{\mu\nu} + \xi^\mu{}_{,\lambda}\vec{\mathcal{F}}^{\lambda\nu} + \xi^\nu{}_{,\lambda}\vec{\mathcal{F}}^{\mu\lambda},$$

where we put $(\vec{\mathcal{W}})_a = \omega_{0a} + \frac{i}{2}\epsilon_{abc}\omega_{bc}$, which is just a rotation vector in a complex 3-dimensional space. Then if we assume $\mathcal{F}_{\mu\nu}$ to be a world tensor just like the ordinary electromagnetic field tensor, then from the definition (2.16) and above relation we can get at once the transformation property of $\vec{\beta}$:

$$\delta\vec{\beta} = -i\vec{\mathcal{W}}\times\vec{\beta}. \tag{2.22}$$

On the other hand, from the transformation property of the Lorentz gauge field $A_{km\mu}$ we find

$$\delta\vec{\mathcal{A}}_\mu = -i\vec{\mathcal{W}}\times\vec{\mathcal{A}}_\mu - \xi^\nu{}_{,\mu}\vec{\mathcal{A}}_\nu - \vec{\mathcal{W}}_{,\mu}$$

and therefore, because of (2.22),

$$\delta\mathcal{A}_\mu = -\xi^\nu{}_{,\mu}\mathcal{A}_\nu + \chi_{,\mu}, \tag{2.23}$$

where we put $\chi = -\vec{\alpha}\cdot\vec{\mathcal{W}}$.

Here it should be remarked that the local Lorentz transformations ($dx^\mu = \xi^\mu = 0$, $\delta b_k{}^\mu = -\omega^m{}_k(x)b_m{}^\mu$) are now reduced to the *complex* phase transformations for the complex Maxwell field $\mathcal{A}_\mu$. Accordingly, this fact leads us such an expectation that we may be able to construct the theory for the Lorentz gauge field being now identified with an complex Maxwell field in terms of a gauge theory based on an extended phase transformation. This problem will be discussed in the next section. Additionally, it should be noted that any local Lorentz transformations can be composed of the following two parts:

(1) the rotation about a parallel axis to $\vec{\beta}$: in this case $\vec{\mathcal{W}}\times\vec{\beta} = 0$, so that $\delta\vec{\beta} = 0$ and $\delta\mathcal{A}_\mu = \chi_{,\mu}$.

(2) the rotation about an axis perpendicular to $\vec{\beta}$: in this case $\chi = -\vec{\alpha}\cdot\vec{\mathcal{W}} = 0$, and $\delta\vec{\beta} = -i\vec{\mathcal{W}}\times\vec{\beta}$ and $\delta\mathcal{A}_\mu = 0$.

Finally, we note that because of the transformation property(2.22) of $\vec{\beta}$, the quantity $\vec{\beta}\cdot\vec{\beta}$ (and therefore its real part β_1 and imaginary part β_2 too) is invariant under any Poincaré gauge transformations.

3 Gauge theory based on extended phase transformations

3.1 Extended phase transformation

Let us consider a set of two fields Ψ_1 and Ψ_2, each (both) of which may be a Dirac or Majorana field or others:

$$\Psi = \begin{pmatrix}\Psi_1 \\ \Psi_2\end{pmatrix}.$$

And we assume that the free field action

$$I_\Psi = \int d^4x\,\mathcal{L}(\partial_m\Psi^\dagger,\,\partial_m\Psi,\,\Psi^\dagger,\,\Psi) \tag{3.1}$$

is invariant under the following phase transformations:

$$\Psi' = e^{-i\gamma}\Psi. \tag{3.2}$$

Here γ is a 2×2 hermitian matrix defined as

$$\gamma = aE + ibI,$$

where a and b are real parameters, and E and I are 2×2 unit matrix (will be omitted hereafter) and the 2×2 representation of imaginary unit i, respectively. Mathmatically speaking, those transformations belong to an Abelian subgroup of U(2).

Incidentally, any quantities like $\Psi^\dagger\Psi$ and $\Psi^\dagger\partial_m\Psi, \cdots$ are clearly invariant under above transformations, since the generators E and I commute.

3.2 The gauge theory

We consider here the gauged infinitesimal transformations

$$\delta\Psi = -i\gamma\Psi, \qquad \delta\Psi^\dagger = i\Psi^\dagger\gamma,$$

where

$$\gamma = \gamma(x) \Longrightarrow a = a(x),\ b = b(x).$$

The action (3.1) is no longer invariant under these transformations. However, we can get the invariant action by means of the replacement of the ordinary derivative $\partial_m\Psi$ in the original action by the covariant one $\nabla_m\Psi$. Here the covariant derivative $\nabla_m\Psi$ is given by

$$\nabla_m\Psi = \partial_m\Psi + i\mathcal{A}_m\Psi$$

with

$$\mathcal{A}_m = A^1_m + iA^2_m I,$$

where A^1_m and A^2_m are two real gauge vector fields. In order to keep the action invariance under the gauged transformations the gauge field $\mathcal{A}_m$must have the transformation property

$$\delta\mathcal{A}_m = \partial_m\gamma.$$

3.3 Invariant action

The field strength $\mathcal{F}_{km}$ for the gauge field $\mathcal{A}_m$ can be gotten by calculating the commutator

$$[\nabla_k, \nabla_m]\Psi = i\mathcal{F}_{km}\Psi.$$

Then $\mathcal{F}_{km}$ can be written as

$$\mathcal{F}_{km} = F^1_{km} + iF^2_{km}I$$

with

$$F^1_{km} = \partial_k A^1_m - \partial_m A^1_k$$
$$F^2_{km} = \partial_k A^2_m - \partial_m A^2_k$$

From this field strength $\mathcal{F}_{km}$ and its complex conjugate $\mathcal{F}^*_{km}$ we can make the following invariants:

$$\mathcal{F}_{km}\mathcal{F}^{km},\ \ \mathcal{F}^*_{km}\mathcal{F}^{km},\ \ \mathcal{F}^*_{km}\mathcal{F}^{*km}.$$

And using these invariants we construct the invariant action $I_\mathcal{A}$ for the field $\mathcal{A}_k$. Taking the reality of the action into account we have

$$I_\mathcal{A} = \int d^4x\sqrt{-g}\mathcal{L}_\mathcal{A} \tag{3.3}$$

where[2]

$$\begin{aligned}\mathcal{L}_\mathcal{A} = (\frac{K_1}{2}\mathcal{F}^*_{km}\mathcal{F}_{np} &+ \frac{\mathcal{K}}{4}\mathcal{F}_{km}\mathcal{F}_{np} \\ &+ \frac{\mathcal{K}^*}{4}\mathcal{F}^*_{km}\mathcal{F}^*_{np})g^{kn}g^{mp}.\end{aligned}$$

K_1 is a real constant and $\mathcal{K} = K_2 + iK_3 I$ for reals K_2 and K_3.

Next, using the action (3.3) we are going to calculate the energy-momentum tensor for the field $\mathcal{A}_k$. Accoding to the ordinary procedure[9] we can get easily the result

$$\begin{aligned} T_{ik} &= 2K_1(\mathcal{F}^*_{im}\mathcal{F}_k{}^m + \mathcal{F}^*_{km}\mathcal{F}_i{}^m) \\ &+ \mathcal{K}(\mathcal{F}_{im}\mathcal{F}_k{}^m + \mathcal{F}_{km}\mathcal{F}_i{}^m) \\ &+ \mathcal{K}^*(\mathcal{F}^*_{im}\mathcal{F}^{*m}_k + \mathcal{F}^*_{km}\mathcal{F}^{*m}_i) - 2g_{ik}\mathcal{L}_\mathcal{A},\end{aligned}$$

or

$$\begin{aligned} T_{ik} &= 4(K_1+K_2)(F^1_{im}F^{1m}_k - \frac{1}{4}g_{ik}F^1_{mn}F^{1mn}) \\ &- 4(K_1-K_2)(F^2_{im}F^{2m}_k - \frac{1}{4}g_{ik}F^2_{mn}F^{2mn}) \\ &+ 4K_3(F^1_{im}F^{2m}_k + F^1_{km}F^{2m}_i \\ &- \frac{1}{2}g_{ik}F^1_{mn}F^{2mn})\end{aligned} \tag{3.4}$$

This expression should be compared with the energy-momentum tensor for the Lorentz gauge field (2.21), then we find that both energy-momentum tensors coincide if

$$K_1 = 2a_1\beta_1,\ K_2 = 0,\ K_3 = -2a_1\beta_2. \tag{3.5}$$

[2]Here we are treating the field in a flat spacetime background. But we use formally the metric g_{km}, $g^{km}$to get a symmetric energy-momentum tensor,

3.4 Dirac field as a source

We here assume that the field Ψ is a set of two Dirac fields ψ^1 and ψ^2. Then we have the following invariant action for the total system:

$$I_T = \int d^4x\left(\frac{i}{2}\overline{\Psi}\gamma^k\nabla_k\Psi - \frac{i}{2}\nabla_k\overline{\Psi}\gamma^k\Psi - m\overline{\Psi}\Psi\right) + \int d^4x\left(\frac{K_1}{4}\mathcal{F}^*_{km}\mathcal{F}^{km} + \frac{\mathcal{K}}{4}\mathcal{F}_{km}\mathcal{F}^{km} + c.c\right) \tag{3.6}$$

where Ψ and γ^k are given in terms of two 4-component Dirac fields ψ^1 and ψ^2 and Dirac's gamma matrices γ_D by

$$\Psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} \quad \text{and} \quad \gamma^k = \begin{pmatrix} \gamma^k_D & 0 \\ 0 & \gamma^k_D \end{pmatrix},$$

respectively.

By the variational principle we can derive the following equations:

$$\left(i\gamma^k\nabla_k - m\right)\Psi = 0,$$

or, in terms of each field ψ^1, ψ^2

$$i\gamma^k(\partial_k + iA^1_k)\psi^1 - m\psi^1 = iA^2_k\gamma^k\psi^2, \tag{3.7}$$

$$i\gamma^k(\partial_k + iA^1_k)\psi^2 - m\psi^2 = -iA^2_k\gamma^k\psi^1, \tag{3.8}$$

and also for the component fields of complex Maxwell field

$$\begin{pmatrix} F^{1km}{}_{,m} \\ F^{2km}{}_{,m} \end{pmatrix} = \frac{-1}{\det K} K \begin{pmatrix} J^{1k} \\ J^{2k} \end{pmatrix}, \tag{3.9}$$

where K is a 2×2 matrix

$$K = \begin{pmatrix} K_1 - K_2 & K_3 \\ K_3 & -(K_1 + K_2) \end{pmatrix},$$

and the currents $J^{1k}$and J^{2k} are put as

$$J^{1k} = \frac{1}{2}(\overline{\psi}^1\gamma^k_D\psi^1 + \overline{\psi}^2\gamma^k_D\psi^2), \tag{3.10}$$

$$J^{2k} = \frac{i}{2}(\overline{\psi}^1\gamma^k_D\psi^2 - \overline{\psi}^2\gamma^k_D\psi^1). \tag{3.11}$$

In particular, if we adopt the values (3.5) for K's and put

$$J^{1\mu} = \frac{1}{2}(\vec{\beta}_R \cdot \vec{S}_{MR}{}^{\mu} - \vec{\beta}_I \cdot \vec{S}_{MI}{}^{\mu})$$

$$J^{2\mu} = -\frac{1}{2}(\vec{\beta}_R \cdot \vec{S}_{MI}{}^{\mu} + \vec{\beta}_I \cdot \vec{S}_{MR}{}^{\mu}),$$

then both equations (2.19,2.20) and (3.9) coincide in flat space-time ($b_k{}^\mu = \delta_k{}^\mu$), where $\vec{\beta}_R$, $\vec{\beta}_I$ and $\vec{S}_{MR}{}^\mu$, $\vec{S}_{MI}{}^\mu$ are the real and imaginary parts of $\vec{\beta}$ and $\vec{S}_M{}^\mu$, respectively.

This coincidence, together with the coincidence of the energy-momentum tensor, appears to imply an equivalency of the complex Maxwell theory reduced from PGT and one derived from the gauge theory based on extended phase transformations. If so, we can conclude that the Lorentz gauge field must be generated by a pair of at least two Dirac fields. Here the term "at least" means the fact that since the complex Maxwell field $\mathcal{A}_\mu$ is a special one of the Lorentz gauge field: $A_{km\mu} \leftrightarrows \vec{\mathcal{A}}_\mu \Longrightarrow \vec{\beta}\mathcal{A}_\mu$, we may need generally more than two Dirac fields.

4 Summary and conclusion

In this paper we treate with a model of Poincaré gauge theory which is reducible to a complex Einstein-Maxwell theory. On the other hand, we propose a gauge theory based on an extended phase transformation. And we have shown that both theories can coincide at least formally. Accoding to this coincidence, we infer that the Lorentz gauge field should be created by a pair of at least two Dirac fields. However, we do not know at this stage how many Dirac fields are needed. This problem may be solved by researching a gauge theory equivalent to a complex Yang-Mills theory. Anyway, it seems to be interesting to discuss about a black hole of such sources as being able to create the Lorentz gauge field.

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