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Significance of the companion matrix in homogeneous linear ordinary differential equation

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Abstract Equivalence of a linear system of n differential equations $\dot{\mathbf{x}} = A\mathbf{x}$ with a non-singular matrix A, to a one-variable homogeneous ordinary linear differential equation of rank n, is discussed by means of the companion matrix of the Frobenius form. A Krylov sequence of vectors is involved in the construction of a matrix to transform the system of differential equations to a one-variable differential equation. The discussion is based on the linear and combinatorial algebra. Solution of the homogeneous linear ordinary differential equation is derived through the companion matrix.
Keywords: Companion matrix; Frobenius form; Krylov sequence; Combinatorics.

1. Introduction

The Newton equation of motion gives Hamilton equations. The Hamilton equations are equivalently represented as Lagrange equations which yield an Euler-Lagrange equation [1]. In this statement, it is worthwhile to note that there exists a certain equivalence between the homogeneous linear differential equation of a variable of rank n and a linear system of n differential equations $\dot{\mathbf{x}} = A\mathbf{x}$ with a coefficient matrix A of rank n. Another example of this kind is found in the theory of relaxation as the relation of differential general linear equation of a pair of macroscopic conjugate variables to the linear system of differential equations on n pairs of microscopic conjugate variables [4]. The macroscopic variables are usually observable physical quantities, while the microscopic variables difficult to observe, consist of n pairs of conjugate variables corresponding to n different relaxation times. Conjugate variables are, for instance, strain vs. stress, temperature vs. entropy, electric displacement vs. electric field, magnetic flux density vs. magnetic field, chemical potential vs. concentration and so on.

The above relation is summarized to an equivalent relation between a one-variable linear differential equation of rank n and a system $\dot{\mathbf{x}} = A\mathbf{x}$ with $A \in \mathrm{GL}(n;K)$ (K: Field), where $\mathrm{GL}(n;K)$ is the group of all general linear transformations of n-dimensional vector space over K or all non-singular matrices of order n with K components. Let $f(z)$ be a polynomial of degree n and $d_t \overset{\text{def}}{=} \frac{d}{dt}$. For a given $f(d_t)x = 0$, a companion matrix of f gives a linear system $\dot{\mathbf{x}} = A\mathbf{x}$ with $\mathbf{x} = (x_i)$, $x_1 = x$ and $x_i = \dot{x}_{i-1}$ $(i = 2,3,\cdots,n)$. The converse does not always hold. For instance, a symmetric matrix with a 2-folded eigenvalue is not similar to any companion matrix. This paper deals with such converse problem and discussion leads to solution of the homogeneous linear ordinary equation through the companion matrix.

2. Companion Matrix

Let $f(z)$ be a monic polynomial in a K coefficient polynomial ring $K[z]$ with a field K:

$$(1) \qquad f(z) = \sum_{i=0}^{n} a_i z^{n-i} \quad (a_i \in K, a_0 = 1).$$

In practice, K is the real or complex number field. The companion matrix of f is defined as a square matrix A of order n whose characteristic polynomial is f; that is, $\Phi_A(z) \overset{\text{def}}{=} |zE - A| = f(z)$, where $\Phi_A(z)$ is the characteristic polynomial of A, and E the unit matrix. Matrices

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

are often cited as companion matrices of the Frobenius form [2, 3]. Hereafter, A_1 is denoted by A_f. Let P be a non-singular matrix, i.e., $P \in \mathrm{GL}(n;K)$. Since $\Phi_{P^{-1}AP}(z) = \Phi_A(z)$, $P^{-1}AP$ with a companion matrix A of f is also a companion matrix of f. The converse does not hold; in fact, for $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, A and B are companion matrices of $(z-1)^2$, although A is not similar to B. By Hamilton-Cayley's theorem, $\Phi_A(A) = \mathrm{O}$. Then,

Proposition 1. $\exists P \in \mathrm{GL}(n:K);\ P^{-1}AP = A_f \Rightarrow f(A) = \mathrm{O}$.

Let $N = \{1, 2, \cdots, n\}$ and $\Omega \subset N$ with $|\Omega| = i$ where $|\Omega|$ stands for the order of $|\Omega|$. A submatrix of A associated with Ω , denoted by A_Ω, is defined as a matrix whose jth rows and columns are deleted from A for all $j \in \Omega$; $A_N \overset{\text{def}}{=} (1)$.

Theorem 2. *If A is a companion matrix of f, then $a_i = (-1)^i \sum_{|\Omega|=n-i} |A_\Omega|$, where the summation ranges over all Ω with $|\Omega| = n - i$ and $|A_\Omega|$ exhibits the determinant of A_Ω.*

Proof. Since A is a companion matrix of f,

(2) $\qquad |zE - A| = \sum_{i=0}^{n} a_i z^{n-i} \quad (a_0 = 1)$.

By definition of the determinant, $|zE - A| = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}\,\sigma \prod_{j=1}^{n} \left(\delta_{j\sigma(j)} z - a_{j\sigma(j)}\right)$. Here δ_{ij} is Kronecker's delta and $\mathrm{sgn}\,\sigma$ the signature of a permutation σ in the symmetric group $\mathfrak{S}_n$ of order n. By comparing the coefficients of degree $n-i$ in (2),

$$a_i = \sum_{|\Omega|=n-i} \sum_{\sigma \in \mathfrak{S}(N-\Omega)} \mathrm{sgn}\,\sigma \prod_{j \in N-\Omega} \left(-a_{j\sigma(j)}\right) .$$

Here, $\mathfrak{S}(N-\Omega)$ is the set of all bijective transformations of $N - \Omega$, and the first summation is carried out over all subsets of N of order $n-i$. Then, $a_i = (-1)^i \sum_{|\Omega|=n-i} |A_\Omega|$. ■

Especially for $i = 1$ and n, it follows directly from $\mathrm{Tr}(P^{-1}AP) = \mathrm{Tr}A$ and $|P^{-1}AP| = |A|$ that $a_1 = -\mathrm{Tr}A$ and $a_n = (-1)^n |A|$. The following is readily deduced from Theorem 2.

Corollary 3. $\exists P \in \mathrm{GL}(n;K);\ P^{-1}AP = A_f \quad \Rightarrow a_i = (-1)^i \sum_{|\Omega|=n-i} |A_\Omega|$.

3. Homogeneous Linear Ordinary Differential Equation (HLODE)

Let K be a topological field, $C^\infty(K)$ the set of all infinitely differentiable functions. Substitution of d_t for z in (1) yields a differential operator $f(d_t)$ of $C^\infty(K)$ to $C^\infty(K)$. Let x be a function of t $(\in K)$ and consider the homogeneous linear ordinary differential equation:

(3) $\qquad f(d_t)x = 0$.

Here, $d_t{}^0 \overset{\text{def}}{=} I$ with the identity operator I. Equation (3) is written in the form of $\dot{\mathbf{x}} = A_f \mathbf{x}$ with $\mathbf{x} = (x_i)$, $x_1 = x$, $x_2 = \dot{x}$, $\cdots$, $x_n = x^{(n-1)} = d_t{}^{n-1}x$ as mensioned in Introduction.

Now, consider the converse problem to find a representation of (3) equivalent to a given system of linear differential equations $\dot{\mathbf{x}} = A\mathbf{x}$ $(A \in \mathrm{GL}(n;K))$.

Let $P = (\mathbf{p}_1 \mathbf{p}_2 \cdots \mathbf{p}_n)$ with column vectors $\mathbf{p}_i$. $AP = PA_f$ gives $A\mathbf{p}_1 = -a_n \mathbf{p}_n$, $A\mathbf{p}_2 = \mathbf{p}_1 - a_{n-1}\mathbf{p}_n$, $\cdots$, $A\mathbf{p}_n = \mathbf{p}_{n-1} - a_1 \mathbf{p}_n$. Then, $\mathbf{p}_i = \left(A^{n-i} + a_1 A^{n-i-1} + \cdots + a_{n-i}E\right)\mathbf{p}_n$. Thus, the following proposition holds.

Proposition 4. $AP = PA_f \Rightarrow P = (\mathbf{p}_i)$, $\mathbf{p}_i = \left(A^{n-i} + a_1 A^{n-i-1} + \cdots + a_{n-i}E\right)\mathbf{p}_n$.

The converse of Prop. 4 holds for $P \in \mathrm{GL}(n;K)$.

Proposition 5. *Let* $P \in \mathrm{GL}(n;K)$.

$$P = (\mathbf{p}_i),\ \ \mathbf{p}_i = \left(A^{n-i} + a_1 A^{n-i-1} + \cdots + a_{n-i}E\right)\mathbf{p}_n \ \ \Rightarrow \ \ AP = PA_f$$

Proof. It suffices to show $A\mathbf{p}_1 = -a_n\mathbf{p}_n$. By Prop. 1, $A^n + a_1A^{n-1} + \cdots + a_{n-1}A = -a_nE$.
Since $\mathbf{p}_1 = \left(A^{n-1} + a_1A^{n-2} + \cdots + a_{n-1}E\right)\mathbf{p}_n$, $A\mathbf{p}_1 = -a_n\mathbf{p}_n$. ■

Let $\mathbf{x}_0$ be a given vector and $\dot{\mathbf{x}} = A\mathbf{x}$ with $\mathbf{x}(0) = \mathbf{x}_0$. Then, $\mathbf{x} = (\exp tA)\mathbf{x}_0$ is the unique solution of the initial value problem of $\dot{\mathbf{x}} = A\mathbf{x}$ with $\mathbf{x}(0) = \mathbf{x}_0$. In the case of $AP = PA_f$ with $P \in \mathrm{GL}(n;K)$ and by setting $\mathbf{x} = P\mathbf{y}$, $\mathbf{y} = \left(\exp tA_f\right)P^{-1}\mathbf{x}_0$ is the solution of $\dot{\mathbf{y}} = A_f\mathbf{y}$, $\mathbf{y}(0) = P^{-1}\mathbf{x}_0$.

4. Jordan Canonical Form

Let J be a matrix of the Jordan canonical form similar to A, i.e.,$\exists U \in \mathrm{GL}(n,K)$; $J = U^{-1}AU$. Suppose $\exists V \in \mathrm{GL}(n,K)$; $JV = VA_f$. Then, $A(UV) = (UV)A_f$. Now, let $P = (\mathbf{p}_1\,\mathbf{p}_2 \cdots \mathbf{p}_n)$ with $\mathbf{p}_j \neq \mathbf{0}$ $(j = 1,2,\cdots,n)$ and $\mathbf{p}_j = {}^t\left(p_{1j}p_{2j}\cdots p_{nj}\right)$ satisfying $A_fP = PJ$. Two cases are considered according to diagonal and nondiagonal J.

Case 1. J: Diagonal. Let λ_i $(i = 1,2,\cdots,n)$ be eigenvalues A_f. By the assumption of $A \in \mathrm{GL}(n;K)$ or $A_f \in \mathrm{GL}(n,K)$, all λ_i's are nonzero. P is assumed to be a matrix related to A_f as

$$A_fP = P\begin{pmatrix} \lambda_1 & & & \mathrm{O} \\ & \lambda_2 & & \\ & & \ddots & \\ \mathrm{O} & & & \lambda_n \end{pmatrix}.$$

Hence, $A_f\mathbf{p}_j = \lambda_j\mathbf{p}_j$ $(j = 1,2,\cdots,n)$. Thus, $p_{i+1\,j} = \lambda_j p_{ij} = \lambda_j{}^i p_{1j}$ $(i = 1,2,\cdots,n-1)$. Then, $\mathbf{p}_j = p_{1j}{}^t\left(1\lambda_j\cdots\lambda_j{}^{n-1}\right) = \mathbf{0}$.
Therefore,

$$|P| = p_{11}p_{12}\cdots p_{1n}\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & & & \vdots \\ \lambda_1{}^{n-1} & \lambda_2{}^{n-1} & \cdots & \lambda_n{}^{n-1} \end{vmatrix} = \prod_{j=1}^{n} p_{1j}\cdot\prod_{i>j}\left(\lambda_i - \lambda_j\right).$$

Hence, it follows that

Proposition 6. (1) $\lambda_i \neq \lambda_j\ (i \neq j) \Rightarrow |P| \neq 0$. (2) $\exists i,j\ (i \neq j); \lambda_i = \lambda_j \ \Rightarrow \ |P| = 0$.

Corollary 7. $\exists P \in \mathrm{GL}(n;K); AP = PA_f \ \Leftrightarrow \ \lambda_i \neq \lambda_j\ (i \neq j)$.

Corollary 7 implies that all eigenvalues of a matrix similar to a non-singular A_f are different one another, in case the Jordan canonical form of A_f is diagonal.

Case 2. J: Nondiagonal, i.e., there exists a Jordan block of J of order larger than 1.

$$J = \begin{pmatrix} J_1 & & & \mathrm{O} \\ & J_2 & & \\ & & \ddots & \\ \mathrm{O} & & & J_r \end{pmatrix}, \ J_i\text{: } n_i \text{ Jordan block such that } \begin{pmatrix} \lambda_i & 1 & & \mathrm{O} \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ \mathrm{O} & & & \lambda_i \end{pmatrix}.$$

J is denoted by $\bigoplus_{i=1}^{r} J_i$ or $J_1 \oplus J_2 \oplus \cdots \oplus J_r$. The characteristic polynomial of J is $f(z) = \prod_{i=1}^{r}\left(z - \lambda_i\right)^{n_i}$, with $\lambda_i \neq \lambda_j$ $(i \neq j)$, $\sum_{i=1}^{r} n_i = n$. Let $A_f^{(i)}$ denote the companion matrix of $\left(z - \lambda_i\right)^{n_i}$. P_i such that $A_f^{(i)}P_i = P_iJ_i$ $(i = 1,2,\cdots,r)$ results in $\left(\bigoplus_{i=1}^{r} A_f^{(i)}\right)\left(\bigoplus_{i=1}^{r} P_i\right) = \left(\bigoplus_{i=1}^{r} P_i\right)\left(\bigoplus_{i=1}^{r} J_i\right)$. Then, it suffices to discuss the case of $r = 1$; that is, $J = \begin{pmatrix} \lambda & 1 & & \mathrm{O} \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ \mathrm{O} & & & \lambda \end{pmatrix}$.

5. Krylov Sequence of Vectors

Since $A \in \mathrm{GL}(n,K)$ is assumed, $\lambda \neq 0$. With $\hat{A}_f \overset{\text{def}}{=} A_f - \lambda E$, $A_f P = PJ$ yields

$$\hat{A}_f \mathbf{p}_1 = \mathbf{0}\ ,\ \hat{A}_f \mathbf{p}_{i+1} = \mathbf{p}_i \quad (i = 1,2,\cdots,n-1)\ \text{, or }\ \hat{A}_f P = PE_1\ ,\quad E_1 = \begin{pmatrix} 0 & 1 & & \mathrm{O} \\ & 0 & \ddots & \\ & & \ddots & 1 \\ \mathrm{O} & & & 0 \end{pmatrix}.$$

Such a sequence of vectors as $\{\mathbf{p}_i\}$ defined by $\mathbf{p}_i = \hat{A}_f \mathbf{p}_{i+1}$ with $\mathbf{p}_n \neq \mathbf{0}$ is called the Krylov sequence associated with $\hat{A}_f$ [3]. By setting $\mathbf{p}_n = \mathbf{p}$,

$$P = (\mathbf{p}_1 \mathbf{p}_2 \cdots \mathbf{p}_n) = (\hat{A}_f \mathbf{p}_2\ \hat{A}_f \mathbf{p}_3 \cdots \hat{A}_f \mathbf{p}_n\ \mathbf{p}_n) = (\hat{A}_f{}^{n-1}\mathbf{p}\ \hat{A}_f{}^{n-2}\mathbf{p} \cdots \hat{A}_f \mathbf{p}\ \mathbf{p}).$$

Now, consider the determinant of P. Since $\mathbf{p}_i = \hat{A}_f{}^{n-i}\mathbf{p} = (A_f - \lambda E)^{n-i}\mathbf{p}$, $\mathbf{p}_i$ is a linear combination of $A_f{}^{n-i}\mathbf{p}$, $A_f{}^{n-i-1}\mathbf{p},\cdots,\mathbf{p}$. Hence, $|P| = |A_f{}^{n-1}\mathbf{p}\ A_f{}^{n-2}\mathbf{p} \cdots A_f \mathbf{p}\ \mathbf{p}|$.

6. Construction of Non-singular Matrix *P* for Nondiagonal *J*

The problem to solve is to show the existence of $P \in \mathrm{GL}(n;K); A_f P = PJ$ and construct such P. Let A_{ij} and E_i be defined as

$$A_{ij} \overset{\text{def}}{=} \left(a_{\mu\nu}^{(ij)}\right),\ a_{\mu\nu}^{(ij)} = \begin{cases} -a_{n-(\nu-j)+1} & (\mu = n-i) \\ 0 & (\text{otherwise}) \end{cases},\ E_i = \left(e_{\mu\nu}^{(i)}\right),\ e_{\mu\nu}^{(i)} = \begin{cases} 1 & (\nu - \mu = i) \\ 0 & (\text{otherwise}) \end{cases},$$

which are further explicitly represented as

$$A_{ij} = \begin{pmatrix} \overbrace{0 & \cdots & 0}^{j} & & \mathrm{O} & \\ \vdots & & \vdots & & & \\ 0 & \cdots & 0 & -a_n & \cdots & -a_{j+1} \\ 0 & \cdots & 0 & \cdots & \cdots & 0 \\ \vdots & & \vdots & & & \vdots \\ 0 & \cdots & 0 & \cdots & \cdots & 0 \end{pmatrix} \begin{matrix} \\ \\ \\ \\ \\ \end{matrix} \left. \right\}i\ ,\qquad E_i = \begin{pmatrix} \overbrace{0 & \cdots & 0}^{i} & 1 & & \mathrm{O} \\ \vdots & & \vdots & & \ddots & \\ 0 & \cdots & 0 & \mathrm{O} & & 1 \\ 0 & \cdots & 0 & \cdots & \cdots & 0 \\ \vdots & & \vdots & & & \vdots \\ 0 & \cdots & 0 & \cdots & \cdots & 0 \end{pmatrix} \left. \right\}i$$

Then, $A_f = A_{00} + E_1$.

Proposition 8. (1) $A_{ij}A_{kl} = -a_{j+k+1}A_{il}$, (2) $E_i A_{jk} = A_{i+jk}$, $A_{jk}E_i = A_{jk+i}$, (3) $E_i E_j = E_{i+j}$, $E_i{}^m = E_{mi}$.

Proof. (1)

$$A_{ij}A_{kl} = \begin{pmatrix} & & & & \mathrm{O} & & & & \\ \overbrace{0 & \cdots & 0}^{j} & -a_n & -a_{n-1} & \cdots & \overset{n-k}{-a_{j+k+1}} & \cdots & -a_{j+1} \\ & & & & \mathrm{O} & & & & \end{pmatrix} i\{ \begin{pmatrix} & & & & \mathrm{O} & & \\ \overbrace{0 & \cdots & 0}^{l} & -a_n & \cdots & -a_{l+1} \\ & & & & \mathrm{O} & & \end{pmatrix} (n-k$$

$$= -a_{j+k+1} \begin{pmatrix} \overbrace{0 & \cdots & 0}^{l} & & \mathrm{O} & \\ \vdots & & \vdots & & & \\ 0 & \cdots & 0 & -a_n & \cdots & -a_{l+1} \\ \vdots & & \vdots & & & \\ 0 & \cdots & 0 & & \mathrm{O} & \end{pmatrix} (n-i \quad = a_{j+k+1}A_{il}$$

(2) For $i+j<n$,

$$E_iA_{jk}=\begin{pmatrix} \overbrace{0 & \cdots & 0}^{i} & 1 & & O \\ \vdots & & \vdots & & \ddots & \\ 0 & \cdots & 0 & O & & 1 \\ 0 & \cdots & 0 & \cdots & \cdots & 0 \\ \vdots & & \vdots & & & \\ 0 & \cdots & 0 & \cdots & \cdots & 0 \end{pmatrix}\begin{pmatrix} \overbrace{0 & \cdots & 0}^{k} & & O & \\ \vdots & & & & & \\ 0 & \cdots & 0 & -a_n & \cdots & -a_{k+1} \\ \vdots & & \vdots & & & \\ 0 & \cdots & 0 & & O & \end{pmatrix}\left(n-j\right.$$

$$=\begin{pmatrix} \overbrace{0 & \cdots & 0}^{k} & & O & \\ \vdots & & \vdots & & & \\ 0 & \cdots & 0 & -a_n & \cdots & -a_{k+1} \\ \vdots & & \vdots & & & \\ 0 & \cdots & 0 & & O & \end{pmatrix}\left(n-i-j\right. = A_{i+jk}$$

Similarly, $A_{jk}E=A_{jk+i}$. (3) Readily proved. ■

N.B. (2) holds for $i+j<n$ $(k+i<n)$. If $i+j\geq n$ $(k+i\geq n)$, then $E_iA_{jk}=O$ $\left(A_{jk}E_i=O\right)$. The same remarks are requisite for (3).

By Prop. 8, only A_{ij}'s $(i+j\leq m-1)$ appear in $A_f{}^m$. Then, $A_f{}^m$ is written as

$$A_f{}^m=\sum_{i=0}^{m-1}\sum_{j=0}^{m-1-i}c_{ij}^{(m)}A_{ij}+E_m \quad (m\geq 1) \quad \text{and} \quad A_f{}^0\overset{\text{def}}{=}E_0 \ .$$

Proposition 9. (1) $c_{0i+1}^{(m+1)}=c_{0i}^{(m)}$ $(m\geq 1)$, (2) $c_{ij}^{(m)}=c_{0i+j}^{(m)}$ $(i+j\leq m-1)$, (3) $c_{ij}^{(m)}=c_{00}^{(m-i-j)}$ $(i+j\leq m-1)$.

Proof. (3) is easily derived by applying (2) and then (1) to $c_{ij}^{(m)}$.

(1) $A_f{}^{m+1}=A_f{}^m\left(A_{00}+E_1\right)$. Since only A_{0i} in the right can join A_{0i+1} as $A_{0i+1}=A_{0i}E_1$, $c_{0i+1}^{(m+1)}=c_{0i}^{(m)}$.

(2) The condition $i+j\leq m-1$ is required from the definition of A_f^m.

$$\left(A_{00}+E_1\right)^m=\sum_{(e_1e_2\cdots e_m)\in\{0,1\}^m}\left(A_{00}{}^{1-e_1}E_1{}^{e_1}\right)\left(A_{00}{}^{1-e_2}E_1{}^{e_2}\right)\cdots\left(A_{00}{}^{1-e_m}E_1{}^{e_m}\right) , A_f^m$$

where $\{0,1\}^m$ is the product set of $\{0,1\}$. In the right, $c_{ij}^{(m)}$ is related to such terms as $\overbrace{E_1E_1\cdots E_1}^{i}A_{00}\hat{A}A_{00}\overbrace{E_1E_1\cdots E_1}^{j}$ with $\hat{A}=\left(A_{00}{}^{1-e_{i+2}}E_1{}^{e_{i+2}}\right)\left(A_{00}{}^{1-e_{i+3}}E_1{}^{e_{i+3}}\right)\cdots\left(A_{00}{}^{1-e_{m-j-1}}E_1{}^{e_{m-j-1}}\right)$. The value $c_{ij}^{(m)}$ is determined only by $A_{00}\hat{A}A_{00}$ and so equal to $c_{0i+j}^{(m)}$ derived from $A_{00}\hat{A}A_{00}\overbrace{E_1E_1\cdots E_1}^{i+j}$. ■

From Prop. 9, it suffices to derive $c_{00}^{(m)}$. For simplicity, $c_{00}^{(m)}$ is, hereafter, denoted by $c^{(m)}$.

Proposition 10. (1) $c^{(1)}=1$, (2) $c^{(m)}=\sum_{i=1}^{m-1}\left(-a_i\right)c^{(m-i)}$ $(m\geq 2)$.

Proof. (1) $A_f=A_{00}+E_1$. $\therefore$ $c^{(1)}=1$. (2) $c^{(m)}A_{00}=A_{00}\sum_{i=0}^{m-2}c_{i0}^{(m-1)}A_{i0}=\sum_{i=0}^{m-2}\left(-a_{i+1}\right)c_{i0}^{(m-1)}A_{00}=\sum_{i=0}^{m-2}\left(-a_{i+1}\right)c_{00}^{(m-1-i)}A_{00}$ $=\sum_{i=1}^{m-1}\left(-a_i\right)c_{00}^{(m-i)}A_{00}$. ■

Let $Q=\left(q_{ij}\right)=\left(A_f^{n-1}\mathbf{p}\, A_f^{n-2}\mathbf{p}\cdots A_f\mathbf{p}\ \mathbf{p}\right)$, $\mathbf{p}={}^t\left(p_1p_2\cdots p_n\right)$, and $\mathbf{a}_j'=\left(\overbrace{0\cdots 0}^{j}-a_n-a_{n-1}\cdots-a_{j+1}\right)=$(ith row of A_{n-ij}) .

Proposition 11. $q_{ij} = \begin{cases} p_{n+i-j} & (i \le j) \\ \sum_{k=1}^{i-j} c^{(k)} \mathbf{a}'_{i-j-k}\mathbf{p} & (i > j) \end{cases}$.

Proof. $A_f^{n-j} = \sum_{k=0}^{n-j-1} \sum_{l=0}^{n-j-1-k} c_{kl}^{(n-j)} A_{kl} + E_{n-j} = \sum_{k=j+1}^{n} \sum_{l=0}^{k-j-1} c_{n-kl}^{(n-j)} A_{n-kl} + E_{n-j}$.

For $i \le j$, $q_{ij} = \left(E_{n-j}\mathbf{p}\right)_i = p_{n+i-j}$. For $i > j$, $\left(A_{n-kl}\mathbf{p}\right)_i = 0 \quad (k \ne i)$. By Prop. 9 (3), $c_{n-kl}^{(n-j)} = c^{(k-j-l)}$. Hence,

$$q_{ij} = \left(\sum_{l=0}^{i-j-1} c^{(i-j-l)} A_{n-il}\mathbf{p}\right)_i .$$

Let $k = i - j - l$. Then, $q_{ij} = \sum_{k=1}^{i-j} c^{(k)} \mathbf{a}'_{i-j-k}\mathbf{p}$. ∎

From Prop. 11 follows

Corollary 12. For $k \le n - \max\{i,j\}$, $k \le n - \max\{i,j\}$.

Definition 13. $q_{ij}^{(k)} \overset{\text{def}}{=} \begin{cases} q_{ij}^{(k-1)} & (i \le k) \\ q_{ij}^{(k-1)} - \lambda q_{i-1j}^{(k-1)} & (i > k) \end{cases}$ with $q_{ik}^{(0)} = q_{ik}$, $r_{ij} \overset{\text{def}}{=} \sum_{k=j}^{n} \binom{n-j}{k-j} (-\lambda)^{k-j} q_{ik}^{(j)} \quad (i = 1,2,\cdots,n)$.

Proposition 14. (1) $q_{i+lj+l}^{(k)} = q_{ij}^{(k)} \; (i > k)$, (2) $r_{i+lj+l} = r_{ij}$.

Proof. (1) holds for $k = 1$ by Cor. 12. For $i > k$, $q_{i+lj+l}^{(k)} = q_{i+lj+l}^{(k-1)} - \lambda q_{i+l-1j+l}^{(k-1)} = q_{ij}^{(k-1)} - \lambda q_{i-1j}^{(k-1)} = q_{ij}^{(k)}$, where the second equality is asserted by the supposition of mathematical induction on k.

(2) It suffices to show $r_{i+lj+l} = r_{i+l-1\,j+l-1}$. For $i + l \le j + l$, i.e., $i \le j$, Def. 13 yields $q_{i+lk}^{(j+l)} = q_{i+lk}^{(i+l-1)} = q_{i+lk}^{(i+l-2)} - \lambda q_{i+l-1k}^{(i+l-2)}$. Hence,

$$\begin{aligned}
r_{i+lj+l} &= \sum_{k=j+l}^{n} \binom{n-j-l}{k-j-l} (-\lambda)^{k-j-l} \left(q_{i+lk}^{(i+l-2)} - \lambda q_{i+l-1k}^{(i+l-2)}\right) \\
&= \sum_{k=j+l}^{n} \binom{n-j-l}{k-j-l} (-\lambda)^{k-j-l} q_{i+lk}^{(i+l-2)} + \sum_{k=j+l+1}^{n} \binom{n-j-l}{k-j-l-1} (-\lambda)^{k-j-l} q_{i+l-1k-1}^{(i+l-2)} + (-\lambda)^{n-j-l+1} q_{i+l-1n}^{(i+l-2)} \\
&= q_{i+lj+l}^{(i+l-2)} + \sum_{k=j+l+1}^{n} \binom{n-j-l+1}{k-j-l} (-\lambda)^{k-j-l} q_{i+lk}^{(i+l-2)} + (-\lambda)^{n-j-l+1} q_{i+l-1n}^{(i+l-2)} \\
&\qquad \left(\because \binom{n-j-l}{k-j-l} + \binom{n-j-l}{k-j-l-1} = \binom{n-j-l+1}{k-j-l},\ q_{i+l-1k-1}^{(i+l-2)} = q_{i+lk}^{(i+l-2)} \text{ by (1)}\right) \\
&= \sum_{k=j+l-1}^{n} \binom{n-(j+l-1)}{k-(j+l-1)} (-\lambda)^{k-(j+l-1)} q_{i+l-1k}^{(i+l-2)} = r_{i+l-1j+l-1} \quad \left(\because q_{i+l-1k}^{(i+l-2)} = q_{i+l-1k}^{(j+l-1)} \text{ by Def. 13}\right).
\end{aligned}$$

For $i > j$,

$$\begin{aligned}
r_{i+lj+l} &= \sum_{k=j+l}^{n} \binom{n-j-l}{k-j-l} (-\lambda)^{k-j-l} q_{i+lk}^{(j+l)} = \sum_{k=j+l}^{n} \binom{n-j-l}{k-j-l} (-\lambda)^{k-j-l} \left(q_{i+lk}^{(j+l-1)} - \lambda q_{i+l-1k}^{(j+l-1)}\right) \\
&= \sum_{k=j+l}^{n} \binom{n-j-l}{k-j-l} (-\lambda)^{k-j-l} q_{i+lk}^{(j+l-1)} + \sum_{k=j+l+1}^{n} \binom{n-j-l}{k-j-l-1} (-\lambda)^{k-j-l} q_{i+l-1\,k-1}^{(j+l-1)} + (-\lambda)^{n-j-l+1} q_{i+l-1n}^{(j+l-1)} \\
&= q_{i+lj+l}^{(j+l-1)} + \sum_{k=j+l+1}^{n} \binom{n-j-l+1}{k-j-l} (-\lambda)^{k-j-l} q_{i+lk}^{(j+l-1)} + (-\lambda)^{n-j-l+1} q_{i+l-1n}^{(j+l-1)} \quad \left(\because q_{i+l-1\,k-1}^{(j+l-1)} = q_{i+l\,k}^{(j+l-1)}\right) \\
&= \sum_{k=j+l-1}^{n} \binom{n-(j+l-1)}{k-(j+l-1)} (-\lambda)^{k-(j+l-1)} q_{i+l-1k}^{(j+l-1)} = r_{i+l-1j+l-1} .
\end{aligned}$$

∎

For $i > j$, p_{n+i-j} is defined as $p_{n+i-j} \overset{\text{def}}{=} q_{ij}$. Then, $q_{ij} = p_{n+i-j} \; (1 \le i,j \le n)$, $q_{ij}^{(1)} = p_{n+i-j} - \lambda p_{n+i-j-1}$.

The matrix $\left(q_{ij}\right)$ is written as $\left(q_{ij}\right) = \begin{pmatrix} p_n & \cdots & p_2 & p_1 \\ p_{n+1} & \ddots & p_3 & p_2 \\ \vdots & \ddots & \vdots & \vdots \\ p_{2n-1} & \cdots & p_{n+1} & p_n \end{pmatrix}$.

Proposition 15 .(1) $q_{ij}^{(k)}=\sum_{l=0}^{k}\binom{k}{l}(-\lambda)^l p_{n+i-j-l}$ $(k<i\le n, 1\le j\le n)$, (2)*For* $i>j$, $r_{ij}=\sum_{k=0}^{n}a_k p_{n+i-j-k}=\sum_{k=0}^{i-j-1}a_k p_{n+i-j-k}-\mathbf{a}'_{i-j-1}\mathbf{p}$,

(3) *For* $i>j$, $\sum_{\substack{k+l=m\\ k\ge 0, l\ge 1}} c^{(l)}a_k=\begin{cases}1 & (m=1)\\ 0 & (1<m\le i-j)\end{cases}$.

Proof. (1) For $k=1$ and $i>1$, $q_{ij}^{(1)}=q_{ij}-\lambda q_{i-1j}=q_{i-j+11}-\lambda q_{i-j1}=p_{n+i-j}-\lambda p_{n+i-j-1}$.
For $i>k$,

$$q_{ij}^{(k)}=q_{ij}^{(k-1)}-\lambda q_{i-1j}^{(k-1)}=\sum_{l=0}^{k-1}\binom{k-1}{l}(-\lambda)^l p_{n+i-j-l}+(-\lambda)\sum_{l=0}^{k-1}\binom{k-1}{l}(-\lambda)^l p_{n+i-1-j-l}$$

$$=p_{n+i-j}+\sum_{l=1}^{k-1}\left\{\binom{k-1}{l}+\binom{k-1}{l-1}\right\}(-\lambda)^l p_{n+i-j-l}+(-\lambda)^k p_{n+i-j-k}=\sum_{l=0}^{k}\binom{k}{l}(-\lambda)^l p_{n+i-j-l}.$$

(2)
$$r_{ij}=\sum_{k=j}^{n}\binom{n-j}{k-j}(-\lambda)^{k-j}q_{ik}^{(j)}=\sum_{k=j}^{n}\binom{n-j}{k-j}(-\lambda)^{k-j}\sum_{l=0}^{j}\binom{j}{l}(-\lambda)^l p_{n+i-k-l}=\sum_{k=j}^{n}\sum_{l=0}^{j}\binom{n-j}{k-j}\binom{j}{l}(-\lambda)^{k-j+l}p_{n+i-k-l}$$

$$=\sum_{m=0}^{n}\left(\sum_{\substack{k+l=m\\ 0\le k\le n-j, 0\le l\le j}}\binom{n-j}{k}\binom{j}{l}\right)(-\lambda)^m p_{n+i-j-m}.$$

Here, the factor in the parentheses satisfies the following lemma.

Lemma 16. $\sum_{\substack{k+l=m\\ 0\le k\le n-j, 0\le l\le j}}\binom{n-j}{k}\binom{j}{l}=\binom{n}{m}$ $(0\le j\le n)$.

Proof. The right is the number of m-element subsets X of $N=\{1,2,\cdots,n\}$. Let N_1 and N_2 be $N=N_1\cup N_2$ with $N_1\cap N_2=\varnothing$, $|N_1|=n-j$, $|N_2|=j$. $\binom{n-j}{k}$ and $\binom{j}{l}$ are the numbers of k-element subsets X_1 of N_1 and the number of l -element subsets X_2 of N_2, respectively. Any combination of X_1 and X_2 gives an X. Conversely, let k, l be $k=|X\cap N_1|$, $l=|X\cap N_2|$. Then, $k+l=m$. Thus, all the m-element subsets X are counted in the left. ■

Since $f(z)=\sum_{i=0}^{n}a_i z^{n-i}=(z-\lambda)^n$, $a_m=\binom{n}{m}(-\lambda)^m$. By Lemma 16, the first equality of Prop. 15 (2) is shown. The proof of the rest is obvious from the definition of $\mathbf{a}'_j$.

(3) Since $c^{(1)}a_0=1$, the proposition holds for $m=1$. For $m>1$, $\sum_{\substack{k+l=m\\ k\ge 0, l\ge 1}}c^{(l)}a_k=\sum_{k=0}^{m-1}c^{(m-k)}a_k=\sum_{k=1}^{m-1}c^{(m-k)}a_k+c^{(m)}a_0$.
By Prop. 10 (2), the right is equal to 0. ■

Proposition 17. $r_{ij}=0$ $(i>j)$.

Proof. For $i>j$, $r_{ij}=\sum_{k=0}^{i-j-1}a_k p_{n+i-j-k}-\mathbf{a}'_{i-j-1}\mathbf{p}$. Since $p_{n+i-j}=q_{i-j+11}=\sum_{l=1}^{i-j}c^{(l)}\mathbf{a}'_{i-j-l}\mathbf{p}$ (by Prop. 11),

$$r_{ij}=\sum_{k=0}^{i-j-1}a_k\sum_{l=1}^{i-j-k}c^{(l)}\mathbf{a}'_{i-j-k-l}\mathbf{p}-\mathbf{a}'_{i-j-1}\mathbf{p}=\left(\sum_{k=0}^{i-j-1}\sum_{l=1}^{i-j-k}c^{(l)}a_k\mathbf{a}'_{i-j-k-l}-\mathbf{a}'_{i-j-1}\right)\mathbf{p}.$$

The first term in the parentheses is written as $\sum_{m=1}^{i-j}\left(\sum_{\substack{k+l=m\\ k\ge 0, l\ge 1}}c^{(l)}a_k\right)\mathbf{a}'_{n+i-j-m}$. Combined with Prop. 15 (3), the proposition is proved. ■

Definition 18. $R_j\overset{\text{def}}{=}\begin{pmatrix} r_{11} & \cdots & r_{1j} & q_{1j+1}^{(j)} & \cdots & q_{1n}^{(j)}\\ & \ddots & \vdots & \vdots & & \vdots\\ \mathrm{O} & & r_{jj} & q_{jj+1}^{(j)} & \cdots & q_{jn}^{(j)}\\ & & & \vdots & & \vdots\\ & \mathrm{O} & & q_{nj+1}^{(j)} & \cdots & q_{nn}^{(j)}\end{pmatrix}$ $(j=1,2,\cdots,n)$.

By Prop's. 14 and 17, denote

$$R_n = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ & \ddots & \ddots & \vdots \\ & & \ddots & r_{12} \\ O & & & r_{11} \end{pmatrix} .$$

For $j \geq 1$, $r_{1j} = \sum_{k=j}^{n} \binom{n-j}{k-j} (-\lambda)^{k-j} q_{1k}^{(j)} = \sum_{k=j}^{n} \binom{n-j}{k-j} (-\lambda)^{k-j} q_{1k} = \sum_{k=j}^{n} \binom{n-j}{k-j} (-\lambda)^{k-j} p_{n+1-k}$.

Since $|P| = |R_i|$ $(i = 1, 2, \cdots, n)$, the following theorem is obtained.

Theorem 19. $|P| = \left\{ \sum_{k=1}^{n} \binom{n-1}{k-1} (-\lambda)^{n-k} p_k \right\}^n$.

Proof.

$$|P| = r_{11}^n = \left\{ \sum_{k=1}^{n} \binom{n-1}{k-1} (-\lambda)^{k-1} p_{n+1-k} \right\}^n = \left\{ \sum_{k=0}^{n-1} \binom{n-1}{k} (-\lambda)^{k} p_{n-k} \right\}^n = \left\{ \sum_{k=0}^{n-1} \binom{n-1}{n-k-1} (-\lambda)^{k} p_{n-k} \right\}^n = \left\{ \sum_{k=1}^{n} \binom{n-1}{k-1} (-\lambda)^{n-k} p_k \right\}^n . \blacksquare$$

Corollary 20. (1) $\mathbf{p} = {}^t\left(1\,(1+\lambda)\cdots(1+\lambda)^{n-1}\right) \Rightarrow |P| = 1$. (2) $\mathbf{p} = {}^t\left(1\,\lambda \cdots \lambda^{n-1}\right) \Rightarrow |P| = 0$.

Proof. By Theorem 19,

$$|P| = \left\{ \sum_{k=1}^{n} \binom{n-1}{k-1} (-\lambda)^{(n-1)-(k-1)} (1+\lambda)^{k-1} \right\}^n = \left[\{(1+\lambda) - \lambda\}^{n-1} \right]^n = 1 .$$

Similarly, (2) is shown. ■

To construct a non-singular P satisfying $A_f P = PJ$ or $\hat{A}_f P = PE_1$, it suffices to say $\hat{A}_f{}^n \mathbf{p} = \mathbf{0}$. Let $\mathbf{p} = {}^t\left(1\,(1+\lambda)\cdots(1+\lambda)^{n-1}\right)$, and $\mathbf{q}_i = {}^t\left(q_1^{(i)}\, q_2^{(i)} \cdots q_n^{(i)}\right)$ $(i = 1, 2, \cdots n)$ be defined as

$$q_j^{(i)} \stackrel{\text{def}}{=} \begin{cases} 0 & (j \leq n-i) \\ \sum_{k=0}^{i+(j-1)-n} \binom{j-1}{k} \lambda^k & (j > n-i) \end{cases} .$$

Then, $\mathbf{q}_i = {}^t\left(0 \cdots 0\, q_{n-i+1}^{(i)} \cdots q_n^{(i)}\right)$; especially, $\mathbf{q}_1 = {}^t(0 \cdots 0 1)$ and $\mathbf{q}_n = \mathbf{p}$.

Proposition 21.

(1) $\hat{A}_f \mathbf{p} = \mathbf{p} - \mathbf{q}_1$,

(2) $\hat{A}_f \mathbf{q}_i = \mathbf{q}_{i+1} - \mathbf{q}_1$ $(i \leq n-1)$; *especially for* $i = n-1$, $\hat{A}_f \mathbf{q}_{n-1} = \mathbf{p} - \mathbf{q}_1$,

(3) $\hat{A}_f{}^i \mathbf{p} = \mathbf{p} - \mathbf{q}_i$ $(i \leq n)$; *especially for* $i = n$, $\hat{A}_f{}^n \mathbf{p} = \mathbf{0}$.

Proof. (3) follows from (1), (2). In fact,

$$\hat{A}_f{}^2 \mathbf{p} = \hat{A}_f(\mathbf{p} - \mathbf{q}_1) = \hat{A}_f \mathbf{p} - \hat{A}_f \mathbf{q}_1 = (\mathbf{p} - \mathbf{q}_1) - (\mathbf{q}_2 - \mathbf{q}_1) = \mathbf{p} - \mathbf{q}_2 .$$

Recursive operation of $\hat{A}_f$ on $\mathbf{p}$ gives $\hat{A}_f{}^i \mathbf{p} = \mathbf{p} - \mathbf{q}_i$. For $i = n$, $\hat{A}_f{}^n \mathbf{p} = \mathbf{p} - \mathbf{q}_n = \mathbf{0}$.

(1) $f(z) = (z - \lambda)^n = \sum_{i=0}^{n} a_i z^{n-i}$, $a_i = \binom{n}{i} (-\lambda)^i$ and

$$\hat{A}_f = A_f - \lambda E = \begin{pmatrix} -\lambda & 1 & & & O \\ & -\lambda & 1 & & \\ & & \ddots & \ddots & \\ O & & & -\lambda & 1 \\ -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 - \lambda \end{pmatrix} .$$

For $j \leq n-1$, $\left(\hat{A}_f \mathbf{p}\right)_j = -\lambda(1+\lambda)^{j-1} + (1+\lambda)^j = (1+\lambda)^{j-1}$.

For $j = n$, $\left(\hat{A}_f \mathbf{p}\right)_n = \sum_{j=1}^{n} \left(-a_{n-j+1} - \delta_{jn}\lambda\right)(1+\lambda)^{j-1} = -\sum_{j=1}^{n} \binom{n}{n-(j-1)} (-\lambda)^{n-(j-1)} (1+\lambda)^{j-1} - \lambda(1+\lambda)^{n-1}$

$$= -\{(1+\lambda) - \lambda\}^n + (1+\lambda)^n - \lambda(1+\lambda)^{n-1} = (1+\lambda)^{n-1} - 1 .$$

Thus, $\hat{A}_f \mathbf{p} = \mathbf{p} - \mathbf{q}_1$.

(2) For $j \le n-i$, $\left(\hat{A}_f \mathbf{q}_i\right)_j = -\lambda q_j^{(i)} + q_{j+1}^{(i)} = \begin{cases} 0 & (j \le n-i-1) \\ 1 & (j = n-i) \end{cases}$.

For $n-i < j \le n-1$,

$$\left(\hat{A}_f \mathbf{q}_i\right)_j = -\lambda \sum_{k=0}^{i+j-n-1} \binom{j-1}{k} \lambda^k + \sum_{k=0}^{i+j-n} \binom{j}{k} \lambda^k = -\sum_{k=1}^{i+j-n} \binom{j-1}{k-1} \lambda^k + \sum_{k=0}^{i+j-n} \binom{j}{k} \lambda^k = 1 + \sum_{k=1}^{i+j-n} \left\{ \binom{j}{k} - \binom{j-1}{k-1} \right\} \lambda^k$$

$$= 1 + \sum_{k=1}^{i+j-n} \binom{j-1}{k} \lambda^k = \sum_{k=0}^{(i+1)+(j-1)-n} \binom{j-1}{k} \lambda^k = q_j^{(i+1)}.$$

The following lemma is shown before completing the proof for $j = n$.

Lemma 22. $\sum_{k=0}^{p} (-1)^k \binom{n}{p-k} \binom{n-p+k}{k} = (-1)^p \sum_{k=0}^{p} (-1)^k \binom{n-k}{p-k} \binom{n}{k} = 0 \quad (p \le n)$.

Proof. Replacing $p-k$ by k yields the first equality.

$$\binom{n-k}{p-k}\binom{n}{k} = \frac{(n-k)!}{(p-k)!(n-p)!} \cdot \frac{n!}{k!(n-k)!} = \frac{p!}{k!(p-k)!} \cdot \frac{n!}{p!(n-p)!} = \binom{p}{k}\binom{n}{p},$$

whence

$$\sum_{k=0}^{p} (-1)^k \binom{n-k}{p-k}\binom{n}{k} = \binom{n}{p} \sum_{k=0}^{p} (-1)^k \binom{p}{k} = \binom{n}{p} (1-1)^p = 0. \qquad \blacksquare$$

Return to the proof of the proposition.

For $j = n$,

$$\left(\hat{A}_f \mathbf{q}_i\right)_n = \sum_{l=1}^{n} (-a_{n-l+1}) q_l^{(i)} - \lambda q_n^{(i)}$$

$$= \sum_{l=n-i+1}^{n} \left\{ (-1)^{n-l} \binom{n}{n-l+1} \lambda^{n-l+1} \cdot \sum_{k=0}^{i+(l-1)-n} \binom{l-1}{k} \lambda^k \right\} - \lambda \sum_{k=0}^{i-1} \binom{n-1}{k} \lambda^k$$

$$= \sum_{l=n-i+1}^{n} \sum_{k=0}^{i+(l-1)-n} (-1)^{n-l} \binom{n}{n-l+1} \binom{l-1}{k} \lambda^{n-l+k+1} - \sum_{k=0}^{i-1} \binom{n-1}{k} \lambda^{k+1}$$

$$= \sum_{p=1}^{i} \left\{ (-1)^{p+1} \sum_{k=0}^{p-1} (-1)^k \binom{n}{p-k} \binom{n-p+k}{k} - \binom{n-1}{p-1} \right\} \lambda^p$$

$$= \sum_{p=1}^{i} \left\{ (-1)^{p+1} \sum_{k=0}^{p} (-1)^k \binom{n}{p-k} \binom{n-p+k}{k} + \binom{n}{p} - \binom{n-1}{p-1} \right\} \lambda^p.$$

From $\binom{n}{p} = \binom{n-1}{p} + \binom{n-1}{p-1}$ and Lemma 22, follows $\left(\hat{A}_f \mathbf{q}_i\right)_n = \sum_{p=1}^{(i+1)-1} \binom{n-1}{p} \lambda^p = q_n^{(i+1)} - 1$. Thus, $\hat{A}_f \mathbf{q}_i = \mathbf{q}_{i+1} - \mathbf{q}_1$. ■

It is, accordingly, verified that

Theorem 23.

(1) $\forall J$: *Nondiagonal Jordan block with an eigenvalue* λ, $\exists P \in \mathrm{GL}(n;K)$; $A_f P = PJ$.

P is expressed as $P = \left(\hat{A}_f^{\,n-1}\mathbf{p}\ \hat{A}_f^{\,n-2}\mathbf{p} \cdots \hat{A}_f \mathbf{p}\ \mathbf{p}\right)$ *with* $\mathbf{p} = {}^t\left(1\,(1+\lambda) \cdots (1+\lambda)^{n-1}\right)$.

(2) $\forall J = \bigoplus_{i=1}^{r} J_i$, $\exists P_i (i = 1,2,\cdots,r)$;

$$\left(\bigoplus_{i=1}^{r} A_f^{(i)}\right)\left(\bigoplus_{i=1}^{r} P_i\right) = \left(\bigoplus_{i=1}^{r} P_i\right)\left(\bigoplus_{i=1}^{r} J_i\right), \quad P_i = \left(\hat{A}_f^{(i)n_i-1}\mathbf{p}_i\ \hat{A}_f^{(i)n_i-2}\mathbf{p}_i \cdots \hat{A}_f^{(i)}\mathbf{p}_i\ \mathbf{p}_i\right)$$

with $\mathbf{p}_i = {}^t\left(1(1+\lambda_i)\cdots(1+\lambda_i)^{n_i-1}\right)$.

7. Solution of HLODE through Companion Matrix

Let $f(z)$ be a polynomial of (1) and $f(z) = \prod_{i=1}^{r} (z-\lambda_i)^{n_i}$, $\lambda_i \ne \lambda_j \ (i \ne j)$, $\sum_{i=1}^{r} n_i = n$. The general solution of $f(d_t)x = 0$ is given by

(4) $$x = \sum_{i=1}^{r} \sum_{j=0}^{n_i-1} c_{i,j} e_{\lambda_i,j} \quad (c_{i,j} \in K), \quad c_{ij} = \frac{1}{(n_i-j)!} \left(\frac{\partial}{\partial \lambda_i}\right)^{n_i-j} \frac{1}{\prod_{k=1,k\ne i}^{r} (\lambda_i - \lambda_k)^{n_k}},$$

where $e_{\lambda,j} = \frac{t^j}{j!} e^{\lambda_i t}$ $(j \ge 0)$ [7]. For a non-singular matrix A similar to a nondiagonal Jordan canonical form, the general solution of $\dot{\mathbf{x}} = A\mathbf{x}$ is expressed as $\mathbf{x} = P\mathbf{y}$, $\mathbf{y} = {}^t\left(x\, \dot{x} \cdots x^{(n-1)}\right)$, where P is a non-singular matrix such that $AP = PA_f$. Here, $x^{(i)}$ requires calculation of $d_t{}^i e_{\lambda,j}$. The following lemma is readily proved.

Lemma 24. $d_t e_{\lambda,j} = \begin{cases} \lambda e_\lambda & (j=0) \\ e_{\lambda,j-1} + \lambda e_{\lambda,j} & (j \ge 1) \end{cases}$.

Proposition 25. $d_t^m e_{\lambda,j} = \sum_{k=\max\{j-m,0\}}^{j} \binom{m}{m-j+k} \lambda^{m-j+k} e_{\lambda,k}$.

Proof. Since $d_t^0 e_{\lambda,j} = e_{\lambda,j}$, the proposition holds for $m = 0$. In the case of $0 \le m \le j$,

$$d_t^m e_{\lambda,j} = d_t\left(d_t^{m-1} e_{\lambda,j}\right) = \sum_{i=0}^{m-1} \binom{m-1}{i} \lambda^i d_t e_{\lambda,j-(m-1)+i} = \sum_{i=0}^{m-1} \binom{m-1}{i} \lambda^i \left(e_{\lambda,j-m+i} + \lambda e_{\lambda,j-(m-1)+i}\right)$$

$$= e_{\lambda,j-m} + \sum_{i=1}^{m-1} \left(\binom{m-1}{i} + \binom{m-1}{i-1}\right) \lambda^i e_{\lambda,j-m+i} + \lambda^m e_{\lambda,j} = \sum_{i=0}^{m} \binom{m}{i} \lambda^i e_{\lambda,j-m+i} = \sum_{i=j-m}^{j} \binom{m}{m-j+i} \lambda^{m-j+i} e_{\lambda,i} .$$

Similarly to this case, the mathematical induction on m is adapted to $m > j$.

$$d_t^m e_{\lambda,j} = d_t\left(d_t^{m-1} e_{\lambda,j}\right) = \binom{m-1}{m-1-j} \lambda^{m-j} e_{\lambda,0} + \sum_{i=1}^{j} \binom{m-1}{m-1-j+i} \lambda^{m-1-j+i} \left(e_{\lambda,i-1} + \lambda e_{\lambda,i}\right)$$

$$= \left(\binom{m-1}{m-1-j} + \binom{m-1}{m-j}\right) \lambda^{m-j} e_{\lambda,0} + \sum_{i=2}^{j} \left(\binom{m-1}{m-1-j+i} + \binom{m-1}{m-1-j+i-1}\right) \lambda^{m-1-j+i} e_{\lambda,i-1} + \lambda^m e_{\lambda,j}$$

$$= \binom{m}{m-j} \lambda^{m-j} e_{\lambda,0} + \sum_{i=2}^{j} \binom{m}{m-1-j+i} \lambda^{m-1-j+i} e_{\lambda,i-1} + \lambda^m e_{\lambda,j}$$

$$= \binom{m}{m-j} \lambda^{m-j} e_{\lambda,0} + \sum_{i=1}^{j-1} \binom{m}{m-j+i} \lambda^{m-j+i} e_{\lambda,i} + \lambda^m e_{\lambda,j} = \sum_{i=0}^{j} \binom{m}{m-j+i} \lambda^{m-j+i} e_{\lambda,i} .$$ ■

By Theorem 23 (2), it is, therefore, sufficient only to apply the above general solution (4) to the case of $f(z) = (z - \lambda_i)^{n_i}$, in order to solve the homogeneous linear ordinary differential equation.

8. Engineering Application

The automobile car-body requires cold-rolled sheet of low carbon steel with high strength as well as press formability and non-ageing. In the process of production of such steel sheet, the continuous annealing line has been employed since 1970s. On rapid cooling, interstitial atoms of carbon and nitrogen contained in the steel in the order of 10 wt ppm remain in solid solution. These atoms can diffuse to dislocations to relax internal stress even at room temperature, and they deteriorate press formability. The internal friction measurement which is non-destructive and the most precise analysis of each interstitial has played an important role to control the content of interstitials in solid solution. A species of interstitial yields a Snoek peak of internal friction due to stress-induced ordering of interstitial atoms [5]. The Snoek peak is an anelastic relaxation peak of the Debye type. The mechanical behaviour of the interstitials under an applied stress is characteristic of the standard linear solid [6]. Each standard linear solid corresponding to a species of interstitials is represented by an ordinary differential equation of stress and strain. There exist differential equations for each species of interstitials. The behaviour of solid, here steel sheet, is a resultant synthesis of all interstitials in solid solution. The present mathematical consideration links single relaxations to a multiple relaxation to understand mechanical property of the low carbon steel sheet. Furthermore, decomposition of the multiple relaxation into its constituent single relaxations is theoretically asserted. The conjugate variables of the strain and stress are pointed out as the introduction.

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