THE BULLETIN OF OKAYAMA UNIVERSITY OF SCIENCE No. 38 A pp. 5-14 (2002)

Kneading Patterns of Halley Iterations

Yasutoshi NOMURA

Department of Applied Science,

Faculty of Science,

Okayama University of Science,

Ridaicho 1—1, Okayama 700—0005, Japan

(Received November 1, 2002)

1. Introduction

One of the most important problems in one- dimensional dynamics is to study the behavior of periodic orbits. In order to describe iterated interval maps combinatorially, Milnor and Thurston [4] developed the kneading theory for uni-modal continuous maps(see Milnor and Tresser [5] for multi-modal maps), thereby succeeded in refining the naïve argument using patterns in Metropolis et al [3] and Derrida et al [2].

For uni-modal map with critical point c the itinerary of a point a is defined to be the sequence of letters L or R which is determined according as each iterate of a lies in the left or right side of c. But this method of assignment is not so much efficient for discontinuous functions such as rational functions.

In the previous paper [6] I have investigated the order structure of parameter-sequences for which periodic critical orbits arise for Halley iterations of certain polynomials . But I have looked over periodic orbits containing the critical point 0. Here we examine the latter cases by numerical experiment in which there occur strange phenomena that cannot be arise in the case of logistic families or in the case of super-stable periodic orbits for Halley iterations of the same polynomials.

It will turn out that kneading patterns is useful in classifying the parameter-families admitting superstable periodic orbits.

2. Kneading Patterns of Periodic Orbits

Let $F(x)$ be a real-valued function which is smooth in any finite interval except finitely many points. We assume that F has a finitely many critical points in any finite interval.

Let c be a critical point. For a point a we define *itinerary* $I_c(a)$ *of a with respect to c* to be

$$I_c(a) = I_1 I_2 \cdots\ I_k \cdots,$$

where I_k is

$+$ if $F^{k-1}(a) < F^k(a)$ and both iterates lie in the same side of c

$-$ if $F^{k-1}(a) > F^k(a)$ and both iterates lie in the same side of c

$|+$ if $F^{k-1}(a) < c < F^k(a)$

$|-$ if $F^k(a) < c < F^{k-1}(a)$.

In particular, when c is a periodic point of period p, we define ***kneading pattern*** $K(c)$ ***of*** c to be the truncated $I_c(c)$ up to p, that is,

$$K(c) = I_1 I_2 \cdots I_p,$$

where $I_c(c) = I_1 I_2 \cdots$. Further we make abbreviations $(+)^k$ for $++ \cdots +$ (k times) and $(-)^k$ for $- - \cdots -$ (k times).

We list a program (using U-basic) which outputs periodic orbits and kneading patterns of a critical point z_0, in which b denotes the repulsive fixed point given in the next section.

```
10 print "motion of peridic orbits of free critical point z0 of Halley iterations of z^q + (c-1)z-c"
20 point 18
30 input "degree q, period p, c, width m, sign= "; q, p, c, m, e
40      z0 = e*(2.0*(q-2)*(c-1.0) /q/(q+1))^(1/(q-1)): w0 = fnk( z0, c, q)
50 view ( 31, 31 )-( 399, 399 ), 1
60 window (-2.0*m*abs(w0), 2 )-(2.0*m*abs(w0), p+ 4 )
70 cls
80    pset (z0, 3), 6
90    I = 1: u = z0
100     while I < p+1
110        line -( fnk( u, c, q ), I + 3 ), 6
120        u = fnk( u, c, q )
130     I = I + 1: wend
140   u = z0: j =1
150         while j <=p + 1
160              line (u-0.2*w0, j+2 )-( u + 0.2*w0, j+ 2 ), 2
170         u = fnk( u, c, q) : j = j+1: wend
180       line ( z0, 2 )-( z0, p+4 ), 4
190       print  " q, p, c = "; q, p, c ; "     critical point = "; z0
200    x = z0: j = 1
210     while j <=p
220        y = fnk( x, c, q )
230          if y<x then print "-" ;
240          if y>x then print "+" ;
```

```
250        x = y
260    j = j+1: wend
270       b = ((1.0−c)/q)^( 1/(q−1))
280    rem    line (−b, 2)−(−b, p+3), 7: line (b, 2)−(b, p+3), 7
290    rem    print: print "repulsive fixed pt= "; −b; "     "; b
300  end
310   fnk( z, c, q)
320        local r, s, t, w
330        r = z^q + (c−1.0)*z−c: s = q*z^(q−1)+c−1.0: t = q*(q−1)*z^(q−2)
340        if abs(s^2−0.5*r*t) <0.00000000000000001 then 370
350          w = z−r*s/(s^2 − 0.5*r*t)
360   return(w)
370   w=100000000.0: return(w)
```

3. Parameter sequences associated with the periodic orbit of 0 in Halley iterations

Given a smooth function $f(x)$, the *Halley iteration function* $Hf(x)$ is defined by

$$H_f(x) = x - 2f(x)f'(x)/\Phi_f(x) \qquad (3.1)$$

where

$$\Phi_f(x) = 2f'(x)^2 - f(x)f''(x). \qquad (3.2)$$

The derivative is given by

$$(H_f)'(x) = f(x)^2\,\Psi_f(x)\,\Phi_f(x)^{-2} \qquad (3.3)$$

where

$$\Psi_f(x) = 3f''(x)^2 - 2f'(x)f'''(x). \qquad (3.4)$$

From now on we take for $f(x)$ a one-parameter family of polynomials of degree q:

$$f(x) = x^q + (\lambda - 1)x - \lambda \qquad (q \geqq 5)$$

Then a simple calculation shows that

$$H_f(x) = x - 2(x^q + (\lambda-1)x - \lambda)(qx^{q-1} + \lambda - 1)/\Phi_f(x) \qquad (3.5)$$

$$\Phi_f(x) = q(q+1)x^{2q-2} - q(q-5)(\lambda-1)x^{q-1} + q(q-1)\lambda x^{q-2} + 2(\lambda-1)^2 \qquad (3.6)$$

$$\Psi_f(x) = q(q-1)x^{q-3}\{q(q+1)x^{q-1} - 2(q-2)(\lambda-1)\}. \qquad (3.7)$$

Thus *free* critical points of $H_f(x)$ are 0 and

$$\gamma = -[2(q-2)(\lambda-1)q^{-1}(q+1)^{-1}]^{(1/(q-1))} \qquad \text{for odd } q,\ \lambda \geqq 1$$

$$\gamma = -[2(q-2)(1-\lambda)q^{-1}(q+1)^{-1}]^{(1/(q-1))} \qquad \text{for even } q$$

In what follows we assume q is odd. Then the fixed point set consists of the solutions of $f(x) = (x-1)(x^{q-1} + \cdots + x + \lambda) = 0$ and the solutions $\pm\beta$ of $qx^{q-1} + \lambda - 1 = 0$, where $\beta = ((1-\lambda)/q)^{1/(q-1)}$. Note from [7] that $(H_f)'(\pm\beta) = 3$ and that the number of solutions of $x^{q-1}+ \cdots +x+\lambda =0$ is at most 2 (if exist then we denote them by α' and α, $\alpha' \leqq \alpha$). Further observe that $\Phi_f(x)$ has at most two zeros in $x>0$ and we denote them by δ_1 and $\delta_2(\delta_1 \leqq \delta_2)$.

In order to state the results of our numerical experiment we shall use the following notations introduced in [6]. We shall denote the parameter-value λ for which the corresponding Halley iteration function H_f has the periodic critical point c with period p generically by $\lambda(p)$. If there exists a monotone geometrically convergent sequence $\lambda(m), \lambda(m+1), \cdots, \lambda(k)$, $\cdots$ such that the successive ratios of first differences

$$\rho(k)=[\lambda(k+2)-\lambda(k+1)]/[\lambda(k+1)-\lambda(k)]$$

tends to a *asymptotic ratio* $\rho(\infty)$, we express this fact by

$$\mathrm{NS}(m, \lambda(m), \rho(\infty);\ \lambda(\infty)),$$

where $\lambda(\infty)$ denotes the limit of $\lambda(k)$ as k tends to an infinity which is a rough estimate in most cases. In [6] we have made numerical computation of parameters for which γ are superstable periodic orbits but did not such computation for orbits of 0. So we have performed numerical computation for the periodic critical point 0 and have found following geometrically convergent parameter-sequences in which patterns are those for $\lambda(p)$:

1) Parameter-sequences containing $\lambda(3)$ as a term (see Tables 1 and 2):

for $q=5$

NS(3, 0.582862445, 2/3 ; 1) with pattern $-(+)^{p-1}$
NS(3, 0.48156607, 2/3 ; 0.55948038) with pattern $-+-(+)^{p-3}$
NS(2, 0.44686875, 1/3 ; 0.4374605214) with pattern $-\mid+(-)^{p-2}$
NS(3, 0.3904072, 2/3 ; 0.427608592) with pattern $--(+)^{p-2}$
NS(3, 0.3649024915, 2/3 ; 0.387376987) with pattern $---(+)^{p-3}$
NS(2, 0.35020664, 1/3 ; 0.3486429479) with pattern $-\mid+(-)^{p-2}$
NS(3, 0.3353712, 2/3 ; 0.347168858) with pattern $--(+)^{p-2}$
NS(3, 0.3286387, 2/3 ; 0.33500326) with pattern $---(+)^{p-3}$

for $q=7$

NS(3, 0.54532, 3/4 ; 1) with pattern $-(+)^{p-1}$
NS(3, 0.47523782, 3/ 4 ; 0.5337131323) with pattern $-+-(+)^{p-3}$
NS(2, 0.4488294, 1/ 3 ; 0.44416952647) with pattern $-\mid+(-)^{p-2}$
NS(3, 0.4111679, 3/4 ; 0.4394051981) with pattern $--(+)^{p-2}$
NS(3, 0.3928835, 3/4; 0.4096437096) with pattern $---(+)^{p-3}$
NS(2, 0.3819007, 1/3 ; 0.381113223481) with pattern $-\mid+(-)^{p\ \ 2}$
NS(3, 0.37144282, 3/4 ; 0.380364186) with pattern $--(+)^{p-2}$
NS(3, 0.36651422, 3/4; 0.3712528241) with pattern $---(+)^{p-3}$

2) **Monotone-decreasing parameter-sequences which begin with $\lambda(3)$ in the last sequences listed in 1) for each q (see Table 3) :**

NS(3, 0.3286387, 1/9 ; 0.32644677652358999) for $q=5$

NS(3, 0.36651422, 1/9 ; 0.36490610602825847673) for $q=7$

NS(3, 0.38969882 , 1/9 ; 0.38843330942O4250803) for $q=9$

where the all sequences have patterns $(-)^{p-1}+$.

3) Each term in the sequences shown in 1) and 2) has sequences as offshoots. Further each term in these offshoots has sequences as offshoots. It seems that such hierarchies continue indefinitely. For example,

for $q=5$

NS(3, 0.3904072, 1/3 ; 0.388948) with pattern $-|+(-)^{p-3}$

NS(3, 0.3904072, 1/3; 0.401156) with pattern $-(+)^{p-2}$

for $q=7$

NS(3,0.38122575, 3/4 ; 0.39198) with pattern $-+-(+)^{p-3}$

NS(4, 0.3871682794, 1/3 ; 0.3870555435) with pattern $-+-|+(-)^{p-4}$

These numerical results seem to support:

Conjecture 1. *For each odd $q \geqq 5$ there are six geometrically convergent sequences increasing as period p*

$$S_{q,k} = (\lambda_k(3),\ \lambda_k(4),\ \lambda_k(5),\ \cdots)$$

with asymptotic ratio $(q-1)/(q+1)$, where $\lambda_k(p)$ denote a parameter for which 0 is a periodic point with period p and $\lambda_1(3) > \lambda_2(3) > \cdots > \lambda_6(3)$ and the kneading patterns are $-(+)^{p-1}$, $-+-(+)^{p-3}$, $--(+)^{p-2}$, $---(+)^{p-3}$, $--(+)^{p-2}$, $---(+)^{p-3}$ *for $k=1, 2, \cdots, 6$ respectively. Further there are two geometrically convergent sequences decreasing as period*

$$S^{(1)} = (\lambda^{(1)}(2), \lambda^{(1)}(3), \lambda^{(1)}(4), \cdots) \quad \text{and} \quad S^{(2)} = (\lambda^{(2)}(2), \lambda^{(2)}(3), \lambda^{(2)}(4), \cdots)$$

with asymptotic ratio 1/3 and with patterns $-\,|\,+(-)^{p-2}$, where

$$\lambda_2(3) < \lambda^{(1)}(2) < \lambda_3(3) \quad \text{and} \quad \lambda_4(3) < \lambda^{(2)}(2) < \lambda_5(3).$$

(Sequences in this conjecture may be called *primary sequences*).

Conjecture 2. *Multipliers of periodic orbits of 0 are non-negative and therefore period-doubling bifurcations cannot occur.*

Conjecture 3. *For each odd $q \geqq 5$ there is a geometrically convergent sequence*

$$T_q = (\ \lambda_6(3),\ \lambda(4),\ \lambda(5),\ \cdots)$$

with asymptotic ratio 1/9 and which is decreasing as period p, and whose $\rho(5), \rho(6), \rho(7), \cdots$ forms a sequence with asymptotic ratio 1/3. There is no parameter-value in the left side of $\lambda(p)$ for which periodic orbits of 0 with period p. (Sequences in this conjecture may be called *tail* *sequences*).

Conjecture 4. $\delta_1 \leqq -\beta \leqq \delta_2$ *if* $\Phi_f(x)$ *has a zero.*

The fact that $(H_f)'(\beta) = 3$ and the following propositions seem to give us reason for asymptotic ratio values stated in conjectures, because periodic orbits starting from a critical point return back *after traveling near repulsive fixed points.*

Proposition 1. *The curve* $y = H_f(x)$ *has a asymptotic line* $y = (q-1)(q+1)^{-1}x$.

Proof. Since, by (3.5) and (3.6), we have

$$H_f(x) = \Phi_f()^{-1}\{q(q-1)x^{2q-1} - (\lambda-1)(q-1)(q-2)x^q + q(q+1)x^{q-1} + 2\lambda(\lambda-1)\},$$

we see that the distance $D_q(x)$ from a point $(x, H_f(x))$ to the line $y = (q-1)(q+1)^{-1}x$ is given by

$$\pm D_q() = [1 + (q-1)^2(q+1)^{-2}]^{-\frac{1}{2}}\{H_f(x) - (q-1)(q+1)^{-1}x\},$$

in which a factor of denominator is

$$(q+1)\Phi_f(x) = q(q+1)^2x^{2q-2} + \text{lower terms}$$

and a factor of numerator is

$$(q+1)\{H_f(x) - (q-1)(q+1)^{-1}x\} - (q-1)x\Phi_f(x) = -2(\lambda-1)(q-1)(q^2-3q-1)x^q + \text{lower terms}.$$

Thus $D_q(x)$ tends to 0 as $|x| \to \infty$.

Proposition 2. ∞ *is a fixed point of* H_f *with multiplier* $(q+1)(q-1)^{-1}$.

Proof. Since the degree of the numerator is $2q-1$ and that of the denominator is $2q-2$, ∞ is a fixed point. By a change of variable $x = t^{-1}$, $H_f(x)$ is conjugate to $H_f(t^{-1})^{-1}$ and, by (3.5) and (3.6), we have

$$[H_f(t^{-1})^{-1}]' = t^{-2}(H_f)'(t^{-1})H_f(t^{-1})^{-2}$$
$$= f(t^{-1})^2\Psi_f(t^{-1})t^{-2}[q(q-1)t^{-2q+1} - (\lambda-1)(q-1)(q-2)t^{-q} + \lambda q(q+1)t^{-q+1} + 2\lambda(\lambda-1)]^{-2},$$

in which the factor

$$f(t^{-1})\Psi_f(t^{-1}) = t^{-(4q-4)}[1 + (\lambda-1)t^{q-1} - \lambda]^2 q(q-1)\{q(q+1) - 2(q-2)(\lambda-1)t^{q-1}\}$$

and the reciprocal of the remaining factor is

$$t^{-(4q-4)}[q(q-1) - (\lambda-1)(q-1)(q-2)t^{q+3} + \lambda q(q+1)t^{q+4} + 2\lambda(\lambda-1)t^{2q+3}]^2.$$

Hence the limit of $[H_f(t^{-1})^{-1}]'$ is $(q+1)(q-1)^{-1}$ as $t \to 0$.

References

1. P. Collet and J. P. Eckmann: Iterated maps on the interval as dynamical systems, Prog. Physics edited by A. Jaffe and D. Ruelle. Birkhauser, Boston-Basel-Berlin (1997)

2. B. Derrida, A. Gervois and Y. Pomeau: Iteration of endomorphisms on the real axis and representation of numbers, Ann. Inst. Henri Poincare 26, 305—356 (1978)

3. N. Metropolis, M. L. Stein and P. R. Stein: On finite limit sets for transformations on the unit interval, J. Combinatorial Theory 15, 25—44 (1973)

4. J. Milnor and W. Thurston: On iterated maps of the interval, Springer Lecture Notes 1342, 465—563 (1988)

5. J. Milnor and C. Tresser: On entropy and monotonicity for real cubic maps, Commun. Math. Phys. 209, 123—178 (2000)

6. Y. Nomura: Periodic critical orbits of Newton and Halley iterations, The Bull. Okayama Univ. Sci. 36, 29 38 (2000)

7. Y. Nomura and Y. Furusawa: Attractive basins of fixed points of Halley iterations, The Bulletin Okayama Univ. Sci. 37, 23—28 (2001)

Table 1 Primary sequences $S_{q,k}$

$q=5$

p	$\lambda_1(p)$	$\rho(p)$	$\lambda_2(p)$	$\rho(p)$
3	0.58286 2445		0.48156 6071	
4	0.69425 9908		0.52192 9513	
5	0.78144 49855	0.7826486766	0.54025 96651 8	0.45412 75966
6	0.84688 9762	0.7506419590	0.54879 09356 9	0.4654227913
…			…	
18	0.99868 79578 2313	0.6673866670 29	0.55942 10672 6458467	0.6647462474 31
19	0.99912 50532 77137	0.6671467597 58	0.55944 08635 0461388	0.6653868285 35
20	0.99941 65901 62165	0.6669867699 607	0.55945 40441 1120915	0.6658136381 36

p	$\lambda_3(p)$	$\rho(p)$	$\lambda_4(p)$	$\rho(p)$
3	0.39040 72		0.36490 24915	(pattern $-++$)
4	0.40950 66352		0.37829 66321 4	
5	0.41791 19109 7	0.44007	0.38338 61395 7	0.37998
6	0.62198 08795 77	0.484097	0.38537 24898 41	0.39028 3
…			…	
18	0.42757 45585 66924 17	0.6652472027	0.38736 73714 69459 10	0.6640992269
19	0.42758 59135 31861 058	0.6657208009	0.3873705805 46530735	0.6649565496
20	0.42759 34763 50542 616	0.6660362866	0.3873727162 74798449	0.6655272590

p	$\lambda_5(p)$	$\rho(p)$	$\lambda_6(p)$	$\rho(p)$
3	0.33537 12		0.32863 87	
4	0.34196 43641		0.33234 60497	
5	0.34468 46318 6	0.412589	0.33395 61944 5	0.434317
6	0.34583 68182 9	0.423556	0. 33454 73421 04	0.367139
…			…	
17	0.34716 16702 51070 6	0.6636745457 3	0.33500 00794 66260 424	0.6610260367
18	0.34716 52715 96019 5	0.6646744463 7	0.33500 09130 15414 769	0.6629148255
19	0.34716 76677 13574 45	0.665339640856	0.33500 14666 33111 447	0.7041188796

$q=7$

p	$\lambda_1(p)$	$\rho(p)$	$\lambda_2(p)$	$\rho(p)$
3	0.54532 39		0.47523 782	
4	0.62913 40		0.50476 0448	

5	0.70155 06	0.8640557	0.51816 48487	0.45403829
6	0.76299 68	0.84850987	0.52449 31381	0.47210538
…			…	
17	0.98793 64705 8	0.7552695152 4	0.53348 34568 2119	0.7390038374 2
18	0.99093 18887 8	0.7539598572 3	0.53354 16602 5646	0.7414833415 9
19	0.99318 73614 9	0.7529742291 07	0.53358 49551 4961	0.7438540061 9

p	$\lambda_3(p)$	$\rho(p)$	$\lambda_4(p)$	$\rho(p)$
3	0.41116 790	(pattern −++)	0.39288 35	(pattern −++)
4	0.42509 778		0.40277 0988	
5	0.43125 7160	0.44217	0.40651 7179	0.37888 5
6	0.43431 02065 9	0.495674	0.40798 1084	0.3907716
…			…	
13	0.43893 73830 676	0.7216646126	0.40952 68388 8385	0.7061750436
14	0.43905 85430 04771	0.7289824577 9	0.40955 76284 56714	0.7176932645
15	0.43914 75187 915023	0.7343663991 6	0.40957 99841 09246	0.7260786836

p	$\lambda_5(p)$	$\rho(p)$	$\lambda_6(p)$	$\rho(p)$
3	0.37144 282		0.36651 422	
4	0.37631 731		0.36925 6553	
5	0.37833 0193	0.41294166	0.37045 20172	0.43593003
6	0.37919 26185	0.42845306	0.37089 04324	0.36673208
…			…	
21	0.38035 12382 66386 8	0.7460030044 36	0.37125 10832 02688	0.7446693654 79
22	0.38035 34796 290314	0.74745 6043539	0.37125 15197 63670 1	0.7460111881 417
23	0.38035 515637933348	0.7480941587206	0.37125 18458 80624 1	0.7470135157429

Table 2 Primary sequences $S^{(1)}$ and $S^{(2)}$

$q = 5$

p	$\lambda^{(1)}(p)$	$\rho(p)$	$\lambda^{(2)}(p)$	$\rho(p)$
2	0.44686 875		0.35020 644	(pattern −+)
3	0.43904 5939		0.34890 69943	
4	0.43790 25506	0.14616072	0.34871 56963	0.14719
5	0.43759 98201 8	0.26476597	0.34866 57748	0.26096
6	0.43750 60849 7	0.309632527	0.34865 04074 4	0.30783

…			…	
12	0.43746 05421 17357	0.333300207	0.34864 29581 00663	0.333297331
13	0.43746 05007 88304	0.3333222546	0.34864 29513 4583589	0.3333213752
14	0.43746 04870 121041	0.3333296797	0.34864 29490 9425384	0.3333292942

$q=7$

p	$\lambda^{(1)}(p)$	$\rho(p)$	$\lambda^{(2)}(p)$	$\rho(p)$
2	0.44882 94	(pattern −+)	0.38190 07	(pattern −+)
3	0.44483 08	(pattern − | +−)	0.38122 575	(pattern − | +−)
4	0.44434 89802	0.120517	0.38114 352	0.12182
5	0.44422 56879	0.255846	0.38112 2677	0.25356
…			…	
11	0.44416 96014 0036	0.3332220024 9	0.38111 32360 2205	0.3332170411
12	0.44416 95515 8879	0.3332961083 0	0.38111 32276 4988	0.3332936563 2
13	0.44416 95349 85547 0	0.3333211549 01	0.381113224859268	0.3333203636 4

Table 3 Tail sequences T_q

$q=5$

p	$\lambda(p)$	$\rho(p)$
3	0.32863 87	
4	0.32667 92778	
5	0.32647 22480 9109	0.1058544016
6	0.32644 95938 19155	0.1094251742 5
…		
19	0.32644 67765 23589 99220 53842 97785 25363	0.1111111100 93771097
20	0.32644 67765 23589 99122 23963 58552 35965 1	0.1111111107 71997778
21	0.32644 67765 23589 99111 31754 76526 48620 26	0.1111111109 98073334

$q-7$

p	$\lambda(p)$	$\rho(p)$
3	0.36651 422	
4	0.36507 7780	
5	0.36492 49511 7	0.1063942803
…		
18	0.36490 61060 28258 49065 31591 48002	0.1111111084 9399838435
19	0.36490 61060 28258 48401 14466 70759 952	0.1111111102 38748 48671
20	0.36490 61060 28258 48337 34786 19649 32852	0.1111111108 2030700523