THE BULLETIN OF OKAYAMA UNIVERSITY OF SCIENCE No. 38 A pp. 1-4 (2002)

Ideals generated by $\alpha - a$ and ideals generated by $a\alpha - 1$ of a Laurent extension $R[\alpha, \alpha^{-1}]$ of a Noetherian domain

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(Received November 1, 2002)

Abstract

Let R be a Noetherian domain and α an anti-integral element over R. Set $A = R[\alpha]$. Let a be an element of R. Then we investigate the contraction ideals $(\alpha - a)R[\alpha, \alpha^{-1}] \cap A$ and $(a\alpha - 1)R[\alpha, \alpha^{-1}] \cap A$, which are described in terms of the denominator ideals $I_{[\alpha]}$ and $I_{[\alpha^{-1}]}$, and the monic minimal polynomials $\varphi_\alpha(X)$ and $\varphi_{\alpha^{-1}}(X)$.

Let R be a Noetherian domain with quotient field K and $R[X]$ a polynomial ring over R in an indeterminate X. Let α be an element of an algebraic field extension of K and $\pi : R[X] \longrightarrow R[\alpha]$ the R-algebra homomorphism defined by $\pi(X) = \alpha$. Let $\varphi_\alpha(X)$ be the monic minimal polynomial of α over K with $\deg \varphi_\alpha = d$, and write $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$, $(\eta_1, \dots, \eta_d \in K)$. We define $I_{[\alpha]} := \bigcap_{i=1}^d I_{\eta_i}$ and $J_{[\alpha]} := I_{[\alpha]}(1, \eta_1, \dots, \eta_d)$ where $I_{\eta_i} = \{c \in R; c\eta_i \in R\}$ and $(1, \eta_1, \dots, \eta_d)$ is an R-module generated by $1, \eta_1, \dots, \eta_d$. An element α is called an anti-integral element of degree d over R if $\operatorname{Ker} \pi = I_{[\alpha]}\varphi_\alpha(X)R[X]$. An element α is said to be a super-primitive element of degree d over R if $J_{[\alpha]} \not\subset p$ for every $p \in \mathrm{Dp}_1(R)$ where $\mathrm{Dp}_1(R) = \{p \in \operatorname{Spec} R; \operatorname{depth} R_p = 1\}$.

Our general reference for unexplained terms is [2].

Lemma 1.([5, Lemma 1]) *Let a be an element of R. Then the following assertions hold:*
(1) $\varphi_{\alpha - a}(X) = \varphi_\alpha(X + a)$.
(2) $I_{[\alpha - a]} = I_{[\alpha]}$.
(3) *If α is an anti-integral element of degree d over R, then so is $\alpha - a$.*

Proposition 2. *Let R be an integral domain and α an anti-integral element of degree d over R. Set $A = R[\alpha]$ and let a be an element of R. Then the following two assertions hold:*
(1) $I_{[\alpha]}\varphi_\alpha(a) = (\alpha - a)A \cap R$.
(2) *If $\alpha - a \neq 0$, then $(\alpha - a)A \cap R = I_{[(\alpha - a)^{-1}]}$.*

Proof. (1) We prove that $I_{[\alpha]}\varphi_\alpha(a) \subset (\alpha - a)A \cap R$. By Lemma 1, we know that $I_{[\alpha]}\varphi_\alpha(a) = I_{[\alpha - a]}\varphi_{\alpha - a}(0)$. Then it is clear that $I_{[\alpha]}\varphi_\alpha(a) \subset (\alpha - a)A \cap R$.

We prove the converse inclusion. Let b be an element of $(\alpha - a)A \cap R$. Then there exists a polynomial $g(X)$ of $R[X]$ such that $b = (\alpha - a)g(\alpha)$. Since $\operatorname{Ker}\pi = I_{[\alpha]}\varphi_\alpha(X)R[X]$, there exists a polynomial $h(X)$ of $I_{[\alpha]}R[X]$ such that $(X - a)g(X) - b = h(X)\varphi_\alpha(X)$. Substituting a for X, we have $-b = h(a)\varphi_\alpha(a)$. Hence b is in $I_{[\alpha]}\varphi_\alpha(a)$ because $h(a)$ is in $I_{[\alpha]}$. This shows that $(\alpha - a)A \cap R \subset I_{[\alpha]}\varphi_\alpha(a)$.

(2) By the assertion (1), we have $(\alpha - a)A \cap R = I_{[\alpha]}\varphi_\alpha(a)$. By Lemma 1, we see that $I_{[\alpha]}\varphi_\alpha(a) = I_{[\alpha-a]}\varphi_{\alpha-a}(0)$. Simple calculation shows that $I_{[\alpha-a]}\varphi_{\alpha-a}(0) = I_{[(\alpha-a)^{-1}]}$ because $I_{[\alpha}\varphi_\alpha(0) = \eta_d I_{[\alpha]} = I_{[\alpha^{-1}]}$. Hence $(\alpha - a)A \cap R = I_{[(\alpha-a)^{-1}]}$. Q.E.D.

Corollary 3. *Let R be an integral domain and α an anti-integral element of degree d over R. Set $A = R[\alpha]$ and let a be an element of R. Then $\alpha - a$ is a unit of A if and only if $I_{[\alpha]}\varphi_a(a) = R$.*

Proof. By Proposition 2 (1), we know that $(\alpha - a)A \cap R = R$ if and only if $I_{[\alpha]}\varphi_a(a) = R$. It is easily seen that $\alpha - a$ is a unit of A if and only if $(\alpha - a)A \cap R = R$. Hence we reach the conclusion. Q.E.D.

We will consider the contraction of an ideal $(\alpha - a)R[\alpha, \alpha^{-1}]$ to $R[\alpha]$ and to R.

Theorem 4. *Let R be a Noetherian domain and α a super-primitive element of degree d over R. Set $A = R[\alpha]$. Assume that* $\operatorname{depth} R_p = 1$ *for every prime divisor P of αA with $p = P \cap R$. Let a be an element of R such that $a \,(\operatorname{mod} I_{[\alpha^{-1}]})$ is a non-zero divisor of $R/I_{[\alpha^{-1}]}$. Then the following two assertions hold:*

(1) $(\alpha - a)R[\alpha, \alpha^{-1}] \cap A = (\alpha - a)A$.

(2) $(\alpha - a)R[\alpha, \alpha^{-1}] \cap R = I_{[\alpha]}\varphi_\alpha(a)$.

Proof. (1) Set $B = R[\alpha, \alpha^{-1}]$. The inclusion $(\alpha - a)B \cap A \supset (\alpha - a)A$ is clear. We will prove the converse inclusioin. Let c be an element of $(\alpha - a)B \cap A$. Then there exists an element b of B such that $c = (\alpha - a)b$. Let P be an element of $\operatorname{Dp}_1(A)$. If α is not in P, then $A_P = B_P$ and c is in $(\alpha - a)B_P = (\alpha - a)A_P$. Hence $c/(\alpha - a)$ is in A_P. If α is in P, then P is a prime divisor of αA by [6, Proposition 1.10]. Set $p = P \cap R$. Then, by the assumption, p is an element of $\operatorname{Dp}_1(R)$. We will prove that $I_{[\alpha^{-1}]} \subset p$. Note that $\eta_d \neq 0$ and $I_{[\alpha^{-1}]} = \cap_{i=0}^{d-1} I_{\eta_d^{-1}\eta_i}$ where $\eta_0 = 1$. Let r be an element of $I_{[\alpha^{-1}]}$, then

$$-r = r\eta_d^{-1}\alpha^d + r\eta_d^{-1}\eta_1\alpha^{d-1} + \cdots + r\eta_d^{-1}\eta_{d-1}\alpha$$

and $r\eta_d^{-1}, r\eta_d^{-1}\eta_1, \ldots, r\eta_d^{-1}\eta_{d-1}$ are in R because r is an element of $I_{[\alpha^{-1}]}$. Hence r is in $P \cap R = p$, and so $I_{[\alpha^{-1}]} \subset p$. Since $I_{[\alpha^{-1}]}$ is a divisorial ideal of R, we see that p is a prime divisor of $I_{[\alpha^{-1}]}$ by [6, Proposition 1.10]. Since $a \,(\operatorname{mod} I_{[\alpha^{-1}]})$ is a non-zero divisor of $R/I_{[\alpha^{-1}]}$, we see that a is not in p. Therefore a is not in P. This implies that $\alpha - a$ is not in P because α is in P. Hence $c/(\alpha - a)$ is in A_P. This implies that $c/(\alpha - a)$ is in $\bigcap_{P \in \operatorname{Dp}_1(A)} A_P = A$. Therefore c is in $(\alpha - a)A$, hence $(\alpha - a)B \cap A \subset (\alpha - a)A$.

(2) The assertion (1) and Proposition 2 (2) show that $(\alpha-a)R[\alpha, \alpha^{-1}] \cap R = (\alpha-a)R[\alpha, \alpha^{-1}] \cap A \cap R = (\alpha - a)A \cap R = I_{[\alpha]}\varphi_\alpha(a)$. Q.E.D.

Proposition 5. *Let R be a Noetherian domain and α an anti-integral element of degree d over R. Set $A = R[\alpha]$ and assume that A/R is a flat extension. Let a be an element of R such that $a \,(\operatorname{mod} I_{[\alpha^{-1}]})$ is a non-zero divisor of $R/I_{[\alpha^{-1}]}$. Then $(\alpha - a)R[\alpha, \alpha^{-1}] \cap R = I_{[\alpha]}\varphi_\alpha(a)$.*

Proof. Since A/R is a flat extension, we get $J_{[\alpha]} = R$ by [3, Proposition 2.6]. Hence α is a super-primitive element of degree d over R by [3, Theorem 1.12]. Let P be a prime divisor of αA. Then $\operatorname{depth} A_P = 1$ by [6, Proposition 1.10]. Flatness of A/R shows that $\operatorname{depth} R_p = 1$ where $p = P \cap R$. So we get the conclusion by Theorem 4. Q.E.D.

Remark 6. Let R be a Noetherian domain and α a super-primitive element of degree d over R. Set $A = R[\alpha]$. Assume that A/R is LCM-stable. Let a be an element of R such that $a\,(\mathrm{mod} I_{[\alpha^{-1}]})$ is a non-zero divisor of $R/I_{[\alpha^{-1}]}$. Then $(\alpha - a)R[\alpha, \alpha^{-1}] \cap R = I_{[\alpha]}\varphi_\alpha(a)$.

Proof. Let P be a prime divisor of αA. Then depth$A_P = 1$ by [6, Proposition 1.10]. Since A/R is LCM-stable, we know that depth$R_p = 1$ where $p = P \cap R$ by [4, Lemma 1]. Therefore Theorem 4 (2) implies that $(\alpha - a)R[\alpha, \alpha^{-1}] \cap R = I_{[\alpha]}\varphi_\alpha(a)$. Q.E.D.

Next we will consider the contraction of an ideal $(a\alpha - 1)R[\alpha, \alpha^{-1}]$.

Lemma 7. *Let R be a Noetherian domain and a an element of R. Then* $(a\alpha - 1)R[\alpha, \alpha^{-1}] \cap R[\alpha] = (a\alpha - 1)R[\alpha]$.

Proof. Set $A = R[\alpha]$ and $B = R[\alpha, \alpha^{-1}]$. The inclusion $(a\alpha - 1)B \cap A \supset (a\alpha - 1)A$ is clear. We will prove the converse inclusion. Let c be an element of $(a\alpha - 1)B \cap A$. Then there exists an element b of B such that $c = (a\alpha - 1)b$. Let P be an element of $\mathrm{Dp}_1(A)$. If α is not in P, then $A_P = B_P$ and $(a\alpha - 1)b$ is in $(a\alpha - 1)B_P = (a\alpha - 1)A_P$. Hence $c/(a\alpha - 1)$ is in A_P. If α is in P, then $a\alpha - 1$ is not in P. Hence $c/(a\alpha - 1)$ is in A_P. This shows that $c/(a\alpha - 1)$ is in $\bigcap_{P \in \mathrm{Dp}_1(A)} A_P = A$. Hence we see that c is in $(a\alpha - 1)A$, and so $(a\alpha - 1)B \cap A \subset (a\alpha - 1)A$. Q.E.D.

Prposition 8. *Let R be a Noetherian domain and α an anti-integral element of degree $d \geq 2$ over R. Assume that $R[\alpha]/R$ is a flat extension. Let a be an elemet of R such that* grade$(I_{[\alpha]} \cap I_{[\alpha^{-1}]} + aR) > 1$. *Then* $(a\alpha - 1)R[\alpha] \cap R = I_{[\alpha^{-1}]}\varphi_{\alpha^{-1}}(a)$.

Proof. Set $A = R[\alpha]$. We will prove that $(a\alpha - 1)A \cap R \supset I_{[\alpha^{-1}]}\varphi_{\alpha^{-1}}(a)$. Let b be an element of $I_{[\alpha^{-1}]}\varphi_{\alpha^{-1}}(a)$. We have only to prove that $b/(a\alpha - 1)$ is in A. Let P be an element of $\mathrm{Dp}_1(A)$ and set $p = P \cap R$. Since A/R is a flat extension, we see that p is in $\mathrm{Dp}_1(R)$. Hence $p \not\supset I_{[\alpha]} \cap I_{[\alpha^{-1}]}$ or $p \not\ni a$ because grade$(I_{[\alpha]} \cap I_{[\alpha^{-1}]} + aR) > 1$. If $p \not\supset I_{[\alpha]} \cap I_{[\alpha^{-1}]}$, then both α and α^{-1} are integral over R_p by [3, Corollary 2.3]. Hence $R_p[\alpha] = R_p[\alpha^{-1}]$ and α is a unit of $R_p[\alpha]$. Therefore $(\alpha^{-1} - a)A_p = (a\alpha - 1)A_p$. Note that α^{-1} is an anti-integral element over R by [1, Theorem 6]. Hence Proposition 2 (1) implies that $(\alpha^{-1} - a)A_p \cap R_p = I_{[\alpha^{-1}]}\varphi_{\alpha^{-1}}(a)R_p$. Since b is in $I_{[\alpha^{-1}]}\varphi_{\alpha^{-1}}(a)R_p$, we see that b is in $(a\alpha - 1)A_p$, and so $b/(a\alpha - 1)$ is in A_p. If $p \not\ni a$, then a is a unit of R_p. Proposition 2 (1) asserts that $(\alpha - a^{-1})A_p \cap R_p = I_{[\alpha]}\varphi_\alpha(a^{-1})R_p$. Besides, we have $a^d I_{[\alpha]}\varphi_\alpha(a^{-1}) = I_{[\alpha^{-1}]}\varphi_{\alpha^{-1}}(a)$. Hence b is in $(\alpha - a^{-1})A_p = (a\alpha - 1)A_p$, and so $b/(a\alpha - 1)$ is in A_p. Therefore $b/(a\alpha - 1)$ is in $\bigcap_{P \in \mathrm{Dp}_1(A)} A_P = A$.

We will prove that $(a\alpha - 1)A \cap R \subset I_{[\alpha^{-1}]}\varphi_{\alpha^{-1}}(a)$. Let c be an element of $(a\alpha - 1)A \cap R$. Then $c/(a\alpha - 1)$ is in A. Since $c/(\alpha^{-1} - a) = \alpha(c/(1 - a\alpha))$, we know that $c/(\alpha^{-1} - a)$ is in A. Hence c is in $(\alpha^{-1} - a)A \cap R = I_{[\alpha^{-1}]}\varphi_{\alpha^{-1}}(a)$. Q.E.D.

Theorem 9. *Let R be a Noetherian domain and α an anti-integral element of degree $d \geq 2$ over R. Assume that $R[\alpha]/R$ is a flat extension. Let a be an element of R such that* grade$(I_{[\alpha]} \cap I_{[\alpha^{-1}]} + aR) > 1$. *Then* $(a\alpha - 1)R[\alpha, \alpha^{-1}] \cap R = I_{[\alpha^{-1}]}\varphi_{\alpha^{-1}}(a)$.

Proof. By Lemma 7 we have $(a\alpha - 1)R[\alpha, \alpha^{-1}] \cap R[\alpha] = (a\alpha - 1)R[\alpha]$. Then Proposition 8 implies that $(a\alpha - 1)R[\alpha, \alpha^{-1}] \cap R = (a\alpha - 1)R[\alpha, \alpha^{-1}] \cap R[\alpha] \cap R = (a\alpha - 1)R[\alpha] \cap R = I_{[\alpha^{-1}]}\varphi_{\alpha^{-1}}(a)$. Q.E.D.

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