THE BULLETIN OF OKAYAMA UNIVERSITY OF SCIENCE No.37 A pp.29-34 (2001)

# Elliptic graded commutative Q-algebras of dimensions 11 and 13 over Q

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(Received November 1, 2001)

## 1. Introduction

Let $A$ be a graded commutative $\mathbf{Q}$-algebra with $\dim_{\mathbf{Q}} A < \infty$. Then $A$ is isomorphic to a truncated weighted polynomial ring $\mathbf{Q}[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$, where $m \geq n$ and $f_i$ is a homogeneous element $(i = 1, \ldots, m)$ (cf. [2]). As defined in [4], we call a graded commutative $\mathbf{Q}$-algebra $A$ *elliptic* if $\dim_{\mathbf{Q}} A < \infty$ and

$$A \cong \mathbf{Q}[x_1, \ldots, x_n]/(f_1, \ldots, f_n),$$

where $f_i$ is a homogeneous element $(i = 1, \ldots, n)$.

According to [1], the rational cohomology algebra of an elliptic space with positive Euler characteristic is an elliptic graded commutative $\mathbf{Q}$-algebra. The classification of isomorphism types of elliptic graded commutative $\mathbf{Q}$-algebras for the cases $1 \leq \dim_{\mathbf{Q}} A \leq 7$ and $\dim_{\mathbf{Q}} A = 10$ are completed in [3] and [4] respectively. The purpose of this paper is to classify the isomorphism type for the cases $\dim_{\mathbf{Q}} A = 11$ and 13. In order to state the following theorems we set

$$|f| = \deg f$$

for a homogeneous element $f$ of weighted polynomial ring $\mathbf{Q}[x_1, \ldots, x_n]$.

**Theorem 1.** *Let $A$ be an elliptic graded commutative $\mathbf{Q}$-algebra. If $\dim_{\mathbf{Q}} A = 11$, then $A$ is isomorphic to one of the following:*

(1) $$\mathbf{Q}[x]/(x^{11}).$$

(2) $$\mathbf{Q}[x_1, x_2]/(x_1 x_2^3, x_1^3 - x_2^2) \qquad (|x_1| = (2/3)|x_2|).$$

(3) $$\mathbf{Q}[x_1, x_2]/(x_1^3 x_2, x_1^5 - x_2^2) \qquad (|x_1| = (2/5)|x_2|).$$

(4) $$\mathbf{Q}[x_1, x_2]/(x_1^2 x_2, x_1^7 - x_2^2) \qquad (|x_1| = (2/7)|x_2|).$$

(5) $$\mathbf{Q}[x_1, x_2]/(x_1 x_2, x_1^9 - x_2^2) \qquad (|x_1| = (2/9)|x_2|).$$

(6) $$\mathbf{Q}[x_1, x_2]/(x_1 x_2^2, x_1^4 - x_2^3) \qquad (|x_1| = (3/4)|x_2|).$$

(7) $$\mathbf{Q}[x_1, x_2]/(x_1^2 x_2, x_1^5 - x_2^3) \qquad (|x_1| = (3/5)|x_2|).$$

(8) $\mathbf{Q}[x_1, x_2]/(x_1x_2, {x_1}^8 - {x_2}^3)$ $(|x_1| = (3/8)|x_2|)$.

(9) $\mathbf{Q}[x_1, x_2]/(x_1x_2, {x_1}^7 - {x_2}^4)$ $(|x_1| = (4/7)|x_2|)$.

(10) $\mathbf{Q}[x_1, x_2]/(x_1x_2, {x_1}^6 - {x_2}^5)$ $(|x_1| = (5/6)|x_2|)$.

(11) $\mathbf{Q}[x_1, x_2, x_3]/(x_1x_3, {x_1}^2 - x_2x_3, {x_2}^3 - {x_3}^2)$ $(|x_1| = (5/4)|x_2| = (5/6)|x_3|)$.

(12) $\mathbf{Q}[x_1, x_2, x_3]/(x_1x_2, {x_1}^3 - x_2x_3, {x_2}^3 - {x_3}^2)$ $(|x_1| = (5/6)|x_2| = (5/9)|x_3|)$.

(13) $\mathbf{Q}[x_1, x_2, x_3]/(x_1x_2, {x_1}^2 - {x_2}^2x_3, {x_2}^3 - {x_3}^2)$ $(|x_1| = (7/4)|x_2| = (7/6)|x_3|)$.

(14) $\mathbf{Q}[x_1, x_2, x_3]/(x_1x_2, {x_1}^2 - x_2x_3, {x_2}^5 - {x_3}^2)$ $(|x_1| = (7/4)|x_2| = (7/10)|x_3|)$.

**Theorem 2.** *Let $A$ be an elliptic graded commutative $\mathbf{Q}$-algebra. If $\dim_{\mathbf{Q}} A = 13$, then $A$ is isomorphic to one of the following:*

(1) $\mathbf{Q}[x]/(x^{13})$.

(2) $\mathbf{Q}[x_1, x_2]/({x_1}^2{x_2}^3, {x_1}^3 - {x_2}^2)$ $(|x_1| = (2/3)|x_2|)$.

(3) $\mathbf{Q}[x_1, x_2]/({x_1}^4x_2, {x_1}^5 - {x_2}^2)$ $(|x_1| = (2/5)|x_2|)$.

(4) $\mathbf{Q}[x_1, x_2]/({x_1}^3x_2, {x_1}^7 - {x_2}^2)$ $(|x_1| = (2/7)|x_2|)$.

(5) $\mathbf{Q}[x_1, x_2]/({x_1}^2x_2, {x_1}^9 - {x_2}^2)$ $(|x_1| = (2/9)|x_2|)$.

(6) $\mathbf{Q}[x_1, x_2]/(x_1x_2, {x_1}^{11} - {x_2}^2)$ $(|x_1| = (2/11)|x_2|)$.

(7) $\mathbf{Q}[x_1, x_2]/({x_1}^3x_2, {x_1}^4 - {x_2}^3)$ $(|x_1| = (3/4)|x_2|)$.

(8) $\mathbf{Q}[x_1, x_2]/(x_1{x_2}^2, {x_1}^5 - {x_2}^3)$ $(|x_1| = (3/5)|x_2|)$.

(9) $\mathbf{Q}[x_1, x_2]/({x_1}^2x_2, {x_1}^7 - {x_2}^3)$ $(|x_1| = (3/7)|x_2|)$.

(10) $\mathbf{Q}[x_1, x_2]/(x_1x_2, {x_1}^{10} - {x_2}^3)$ $(|x_1| = (3/10)|x_2|)$.

(11) $\mathbf{Q}[x_1, x_2]/({x_1}^2x_2, {x_1}^5 - {x_2}^4)$ $(|x_1| = (4/5)|x_2|)$.

(12) $\mathbf{Q}[x_1, x_2]/(x_1x_2, {x_1}^9 - {x_2}^4)$ $(|x_1| = (4/9)|x_2|)$.

(13) $\mathbf{Q}[x_1, x_2]/(x_1x_2, {x_1}^8 - {x_2}^5)$ $(|x_1| = (5/8)|x_2|)$.

(14) $$\mathbf{Q}[x_1, x_2]/(x_1x_2, {x_1}^7 - {x_2}^6) \qquad (|x_1| = (6/7)|x_2|).$$

(15) $$\mathbf{Q}[x_1, x_2, x_3]/(x_1x_3, {x_1}^2 - {x_2}^2x_3, {x_2}^3 - {x_3}^2) \qquad (|x_1| = (7/4)|x_2| = (7/6)|x_3|).$$

(16) $$\mathbf{Q}[x_1, x_2, x_3]/(x_1x_2, {x_1}^3 - {x_2}^2x_3, {x_2}^3 - {x_3}^2) \qquad (|x_1| = (7/6)|x_2| = (7/9)|x_3|).$$

(17) $$\mathbf{Q}[x_1, x_2, x_3]/(x_1x_2, {x_1}^2 - {x_2}^3x_3, {x_2}^3 - {x_3}^2) \qquad (|x_1| = (9/4)|x_2| = (3/2)|x_3|).$$

(18) $$\mathbf{Q}[x_1, x_2, x_3]/(x_1{x_2}^2, {x_1}^2 - x_2x_3, {x_2}^3 - {x_3}^2) \qquad (|x_1| = (5/4)|x_2| = (5/6)|x_3|).$$

(19) $$\mathbf{Q}[x_1, x_2, x_3]/(x_1x_2, {x_1}^4 - x_2x_3, {x_2}^3 - {x_3}^2) \qquad (|x_1| = (5/8)|x_2| = (5/12)|x_3|).$$

(20) $$\mathbf{Q}[x_1, x_2, x_3]/(x_1x_2, {x_1}^2 - {x_2}^2x_3, {x_2}^5 - {x_3}^2) \qquad (|x_1| = (9/4)|x_2| = (9/10)|x_3|).$$

(21) $$\mathbf{Q}[x_1, x_2, x_3]/(x_1x_2, {x_1}^3 - x_2x_3, {x_2}^5 - {x_3}^2) \qquad (|x_1| = (7/6)|x_2| = (7/15)|x_3|).$$

(22) $$\mathbf{Q}[x_1, x_2, x_3]/(x_1x_2, {x_1}^2 - x_2x_3, {x_2}^7 - {x_3}^2) \qquad (|x_1| = (9/4)|x_2| = (9/14)|x_3|).$$

(23) $$\mathbf{Q}[x_1, x_2, x_3]/(x_1x_2, {x_1}^2 - x_2x_3, {x_2}^4 - {x_3}^3) \qquad (|x_1| = (7/6)|x_2| = (7/8)|x_3|).$$

## 2. Proof of Theorems

Let $A \cong \mathbf{Q}[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ be an elliptic graded commutative $\mathbf{Q}$-algebra. Then according to [1], we have

(2.1) $$\dim_{\mathbf{Q}} A = |f_1| \cdots |f_n| / |x_1| \cdots |x_n|.$$

We assume that each $f_i$ $(i = 1, \ldots, n)$ has no linear terms. It follows from (2.1) and [3, Lemma 2.1] that

(2.2) $$\dim_{\mathbf{Q}} A \geq 2^n.$$

**Lemma 2.3.** *Let $p$ be a prime. If $n = 2$ and $\dim_{\mathbf{Q}} A = p$, then $p \geq 5$ and there exist positive integers $a, b, k$ and $l$ with $2 \leq l < k$, $a < k$, $al + bk = p$ and*

$$A \cong \mathbf{Q}[x_1, x_2]/({x_1}^a{x_2}^b, {x_1}^k - {x_2}^l),$$

*where $|x_2| = (k/l)|x_1|$.*

*proof.* We assume that $|x_1| \leq |x_2|$ and $f_2 \notin (x_1)$. Then there exists an integer $l \geq 2$ with $|f_2| = l|x_2|$. By (2.1), $|f_1| = (p/l)|x_1|$. This implies that $f_1 \in (x_2)$, and hence $f_2 \notin (x_2)$. Then there exists an integer $k > l$ with $|f_2| = k|x_1|$. By (2.1), $|f_1| = (p/k)|x_2|$. There exist non-negative integers $c$ and $d$ with

$$c|x_1| + d|x_2| = |f_1|.$$

This implies that $cl + dk = p$, and hence $(cl, dk) = 1$. Set $a = c - k[c/k]$ and $b = d + l[c/k]$. Then $al + bk = p$, $1 \leq a < k$ and $1 \leq b$.

Suppose that there exist non-negative integers $s$ and $t$ with $s|x_1| + t|x_2| = |f_2|$. This implies that $sl + tk = kl$. Since $(l, k) = 1$ and $0 \le t \le l$, $t = s - k = 0$ or $t - l = s = 0$. So we can assume that $f_2 = x_1{}^k - rx_2{}^l$ for $r \neq 0$. Since $(l, k) = 1$, there exist integers $u$ and $v$ with $ku + lv = 1$. Put $y_1 = r^{-u}x_1$ and $y_2 = r^v x_2$. Then $f_2 = r^{ku}(y_1{}^k - y_2{}^l)$. So we can assume that $f_2 = x_1{}^k - x_2{}^l$.

Suppose that there exist non-negative integers $s$ and $t$ with $s|x_1| + t|x_2| = |f_1|$. This implies that $sl + tk = p = al + bk$. Since $(l, k) = 1$, there exists an integer $i$ with $s - a = ki$ and $b - t = li$. Then

$$x_1{}^s x_2{}^t = x_1{}^{a+ki} x_2{}^{b-li} \equiv x_1{}^a x_2{}^b \pmod{(f_2)}.$$

So we can assume that $f_1 = x_1{}^a x_2{}^b$. q.e.d.

**Lemma 2.4.** *Let $p$ be a prime. If $n = 3$ and $\dim_{\mathbf{Q}} A = p$, then $p \ge 11$ and there are following two possibilities.*
*(1) There exist non-negative integers $a, b, c, d, e, k, l$ and $m$ with $m \ge 2$, $2 \le l < k$, $1 \le a \le k$, $1 \le b$, $1 \le c < m$, $1 \le d + e$, $d < k$, $(ac + dm)l + (bc + em)k = p$ and*

$$A \cong \mathbf{Q}[x_1, x_2, x_3]/(x_1{}^c x_2{}^d x_3{}^e, x_1{}^m - x_2{}^a x_3{}^b, x_2{}^k - x_3{}^l),$$

*where $|x_2| = (lm/(al + bk))|x_1|$ and $|x_3| = (km/(al + bk))|x_1|$.*
*(2) There exist positive integers $a, b, c, k, l$ and $m$ with $2 \le m < l < k$, $a < k$, $b < l$, $alm + bkm + ckl = p$ and*

$$A \cong \mathbf{Q}[x_1, x_2, x_3]/(x_1{}^a x_2{}^b x_3{}^c, x_1{}^k - x_2{}^l, x_2{}^l - x_3{}^m),$$

*where $|x_2| = (k/l)|x_1|$ and $|x_3| = (k/m))|x_1|$.*

*proof.* Suppose that

$$\{f_1, f_2\} \subset (x_1, x_2) \cap (x_1, x_3) \cap (x_2, x_3).$$

This implies that $f_3 \notin (x_1, x_2) \cup (x_1, x_3) \cup (x_2, x_3)$, and there exist integers $k, l$ and $m$ with $|f_3| = k|x_1| = l|x_2| = m|x_3|$. We assume that $|x_1| \le |x_2| \le |x_3|$. Then $2 \le m \le l \le k$. Since $\{f_1, f_2\} \not\subset (x_1)$, we can assume that $f_2 \notin (x_1)$. There exist positive integers $a$ and $b$ with $a|x_2| + b|x_3| = |f_2|$. By (2.1),

$$|f_1| = (p/(am + bl))|x_1| = (pl/k(am + bl))|x_2| = (pm/k(am + bl))|x_3|.$$

There exist non-negative integers $c, d$ and $e$ with $c + d + e \ge 2$ and $c|x_1| + d|x_2| + e|x_3| = |f_1|$. This implies that $(clm + dkm + ekl)(am + bl) = plm$. Let $g = (m, l)$, $m = gm'$ and $l = gl'$. Then $(cl'm'g + dkm' + ekl')(am' + bl') = pl'm'$. Since $(cl'm'g + dkm' + ekl') > l'm'$, $am' + bl' < p$. So $p|(cl'm'g + dkm' + ekl')$. Let $cl'm'g + dkm' + ekl' = pj$. Then $(l' - ja)m' = jbl'$. Since $(l', m') = 1$, $l'|(l' - ja)$. This contradicts the fact $0 < (l' - ja) < l'$.

Therefore, we can assume that $f_3 \notin (x_1, x_2)$ and $f_2 \notin (x_1, x_3)$. There exist integers $k$ and $l$ with $k \ge 2$, $l \ge 2$, $|f_3| = k|x_3|$ and $|f_2| = l|x_2|$. By (2.1), $|f_1| = (p/kl)|x_1|$. This implies that $f_1 \in (x_2, x_3)$, and hence $\{f_2, f_3\} \not\subset (x_2, x_3)$.

Thus we can assume that $f_3 \notin (x_1, x_2) \cup (x_1, x_3)$ and $f_2 \notin (x_2, x_3)$. There exist integers $k, l$ and $m$ with $|f_3| = l|x_3| = k|x_2|$ and $|f_2| = m|x_1|$. We assume that $|x_2| \le |x_3|$. Then $2 \le l \le k$ and $m \ge 2$. By (2.1),

$$|f_1| = (p/lm)|x_2| = (p/km)|x_3|.$$

This implies that $f_1 \in (x_1)$, and hence $f_2 \notin (x_1)$. There exist non-negative integers $a'$ and $b'$ with $a' + b' \ge 2$ and $a'|x_2| + b'|x_3| = |f_2|$. If $a'b' + [a'/(k+1)] + [b'/(l+1)] \ge 1$, set $a = a' - k[(a'-1)/k]$ and $b = b' + l[(a'-1)/k]$. Then $1 \le a \le k$, $1 \le b$ and $a|x_2| + b|x_3| = |f_2|$. If $a' + [b'/l] = 0$, then $f_2 \notin (x_2, x_3) \cup (x_1, x_2)$, $f_3 \notin (x_1, x_3)$, $m \ge 1$ and $l - b' \ge 1$. If $b' + [a'/k] = 0$, then $f_2 \notin (x_2, x_3) \cup (x_1, x_3)$, $f_3 \notin (x_1, x_2)$, $m \ge 1$ and $k - a' \ge 1$.

Suppose that $(a'^2 + (b' - l)^2)(b'^2 + (a' - k)^2) > 0$. Then we can assume that there exist positive integers $a$ and $b$ with $a \le k$ and $a|x_2| + b|x_3| = |f_2|$. By (2.1),

$$|f_1| = (p/(al + bk))|x_1| = (p/ml)|x_2| = (p/km)|x_3|.$$

There exist non-negative integers $c', d'$ and $e'$ with $c'+d'+e' \geq 2$ and $c'|x_1| + d'|x_2| + e'|x_3| = |f_1|$. This implies that $c'(al+bk)+d'lm+e'km = p$. Then $(c'(al+bk), m(d'l+e'k)) = 1$. This implies that $c' > 0$ and $d'+e' > 0$. Set $c = c' - m[c'/m]$, $d = d' + a[c'/m] - k[(d'+a[c'/m])/k]$ and $e = e' + b[c'/m] + l[(d'+a[c'/m])/k]$. Then $1 \leq c < m$, $d < k$, $d+e \geq 1$ and $c(al+bk)+dlm+ekm = p$.

Suppose that there exist non-negative integers $s, t$ and $u$ with

$$s|x_1| + t|x_2| + u|x_3| = |f_3|.$$

Then $s(al+bk)+tlm+ukm = klm$. Since $(al+bk, m) = 1$, $m|s$. Let $s = ms'$. Then $(t+s'a)l + (s'b+u)k = kl$. Since $(l,k) = 1$ and $0 \leq t+s'a \leq k$, $u = s = t-k = 0$ or $t = s = u-l = 0$. So we can assume that $f_3 = x_2{}^k - rx_3{}^l$ for $r \neq 0$. Since $(l,k) = 1$, there exist integers $u, v$ with $ku+lv = 1$. Put $y_2 = r^{-u}x_2$ and $y_3 = r^v x_3$. Then $f_3 = r^{ku}(y_2{}^k - y_3{}^l)$. So we can assume that $f_3 = x_2{}^k - x_3{}^l$.

Suppose that there exist non-negative integers $s, t$ and $u$ with $s|x_1| + t|x_2| + u|x_3| = |f_2|$. Then $s(al+bk)+tlm+ukm = (al+bk)m$. Since $(al+bk, m) = 1$ and $0 \leq s \leq m$, $t = u = s-m = 0$ or $s = (t-a)l + (u-b)k = 0$. If $s = (t-a)l+(u-b)k = 0$, $k|(t-a)$. Let $t-a = ki$. Then $u-b = -li$ and

$$x_2{}^t x_3{}^u = x_2{}^{a+ki} x_3{}^{b-li} \equiv x_2{}^a x_3{}^b \quad (\mathrm{mod}\ (f_3)).$$

So we can assume that $f_2 = x_1{}^m - rx_2{}^a x_3{}^b$ for $r \neq 0$. Since $(al+bk, m) = 1$, there exist integers $s$ and $t$ with $sm + t(al+bk) = 1$. Put $y_1 = r^{-s}x_1$, $y_2 = r^{tl}x_2$ and $y_3 = r^{tk}x_3$. Then $f_3 = r^{-tlk}(y_2{}^k - y_3{}^l)$ and $f_2 = r^{sm}(y_1{}^m - y_2{}^a y_3{}^b)$. So we can assume that $f_2 = x_1{}^m - x_2{}^a x_3{}^b$.

Suppose that there exist non-negative integers $s, t$ and $u$ with $s|x_1| + t|x_2| + u|x_3| = |f_1|$. Then $s(al+bk)+tlm+ukm = p = c(al+bk)+dlm+ekm$. Since $(al+bk, m) = 1$, $m|(s-c)$. Let $s-c = mi$. Then $(t-d+ai)l = (e-u-bi)k$. Since $(l,k) = 1$, $k|(t-d+ai)$. Let $t-d+ai = kj$. Then $u = e-bi-jl$ and

$$x_1{}^s x_2{}^t x_3{}^u = x_1{}^{c+mi} x_2{}^{d-ai+kj} x_3{}^{e-bi-jl} \equiv x_1{}^c x_2{}^d x_3{}^e \quad (\mathrm{mod}\ (f_2, f_3)).$$

So we can assume that $f_1 = x_1{}^c x_2{}^d x_3{}^e$.

Now we turn to the case $(a'^2 + (b'-l)^2)(b'^2 + (a'-k)^2) = 0$. By (2.1),

$$|f_1| = (p/kl)|x_1| = (p/ml)|x_2| = (p/km)|x_3|.$$

There exist non-negative integers $a', b'$ and $c'$ with $a'+b'+c' \geq 2$ and $a'|x_1| + b'|x_2| + c'|x_3| = |f_1|$. This implies that $a'kl + b'lm + c'km = p$. Then $(a'kl, m) = (b'lm, k) = (c'km, l) = 1$. Put $a = a' - m[a'/m]$, $b = b' - k[b'/k]$ and $c = c' + l([a'/m]+[b'/k])$. Then $0 < a < m$, $0 < b < k$, $0 < c$ and $akl+blm+ckm = p$.

Suppose that there exist non-negative integers $s, t$ and $u$ with

$$s|x_1| + t|x_2| + u|x_3| = |f_2| = |f_3|.$$

Then $skl + tml + umk = klm$. Since $(kl, m) = (k, l) = 1$, $0 \leq s \leq m$ and $0 \leq t \leq k$, $t = u = s-m = 0$, $s = t-k = u = 0$ or $s = t = u-l = 0$. This implies that $f_2$ and $f_3$ are linear combinations of $x_1{}^m, x_2{}^k$ and $x_3{}^l$. There exists $d \in \mathbf{Q}$ such that the coefficient of the term $x_1{}^m$ of the polynomial $f_3' = f_3 + df_2$ is 0. Since $f_3' \notin (x_2) \cup (x_3)$, we can assume that $f_3 = x_2{}^k - sx_3{}^l$ for $s \neq 0$. There exists $e \in \mathbf{Q}$ such that the coefficient of the term $x_3{}^l$ of the polynomial $f_2' = f_2 + ef_3$ is 0. Since $f_2' \notin (x_1) \cup (x_2)$, we can assume that $f_2 = x_1{}^m - rx_2{}^k$ for $r \neq 0$. Since $(m,k) = (km, l) = 1$, there exist integers $t, u, v$ and $w$ with $tm + uk = 1 = vmk + wl$. Set $y_1 = s^{-kv}r^{-k^2uv-t}x_1$, $y_2 = s^{-mv}r^{-kmuv+u}x_2$ and $y_3 = s^w r^{kuw}x_3$. Then

$$f_2 = s^{kmv}r^{m(k^2uv+t)}(y_1{}^m - y_2{}^k) \qquad \text{and} \qquad f_3 = s^{kmv}r^{uk(kmv-1)}(y_2{}^k - y_3{}^l).$$

So we can assume that $f_2 = x_1{}^m - x_2{}^k$ and $f_3 = x_2{}^k - x_3{}^l$.

Suppose that there exist non-negative integers $s, t$ and $u$ with $s|x_1| + t|x_2| + u|x_3| = |f_1|$. Then $skl + tlm + ukm = p = akl + blm + ckm$. Since $(kl, m) = 1$, $m|(s-a)$. Let $s-a = mi$. Then $(t-b)l = (c-u-il)k$. Since $(l,k) = 1$, $k|(t-b)$. Let $t-b = kj$. Then $u = c-(i+j)l$ and

$$x_1{}^s x_2{}^t x_3{}^u = x_1{}^{a+mi} x_2{}^{b+kj} x_3{}^{c-(i+j)l} \equiv x_1{}^a x_2{}^b x_3{}^c \quad (\mathrm{mod}\ (f_2, f_3)).$$

So we can assume that $f_1 = x_1{}^a x_2{}^b x_3{}^c$. q.e.d.

**Remark.** The type (2) of Lemma 2.4 dose not occur for $p < 31$.

**Proof of Theorem 1.** Suppose that $n = 1$. Then $A$ is isomorphic to a type of (1) of Theorem 1.
Suppose that $n = 2$. The possibilities of $(l, k, a, b)$ of Lemma 2.3 are $(2,3,1,3)$, $(2,5,3,1)$, $(2,7,2,1)$, $(2,9,1,1)$, $(3,4,1,2)$, $(3,5,2,1)$, $(3,8,1,1)$, $(4,7,1,1)$ and $(5,6,1,1)$ corresponding to (2), (3), (4), (5), (6), (7), (8), (9) and (10) of Theorem 1 respectively.
Suppose that $n = 3$. The possibilities of $(l, k, a, b, c, d, e, m)$ of (1) of Lemma 2.4 are (2, 3, 1, 1, 1, 0, 1, 2), (2, 3, 1, 1, 1, 1, 0, 3), (2, 3, 2, 1, 1, 1, 0, 2) and (2, 5, 1, 1, 1, 1, 0, 2) corresponding to (11), (12), (13) and (14) of Theorem 1 respectively.

**Proof of Theorem 2.** Suppose that $n = 1$. Then $A$ is isomorphic to a type of (1) of Theorem 2.
Suppose that $n = 2$. The possibilities of $(l, k, a, b)$ of Lemma 2.3 are $(2,3,2,3)$, $(2,5,4,1)$, $(2,7,3,1)$, $(2,9,2,1)$, $(2,11,1,1)$, $(3,4,3,1)$, $(3,5,1,2)$, $(3,7,2,1)$, $(3,10,1,1)$, $(4,5,2,1)$, $(4,9,1,1)$, $(5,8,1,1)$ and $(6,7,1,1)$ corresponding to (2), (3), (4), (5), (6), (7), (8), (9), (10), (11), (12), (13) and (14) of Theorem 2 respectively.
Suppose that $n = 3$. The possibilities of $(l, k, a, b, c, d, e, m)$ of (1) of Lemma 2.4 are (2, 3, 2, 1, 1, 0, 1, 2), (2, 3, 2, 1, 1, 1, 0, 3), (2, 3, 3, 1, 1, 1, 0, 2), (2, 3, 1, 1, 1, 2, 0, 2), (2, 3, 1, 1, 1, 1, 0, 4), (2, 5, 2, 1, 1, 1, 0, 2), (2, 5, 1, 1, 1, 1, 0, 3), (2, 7, 1, 1, 1, 1, 0, 2) and (3, 4, 1, 1, 1, 1, 0, 2) corresponding to (15), (16), (17), (18), (19), (20), (21), (22) and (23) of Theorem 2 respectively.

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