THE BULLETIN OF OKAYAMA UNIVERSITY OF SCIENCE No.37 A pp.23-28 (2001)

Attractive Basins of Fixed Points of Halley Iterations

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(Received November 1, 2001)

1. Introduction

In [1] it was observed by a numerical experiment that, for one-dimensional dynamics arising from Newton and Halley iterations applied to polynomials $x^q+(\lambda-1)x-\lambda$, $q=3,5,\cdots$, there are geometrically convergent λ-sequences. However, in that paper, it remained unsolved that the Halley iteration of $x^4+(\lambda-1)x-\lambda$ admits no periodic orbits of periods $\geqq 2$. In this article we shall confirm this fact by proving that, for the above polynomials of degree 4, attractive basins of attractive fixed points, together with backward orbits of a repulsive fixed point, cover the whole real line. For this purpose we examine the order structure of fixed points and the condition for periodic points of periods $\geqq 2$ to exist generally.

2. Halley Iterations of $x^q+(\lambda-1)x-\lambda$

First we recall some basic notations from [1]. For a smooth function $f(x)$ the *Halley method* of $f(x)$ is defined to be

$$H_f(x)=x-2f(x)f'(x)/\Phi_f(x) \tag{2.1}$$

where

$$\Phi_f(x)=2f'(x)^2-f(x)f''(x). \tag{2.2}$$

It is known (see [2]) that the derivative is given by

$$H_f'(x)=f(x)^2\,\Phi_f(x)^{-2}\,\Psi_f(x) \tag{2.3}$$

where

$$\Psi_f(x)=3f''(x)^2-2f'(x)f'''(x). \tag{2.4}$$

This shows that the critical points of $H_f(x)$ are roots of $f(x)=0$ and roots of $\Psi_f(x)=0$, the latter of which are the so-called *free* critical points of the Halley method. We shall be concerned with the 1-dimensional dynamics arising from iteration of H_f. Let a be an attractive fixed point of this dynamics; then the immediate attractive basin is denoted by $A_*(a)$ and the attractive basin by $A(a)$.

From now on we take for f a one-parameter family of polynomials

$$f(x)=f_\lambda(x)=x^q+(\lambda-1)x-\lambda,\quad q\geqq 3$$

Then one has

$$H_f(x)=x-(x^q+(\lambda-1)x-\lambda)(qx^{q-1}+\lambda-1)/\Phi_f(x) \tag{2.5}$$

where

$$\Phi_f(x)=q(q+1)x^{2q-2}-q(q-5)(\lambda-1)x^{q-1}+q(q-1)\lambda x^{q-2}+2(\lambda-1)^2. \tag{2.6}$$

Since

$$\Psi_f(x)=q(q-1)x^{q-3}\{q(q+1)x^{q-1}-2(q-2)(\lambda-1)\}, \tag{2.7}$$

one has free critical points 0 (if $q>3$) and

$$\gamma=[2(q-2)(\lambda-1)q^{-1}(q+1)^{-1}]^{1/(q-1)}\quad \text{for even } q,$$

or respectively,

$$\gamma = -[2(q-2)(\lambda-1)q^{-1}(q+1)^{-1}]^{1/(q-1)} \quad \text{for odd } q,\ \lambda \geqq 1.$$

Note that $H_f(x)$ takes a local maximum at γ.

Since $H_f'(x)$ tends to $(q-1)/(q+1)$ as $|x| \to \infty$, we see that the graph of H_f has a asymptotic straight line with this limit as slope. It will be interesting that this value happens to coincide with the conjectural asymptotic ratio stated in [1]. Similar phenomena are observed for Newton method of f_λ.

The fixed point set consists of the solutions of $f_\lambda(x)=0$ and of the solution

$$\beta = ((1-\lambda)/q)^{1/(q-1)}$$

of $f_\lambda'(x)=qx^{q-1}+\lambda-1=0$, where we assume $\lambda \leqq 1$ for q odd. Since

$$H_f'(\beta)=3f(\beta)^2 f''(\beta)^2/[f(\beta)f'(\beta)]^2=3,$$

we see that β is a repulsive fixed point. Note that $f_\lambda(x)=(x-1)(x^{q-1}+x^{q-2}+\cdots+1)$.

Lemma 2.1. *The function $\zeta(x)=x^{q-1}+x^{q-2}+\cdots+x+1$ $(q \geqq 2)$ is strictly increasing for even q and convex for odd q.*

Proof. Let q be even; then $\zeta(1)=q>0$. Since $\zeta(x)=(x^q-1)/(x-1)$ if $x \neq 1$, we have

$$\zeta'(x)=[(q-1)x^q-qx^{q-1}+1]/(x-1)^2, \tag{2.8}$$

in which the derivative of the numerator is $q(q-1)x^{q-2}(x-1)$, and therefore the numerator takes the minimum 0 at $x=1$. It follows that $\zeta'(x)>0$.

Next we assume that q is odd. For $x \geqq 0$ we obviously have

$$\zeta'(x)=(q-1)x^{q-1}+\cdots+2x+1>0.$$

Using the above formula of $\zeta'(x)$, one has

$$\zeta''(x)=(N_1+N_2)/(1-x)^2,$$

where $N_1=2(1+x+\cdots x^{q-2})$, $N_2=(q-1)[(q-2)x-q]x^{q-2}$. For x with $-1<x<0$ we have $N_1=(1-x^{q-1})/(1-x)>0$ and $N_2>0$. For x with $x \leqq -1$, since

$$(q-2)x-q \leqq -2q-2 \leqq -2$$

and since N_1+N_2 is the sum of $\{(q-2)x-q\}x^{q-2}+2x^k$ over $0 \leqq k \leqq q-2$, we see that $\zeta''(x)>0$.

In the light of Lemma 2.1 the equation $x^{q-1}+x^{q-2}+\cdots+x+\lambda = \zeta(x)-1+\lambda=0$ has the only one solution α if q is even and at most two roots if q is odd. In case q is odd and if there exist roots, we denote the greatest one by α and the other by α' $(\alpha' \leqq \alpha)$. Note that 1, α and α' are super-attractive fixed points.

Proposition 2.2. *The following inequalities hold:*

$1<\beta<\alpha$ *for q even and* $\lambda<1-q$,

$1=\alpha=\beta$ *for q even and* $\lambda=1-q$,

$\alpha<\beta<1$ *for q even and* $1 \geqq \lambda > 1-q$,

$\alpha'<-\beta<\alpha<\beta<1$ *for q odd and* $\lambda>0$.

Proof. First let q be even. If $\lambda<1-q$ then $\beta>1$ and

$$\zeta(\beta)-1+\lambda=(\beta-1)+\cdots+(\beta^{q-1}-1)+q-1+\lambda$$
$$=(q-1)(\beta^{q-1}-1)+q-1+\lambda=(q-1+\lambda)/q<0,$$

thereby proving $\beta<\alpha$ by Lemma 2.1. Similarly, if $\lambda>1-q$ then $\beta>1$ and $\alpha<\beta$.

Next let q be odd. Note that $0<\lambda$ implies $\lambda>1-q$, thus $\beta>1$. Hence $\alpha<\beta$. On the other hand, we have

$$\zeta(-\beta)-1+\lambda<\zeta(\beta)-1+\lambda<q\beta^{q-1}-1+\lambda=0,$$

which implies $\alpha'<-\beta$.

According to Proposition 2.2, in case q is odd, $\alpha'=\alpha$, i.e., $y=\zeta(x)-1+\lambda$ is tangent to the x-axis if and only if $\zeta(-\beta)-1+\lambda=0$, the latter of which is written as

$$-\beta^{q-1}\beta-1+(1-\lambda)(\beta+1)=0,$$

in other words,

$$(1-\lambda)(q-1)\beta=q\lambda,$$

raising to the $(q-1)$-th power,

$$(1-\lambda)^q(q-1)^{q-1}=q^q\lambda^{q-1}$$

We denote λ with this property by $\lambda_\#$ (see Table 3).

Proposition 2.3. *If q is odd and $\lambda<1$ then $\gamma<-\beta$. If q is even and $\lambda<1$ then $\gamma<\beta$.*

Proof. This follows from

$$\Psi_f(-\beta)=3q(q-1)^2\beta^{q-3}(1-\lambda)>0 \qquad \text{for odd } q,$$
$$\Psi_f(\beta)=3q(q-1)^2\beta^{q-3}(1-\lambda)>0 \qquad \text{for even } q.$$

Proposition 2.4. *For odd q and $\lambda\geqq 0$, $\Phi_f(x)>0\ (x\geqq 0)$ and the curve $y=\Phi_f(x)$ meets the line $x=0$ at most two points. For q even and $\lambda<0$, the curve $y=\Phi_f(x), x<0$, meets the line $y=0$ at most two points.*

Proof. Put $\phi(x)=2(q+1)x^q-(q-5)(\lambda-1)x+(q-2)\lambda$. Then, by (2.6),

$$\Phi_f'(x)=q(q-1)x^{q-3}\phi(x),$$
$$\phi'(x)=2q(q+1)x^{q-1}-(q-5)(\lambda-1).$$

Let ξ denote the root $[(q-5)(\lambda-1)/(2q(q+1))]^{1/(q-1)}$ of $\phi'(x)=0$.

Let q be odd >5 and $\lambda>1$. Then $\phi(0)=(q-5)\lambda>0$. Since $0<\xi<1$, $-1<q^{-1}-1<0$ and since $0<(q-5)(\lambda-1)<(q-2)\lambda$, we have

$$\phi(\xi)=\xi(q-5)(\lambda-1)(q^{-1}-1)+(q-2)\lambda>0.$$

Since $\phi(x)$ has a local minimum at ξ and a local maximum at $-\xi$, we see that $\phi(x)>0$ for $x\geqq-\xi$, hence $\Phi_f'(x)\geqq 0$ for $x>z$ where $z<0$ denotes the only one root of $\phi(x)=0$. This settles the case q odd.

Next assume that q is even and $\lambda<0$. Then $\xi<0$ and $\phi(x)$ is increasing for $x>\xi$ and decreasing for $x<\xi$.

If $\phi(x)\geqq 0$ for all x, then $\Phi_f(x)>0$ and we are done. Therefore we may assume that there exist z and w such that $\phi(z)=0=\phi(w)$, $w<z$. Since $\phi(0)=(q-2)\lambda<0$, we have $w<\xi<0<z$ and we see from $\phi(1)=3(q-1+\lambda)$ that $z>1$ or <1 according as $\lambda<1-q$ or $>1-q$. These imply that Φ_f takes local minimum, maximum and minimum respectively at w, 0 and z. This establishes our assertion for q even.

Remark. Set $t=z^q$; then $2(q+1)t=(q-5)(\lambda-1)z-(q-2)\lambda$. A little algebra shows that

$$4(q+1)z^2\Phi_f(z)=-(\lambda-1)^2(q-1)^2(q-8)z^2+2\lambda(\lambda-1)q(q-1)(q-5)z-q^2(q-2)\lambda^2$$

and hence

$$\begin{aligned}4(q+1)(q-8)z^2\Phi_f(z)&=-[(\lambda-1)(q-1)(q-8)z-\lambda q(q-5)]^2+9\lambda^2q^2\\&=[3\lambda q-(\lambda-1)(q-1)(q-8)z+\lambda q(q-5)]\\&\qquad\cdot[3\lambda q+(\lambda-1)(q-1)(q-8)z-\lambda q(q-5)]\\&=[\lambda q(q-2)-(\lambda-1)(q-1)(q-8)z]\cdot(q-8)[-\lambda q+(\lambda-1)(q-1)z]\end{aligned}$$

$(q-5)$ times the first factor is equal to $4q(q-2)^2\lambda-2(q^2-1)(q-8)t<0$. On the other hand, $(q-5)$ times the third factor equals $2(q^2-1)t+2\lambda(q+1)=2(q+1)[(q-1)t+\lambda]$, which is negative iff

$$z<(-\lambda/(q-1))^{1/q}.$$

This leads to the following

Conjecture 1. $\phi([-\lambda/(q-1)]^{1/q})\geqq 0$, *the equality holds only if* $\lambda=1-q$.

Since numerical experiments support this conjecture, it might be much likely that, for $x>0$, q even and $\lambda\neq 1-q$, $\Phi_f(x)>0$ for $x\geqq 0$, q even and $\lambda\neq 1-q$.

Since $f(x)f'(x)$ is of odd degree, it follows from Proposition 2.2 that

$$f(x)f'(x)>0 \text{ for } x>\max(1,\alpha),$$
$$f(x)f'(x)<0 \text{ for } \min(1,\alpha,\alpha'),$$
$$A_*(\max(1,\alpha))=(\beta,\infty).$$

If $\Phi_f(x)>0$ then $H_f(x)$ is continuous and $H_f(x)<x$ and

$$H_f(x)>x \text{ and } A_*(\min(1,\alpha))=(-\infty,\beta) \qquad \text{for even } q,$$
$$H_f(x)>x \text{ and } A_*(\alpha)=(-\beta,\beta),\ A_*(\alpha')=(-\infty,-\beta) \text{ for odd } q$$

If Φ_f takes a negative value, that is, the curve $y=\Phi_f(x)$ meets $y=0$ at two points z_1, z_2 with $z_1<z_2$, then *it might be likely that* $z_1<\gamma<z_2$ *and, if* $H_f(\gamma)<\gamma$ *then the critical point* γ *gives rise to periodic orbits other than fixed points and* H_f *takes a local maximum at* γ. Let

$$\lambda_\#=\inf\{\lambda:\Phi_f(\gamma)<0\},\ \lambda^\#=\sup\{\lambda:\Phi_f(\gamma)<0\} \text{ for even } q,$$
$$\lambda^\#=\{\lambda:\Phi_f(\gamma)<0\} \qquad \text{for odd } q.$$

These values $\lambda_\#$ and $\lambda^\#$ can be computed by finding λ at which

$$\Phi_f(\gamma)=-2(\lambda-1)^2(q-1)^2(q-8)/[q(q+1)]-eq(q-1)\lambda,\ e=(-1)^{q+1}$$

changes its sign as λ varies.

We have obtained by a numerical computation, in the notation of [1],

NS(3, −1.049117, 9/11; 0.15134…) with $\rho_\delta=11/9$ for $q=10$,
NS(3, −0.397917519, 11/13; 0.22904…) with $\rho_\delta=13/11$ for $q=12$,
NS(3, −0.147938843, 13/15; 0.2770…) with $\rho_\delta=15/13$ for $q=14$,
NS(3, −0.0111889, 15/17; 0.309…) with $\rho_\delta=17/15$ for $q=16$,
NS(3, 0.07614197, 17/19; 0.3339…) with $\rho_\delta=19/17$ for $q=18$,
NS(3, 0.455116098, 127/129; 0.47…) with $\rho_\delta=129/127$ for $q=128$

(as for $q=6,8$, see [1]. We must withdraw the statements "We could not find stable periodic points" in [1, p. 33]).

Thus it will be natural to make

Conjecture 2. *For each even* $\geqq 6$ *there exists the parameter sequence geometrically converging to* $\lambda^\#$ *at which a critical orbit of corresponding periods consisting of a segment of natural numbers* $\geqq 3$ *appears. For even* $q\geqq 8$ *one has* $\lambda_\#=-\infty$.

Conjecture 3. *For the dynamics of* H_f, *where* $f(x)=x^q+(\lambda-1)x-\lambda$, *to admit orbits of periods* $\geqq 2$, *it is necessary that* $\lambda_\#<\lambda<\lambda^\#$. *For odd q the dynamics of* H_f *with* $\lambda_\#<\lambda<1$ *cannot admit any periodic points of non-positive multiplier.*

3. Attractive basins of fixed points of $x^4+(\lambda-1)x-\lambda$

In this section we consider attractive basins of fixed points of H_f for $f(x)=x^4+(\lambda-1)x-\lambda$. Then 1 and α are attractive fixed points and $\beta=[(1-\lambda)/4]^{1/3}$ is repulsive. From now on $f(x)$ is assumed to be the above polynomial of fourth degree.

Lemma 3.1. *The value of*

$$\tfrac{1}{2}\Phi_f(x)=10x^6+2(\lambda-1)x^3+6\lambda x^2+(\lambda-1)^2$$

is always positive, except for $\lambda=1$ *or* -3.

Proof. The substitution $\mu=\lambda+3$ yields

$$\tfrac{1}{2}\Phi_f(x)=\mu^2+2(x-1)(x+2)^2\mu+2(x-1)^2(5x^4+10x^3+15x^2+16x+8)$$

whose discriminant is

$$\tfrac{1}{4}D=-3(x-1)^2x^2(3x^2+4x+2)$$

Since $3x^2+4x+2>0$ for real x, it follows that $\frac{1}{4}D\leqq 0$, the equality being valid only if $x=1$ or 0, according to which $\mu=-(x-1)(x+2)^2=4$ or 0, that is, $\lambda=1$ or -3. It follows that, for real $\lambda\neq 1,-3$, $\Phi_f(x)$ is positive.

Corollary 3.2. *For* $\lambda\neq 1$, *the function* $H_f(x)$ *is continuous on the whole real line.*

Lemma 3.3. *Let $g(x)$ be a smooth function defined on* $\mathbb{R}$ *whose iterations give rise to the dynamical system having super-attractive fixed points a_1 and a_2 and a repulsive one b with $a_2 < b < a_1$. Further assume that $g(x)$ has other critical points c_1 and c_2 with $c_2 < c_1$ such that*

$$g'(x) > 0 \quad \text{on } (\max(c_1, a_1),\ \infty) \cup (-\infty,\ \min(c_2, a_2))$$

and $c_1 < b$. Then attractive basins of a_1, a_2, together with b and the backward orbit of b, cover the whole real line.

Proof. The argument divides into three cases:

i) $c_2 < a_2 < c_1$. In this case we have $g(c_2) > a_2 > g(c_1)$ and, if $g(c_2) > b$ then put

$$g^{-1}(b) = \{b_1,\ b_2\}, \quad b_2 < b_1$$

One has $A_*(a_1) = (b, \infty)$, $A_*(a_2) = (-\infty, b)$ or (b_1, b).

ii) $a_2 < c_2 < c_1$. In this case we have $g(c_1) < g(c_2) < b$ and

$$A_*(a_1) = (b, \infty), \quad A_*(a_2) = (-\infty, b).$$

iii) $c_2 < c_1 < a_2$. In this case we have $g(c_2) > g(c_1)$, $g(c_1) < a_2$ and, if $g(c_2) > b$ then use the same notation as in i). One has

$$A_*(a_1) = (b, \infty), \quad A_*(a_2) = (-\infty, b) \text{ or } = (b_1, b).$$

In any way, in case $A_*(a_2) = (b_1, b)$ one has $g^{-1}(A_*(a_1)) = (b_2, b_1)$ and the basins of fixed points are

$$A(a_1) = \cup_{k=0}^{\infty} g^{-k} A_*(a_1), \qquad A(a_2) = \cup_{k=0}^{\infty} g^{-k} A_*(a_2),$$

the endpoints of these sum intervals constitute the backward orbit of $\{b\}$. The union of these sets and points fill up the whole real line, since any point lying on the left of b_1 moves to within $[b_1, \infty)$ after some iterations of g.

Applying Lemma 3.3 to $\{a_1, a_2\} = \{1, a\}$, $b = \beta$ and $\{c_1, c_2\} = \{0, \gamma\}$, one obtains

Theorem 3.4. *Attractive basins of attractive fixed points 1 and a of Halley iterations for the polynomial $x^4 + (\lambda - 1)x - \lambda$, $\lambda < 1$, together with the backward orbit of the repulsive fixed point β, fill up the whole real line.*

Corollary 3.5. *Halley iteration dynamics of $x^4 + (\lambda - 1)x - \lambda$, $\lambda < 1$, admits no periodic points of periods* $\geqq 2$.

References

1. Y. Nomura: Periodic Critical Orbits of Newton and Halley Iterations, The Bulletin of Okayama University of Science 36, 29–38 (2000)
2. Q. Yao: On the Halley iteration, Numer. Math. 81, 647–677 (1999)

Table 1 $\lambda^{\#}$ for even q

$q = 6$	−0.47788906
8	0
10	0.151343999
12	0.2290491
14	0.2770661
16	0.30997754
18	0.33393389

Table2 $\lambda_{\#}$ for even q

$q = 6$	−412.20000
$q \geqq 8$	$-\infty$?

20	0.3523056
22	0.3786459
24	0.3786459
26	0.3884383
28	0.3966998
30	0.4037688
32	0.4098902
34	0.4152455
36	0.4199722
38	0.42417681
40	0.42794242
42	0.43133548
128	0.476352743

Table 3 $\lambda_{\#}$ for odd q

$q = 3$	0.250007
5	0.32465
7	0.36490611
9	0.3884333
11	0.4044570
13	0.4161423
15	0.4250778
17	0.432153
19	0.437909
21	0.4426909
23	0.4467330
25	0.45019897
27	0.4532068
29	0.45584414
31	0.458177133

Table 4 $\lambda^{\#}$ for odd q

$q = 3$	2.5
5	16.64115
7	220649.307432
$\geqq 9$	∞?