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Extension of the Catalan number

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Abstract. Two enumerative functions are introduced to give a new solution to the problem of nonassociative binary parenthesization. One of the enumerative functions results in an extension of the Catalan number.
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1. Introduction

In the enumerative combinatorics, the Catalan number has been one of the ubiquitous and fascinating numbers. The number is indebted to a Belgian mathematician Eugène Charles Catalan (1814-1894) for his effort in further development of Lamé's proof of the Euler-Segner proposition [1]. The Catalan number is related to enumeration of trees, parenthesizations, ballot sequences, lattice paths and polygon dissections. The problem on the number goes back to the middle of the 18th century when Segner and Euler solved a problem of the triangulation of a polygon.

The Catalan number $C(n)$ is defined for a positive integer n as

$$C(n) \stackrel{\text{def}}{=} \frac{1}{n+1}\binom{2n}{n} = \frac{1}{2n+1}\binom{2n+1}{n} ,$$

that counts elements of as many as 66 sets, 6 typical sets of which are given below [2].

(1) Plane binary trees with $n+1$ endpoints or $2n+1$ vertices.
(2) Sequences $i_1 i_2 \cdots i_{2n}$ of 1's and -1's with $i_1 + i_2 + \cdots + i_j \geq 0$ for all j and $i_1 + i_2 + \cdots + i_{2n} = 0$.
(3) Nonassociative binary parenthesizations of a string of $n+1$ letters.
(4) Paths in the (x, y) plane from $(0,0)$ to $(2n, 0)$, with steps $(1,1)$ and $(1,-1)$, that never pass below the x-axis.
(5) Paths in the (x, y) plane from $(0,0)$ to (n, n), with steps $(1,0)$ and $(0,1)$, that never pass above the line $y = x$.
(6) Dissection of a convex $(n+2)$-gon into n triangles by drawing $n-1$ diagonals, no two of which intersect in their interior.

The present paper is devoted to the 3rd problem of binary parenthesizaition. Two enumerative functions f, g or φ are introduced; g enumerates elements of a subset of the set whose elements f counts. The Catalan number is shown as a restriction of g to a special case.

2. Enumerative Functions

For two integers k,n such that $k \geq 1, n \geq 0$, an enumerative function $f_k(n)$ is defined as

$$f_k(n) \overset{\text{def}}{=} \#\left\{ \mathbf{e} = {}^t(e_1 e_2 \cdots e_k) \in \mathbf{N}^k \mid e_1 + e_2 + \cdots + e_k = n \right\}$$

with the set of all non-negative integers $\mathbf{N}$. $f_k(n)$ implies the number of non-negative integer points on the $k-1$ dimensional hyperplane $e_1 + e_2 + \cdots + e_k = n$.

Proposition 2.1.

$$f_k(n) = \sum_{i=0}^{n} f_{k-1}(i) = f_k(n-1) + f_{k-1}(n) \ .$$

Proof. Let consider the $\mathbf{Z}$ coefficient k-variable polynomial ring $\mathbf{Z}[x_1, x_2, \cdots, x_k]$.

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{i=0}^{n} \binom{n}{i} (x_1 + x_2 + \cdots + x_{k-1})^i x_k^{n-i} \ .$$

Since $f_k(n)$ implies the number of different terms except for coefficient in the expansion of $(x_1 + x_2 + \cdots + x_k)^n$,

$$f_k(n) = \sum_{i=0}^{n} f_{k-1}(i) \ .$$

The second equality is readily derived by $f_k(n-1) = \sum_{i=0}^{n-1} f_{k-1}(i)$. ∎

Proposition 2.2.

$$f_k(n) = \binom{n+k-1}{k-1} = \binom{n+k-1}{n} = f_{n+1}(k-1) \ .$$

Proof. The solution of $e_1 = n$ leads to $f_1(n) = 1$ for $\forall n \geq 0$. By mathematical induction on k,

$$f_k(n) = \sum_{i=0}^{n} f_{k-1}(i) = \sum_{i=0}^{n} \binom{i+k-2}{k-2} \ .$$

Since

$$\binom{i+k-1}{k-1} = \binom{i+k-2}{k-2} + \binom{i+k-2}{k-1} \ ,$$

$$f_k(n) = \binom{k-1}{k-1} + \sum_{i=1}^{n} \left\{ \binom{i+k-1}{k-1} - \binom{i+k-2}{k-1} \right\} = \binom{n+k-1}{k-1} \ .$$

The proof of the rest is trivial. ∎

The relation of $f_k(n)=f_k(n-1)+f_{k-1}(n)$ in Prop. 2.1 yields Table 2.1. From $f_{k+1}(n-1)=f_n(k)$ (Prop. 2.2), the subdiagonals are resulted from reflexion with respect to the diagonal of the table.

Table 2.1. Table of $f_k(n)$ $(k \geq 1, n \geq 0)$.

k, n	0	1	2	3	4	5
1	1	1	1	1	1	1
2	1	2	3	4	5	6
3	1	3	6	10	15	21
4	1	4	10	20	35	56
5	1	5	15	35	70	126
6	1	6	21	56	126	252

Let $g_k(n)$ be an enumerative function defined as

$$g_k(n) \overset{\text{def}}{=} \#\left\{\mathbf{e} \in \mathbf{N}^k \mid e_1+e_2+\cdots+e_k=n,\ e_i+e_{i+1}+\cdots+e_k \leq n-i+1\ (i=2,3,\cdots,k)\right\}$$

for $k \geq 1, n \geq k-1$, or equivalently $\varphi_k(n)$ as

$$\varphi_k(n) \overset{\text{def}}{=} g_k(n+k-1) = \#\left\{\mathbf{e} \in \mathbf{N}^k \mid e_1+e_2+\cdots+e_k=n+k-1,\ e_i+e_{i+1}+\cdots+e_k \leq n+k-i\ (i=2,3,\cdots,k)\right\}$$

for $k \geq 1, n \geq 0$.

Proposition 2.3.

$$\varphi_k(n)=\sum_{i=1}^{n+1}\varphi_{k-1}(i) \qquad (k \geq 1, n \geq 0)$$

$$=\varphi_k(n-1)+\varphi_{k-1}(n+1) \qquad (k \geq 2, n \geq 1)\ .$$

Proof. The defining conditions of $\varphi_k(n)$ is rewritten as

$$e_1+e_2+\cdots+e_{k-1}=n+k-e_k-1\ ,$$
$$e_i+e_{i+1}+\cdots+e_{k-1} \leq n+k-e_k-i\ \ (i=2,3,\cdots,k)$$

and $\quad e_k \le n$.
Since $0 \le e_k \le n$,

$$\varphi_k(n) = \sum_{e_k=0}^{n} \varphi_{k-1}(n - e_k + 1) = \sum_{i=1}^{n+1} \varphi_{k-1}(i) \ .$$

Then,

$$\varphi_k(n-1) = \sum_{i=1}^{n} \varphi_{k-1}(i) \ .$$

Hence the second equality is shown. ∎

Proposition 2.3 is equivalently represented in terms of $g_k(n)$ as the following corollary.

Corollary 2.4.

(1) $$g_k(n+k-1) = \sum_{i=1}^{n+1} g_{k-1}(i+k-2) \qquad (k \ge 1, n \ge 0)$$

$$= g_k(n+k-2) + g_{k-1}(n+k-1) \qquad (k \ge 2, n \ge 1) \ .$$

(2) $$g_k(n) = \sum_{i=k-1}^{n+k-1} g_{k-1}(i) \qquad (k \ge 1, n \ge k-1)$$

$$= g_k(n-1) + g_{k-1}(n+1) \qquad (k \ge 2, n \ge k) \ .$$ ∎

For $n = 0$, $e_k = 0$. Then, by Prop. 2.3 and Cor. 2.4 (1),

Proposition 2.5. For $k \ge 2$,

(1) $\varphi_k(0) = \varphi_{k-1}(1)$.
(2) $g_k(k-1) = g_{k-1}(k-1)$. ∎

Proposition 2.6.

$$\varphi_k(n) = g_k(n+k-1) = \frac{n+1}{n+k}\binom{n+2(k-1)}{k-1} \qquad (k \ge 1, n \ge 0) \ . \tag{1}$$

Proof. For $k = 1$, $e_1 = n$. Thus, $\varphi_1(n) = 1 \ (n \ge 0)$.

$$\text{(2nd righthand side of Eq.(1))} = \frac{n+1}{n+1}\binom{n}{0} = 1 \ .$$

By mathematical induction on k and n, suppose that the proposition holds for $k-1$ $(k \ge 2)$ and for $\forall n \ge 0$. Now, k (≥ 2) is fixed. Prove Eq.(1) for $\forall n \ge 0$ and for the fixed k. By Prop. 2.5, $\varphi_k(0) = \varphi_{k-1}(1)$. By the supposition,

$$\varphi_{k-1}(1)=\frac{1+1}{1+(k-1)}\binom{1+2\{(k-1)-1\}}{(k-1)-1}=\frac{2}{k}\binom{2(k-1)-1}{(k-1)-1} .$$

Since

$$\binom{2(k-1)-1}{k-1}=\binom{2(k-1)-1}{(k-1)-1},$$

$$\varphi_{k-1}(1)=\frac{1}{k}\left\{\binom{2(k-1)-1}{(k-1)-1}+\binom{2(k-1)-1}{k-1}\right\}=\frac{1}{k}\binom{2(k-1)}{k-1} .$$

Thus, the proposition holds for $n=0$.

By Prop. 2.3 and the supposition of induction for $n-1$ at k and for $\forall n\geq 0$ at $k-1$,

$$\varphi_k(n)=\varphi_k(n-1)+\varphi_{k-1}(n+1)$$

$$=\frac{(n-1)+1}{(n-1)+k}\binom{(n-1)+2(k-1)}{k-1}+\frac{(n+1)+1}{(n+1)+(k-1)}\binom{(n+1)+2\{(k-1)-1\}}{(k-1)-1}$$

$$=\frac{n}{n+k-1}\binom{n+2(k-1)-1}{k-1}+\frac{1+(n+1)}{n+k}\binom{n+2(k-1)-1}{k-2} .$$

$$\frac{n}{n+k-1}\binom{n+2(k-1)-1}{k-1}+\frac{1}{n+k}\binom{n+2(k-1)-1}{k-2}$$

$$=\left(\frac{n}{n+k-1}+\frac{k-1}{(n+k)(n+k-1)}\right)\binom{n+2(k-1)-1}{k-1}=\frac{n+1}{n+k}\binom{n+2(k-1)-1}{k-1} .$$

Thus,

$$\varphi_k(n)=\frac{n+1}{n+k}\left\{\binom{n+2(k-1)-1}{k-1}+\binom{n+2(k-1)-1}{k-2}\right\}=\frac{n+1}{n+k}\binom{n+2(k-1)}{k-1} .$$

∎

Corollary 2.7.

$$g_{n+1}(n)=g_n(n)=C(n) \quad (n\geq 0) .$$

Proof. By Prop. 2.5 (2), $g_{n+1}(n)=g_n(n)$.

$$g_{n+1}(n)=g_n(n)=\varphi_n(1)=\frac{2}{n+1}\binom{2n-1}{n-1} .$$

Since

$$\binom{2n-1}{n-1}+\binom{2n-1}{n}=\binom{2n}{n} \text{ and } \binom{2n-1}{n-1}=\binom{2n-1}{n} ,$$

$$g_n(n)=C(n) .$$

∎

Alternative proof.

$$g_{n+1}(n) = \varphi_{n+1}(0) = \frac{1}{n+1}\binom{2n}{n} = C(n) \ . \qquad \blacksquare$$

From the first proof follows the following result.

Proposition 2.8. For $n \geq 1$,

$$C(n) = \frac{2}{n+1}\binom{2n-1}{n-1} \ . \qquad \blacksquare$$

The following table of $\varphi_k(n) = g_k(n+k-1)$ $(k \geq 1, n \geq 0)$ is obtained by using the basic relation $\varphi_k(n) = \varphi_k(n-1) + \varphi_{k-1}(n+1)$ with $\varphi_k(0) = \varphi_{k-1}(1)$, $\varphi_1(n) = 1$ and $\varphi_k(-1) = 0$; here, $\varphi_k(-1) = 0$ is defined so as to satisfy the basic relation.

Table 2.2. Table of $\varphi_k(n) = g_k(n+k-1)$ $(k \geq 1, n \geq 0)$.

k, n	0	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8
3	2	5	9	14	20	27	35	
4	5	14	28	48	75	110		
5	14	42	90	165	275			
6	42	132	297	572				
7	132	429	1001					
8	429	1430						

Moreover, $g_k(n)$ is related to $f_k(n)$ as follows.

Proposition 2.9.

$$g_k(n) = \frac{(n+1)-(k-1)}{n+1} f_k(n) \quad (k \geq 1, n \geq 0) \ .$$

Proof.

$$g_k(n) = g_k\big((n-(k-1))+k-1\big) = \frac{(n-(k-1))+1}{(n-(k-1))+k}\binom{n-(k-1)+2(k-1)}{k-1}$$

$$= \frac{(n+1)-(k-1)}{n+1}\binom{n+k-1}{k-1} \ .$$

∎

3. Parenthesization

Consider in this section the problem of nonassociative binary parenthesization. The binary parenthesization is determined once all the left parentheses are given; the right parentheses are automatically decided according to the left parentheses for binary parenthesization. Thus, it suffices to discuss positions of left parentheses. Let a word consist of $n+1$ letters, $x_1 x_2 \cdots x_{n+1}$, and e_i denote number of left parentheses posed between x_{i-1} and x_i $(i = 1, 2, \cdots, n)$; here, x_0 is a ghost letter temporarily introduced to explain e_1, and e_{n+1} is omitted with inhibition rule of parenthesizing a letter such as (x_{n+1}). The binary parenthesization, then, requires at least the following two conditions:

(1) $e_1 + e_2 + \cdots + e_n = n$,

(2) $e_n \leq 1$.

Furthermore, similarly to (2) $e_2, e_3, \cdots, e_{n-1}$ satisfy $e_i + e_{i+1} + \cdots + e_n \leq n - i + 1$ $(2 \leq i \leq n-1)$. Then,

(2') $e_i + e_{i+1} + \cdots + e_n \leq n - i + 1 \quad (2 \leq i \leq n)$.

The binary parenthesizations are enumerated under the conditions of (1) and (2'). Thus, $g_n(n)$ counts such parenthesizations. Let $p(n+1)$ be an enumerative function counting the nonassociative binary parenthesizations of a string of length $n+1$.

Proposition 3.1.

$$\begin{aligned} p(n+1) &= \varphi_{n+1}(0) = \varphi_n(1) \\ &= g_{n+1}(n) = g_n(n) \\ &= C(n) \ . \end{aligned}$$

∎

From the above discussion to solve the parenthesization problem it is concluded that $g_k(n)$ or $\varphi_k(n)$ is regarded as an extension of the Catalan number.

References

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[2] R. P. Stanley, *Enumerative Combinatorics*, vol. 2, Cambridge Univ. Press, Cambridge, 1999.