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Extensions $R[\alpha]$ and $R[\alpha^2]$ of a Noetherian domain R

KEN-ICHI YOSHIDA AND KIYOSHI BABA*

Department of Applied Mathematics
Okayama University of Science
Ridai-cho 1-1
Okayama 700-0005
JAPAN
and
**Department of Mathematics*
Faculty of Education and Welfare Science
Oita University
Oita 870-1192
JAPAN
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Abstract

Let α be a super-primitive element over a Noetherian domain R. We will prove that $R[\alpha]/R$ is a flat extension if $R[\alpha] = R[\alpha^2]$ holds. Under the condition $R[\alpha] = R[\alpha^2]$, we will study ideals $I_{[\alpha]}$, $J_{[\alpha]}$ and $J_{[\alpha^2]}$ whose definitions are mentioned below. Finally we will examine a condition that $R[\alpha]/R[\alpha^2]$ is a birational extension.

Let R be an integral domain with the quotient field K and $R[X]$ a polynomial ring over R in an indeterminate X. Let α be an element of an algebraic field extension of K and $\pi : R[X] \longrightarrow R[\alpha]$ the R-algebra homomorphism defined by $\pi(X) = \alpha$. Let $\varphi_\alpha(X)$ be the monic minimal polynomial of α over K with deg $\varphi_\alpha = d$ and write

$$\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d,$$

$$(\eta_1, \ldots, \eta_d \in K).$$

We will define

$$I_{[\alpha]} = \bigcap_{i=1}^{d} (R :_R \eta_i)$$

and

$$J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$$

where $(R :_R \eta_i) = \{c \in R; c\eta_i \in R\}$ and $(1, \eta_1, \ldots, \eta_d)$ is the R-module generated by $1, \eta_1, \ldots, \eta_d$. An element α is called an anti-integral element of degree d over R if

$$\operatorname{Ker} \pi = I_{[\alpha]} \varphi_\alpha(X) R[X].$$

An element α is said to be a super-primitive element of degree d over R if $J_{[\alpha]} \not\subset p$ for every $p \in \mathrm{Dp}_1(R)$ where

$$\mathrm{Dp}_1(R) = \{p \in \mathrm{Spec} R; \mathrm{depth} R_p = 1\}.$$

We denote by $k(p)$ the residue field of $p \in \mathrm{Spec} R$.

Our notation is standard and unexplained one is referred to [1].

Lemma 1. *Let R be a Noetherian domain and α an anti-integral element of degree d over R. Set $A = R[\alpha]$. Let P be an element of* $\operatorname{Spec} A$ *and set $p = P \cap R$. If* $\operatorname{tr.deg}_{k(p)} k(P) > 0$*, then $J_{[\alpha]} \subset p$ and $P = pA$.*

Proof. We will prove that $J_{[\alpha]} \subset p$. Suppose the contrary. Then considering $\varphi_\alpha(X)$, we can easily see that there is a polynomial

$$b_0X^d + b_1X^{d-1} + \cdots + b_d$$

of $R[X]$ satisfying $b_i \notin p$ for some i. Set $A/P = R/p[\overline{\alpha}]$. Then $\overline{\alpha}$ is a root of a non-zero polynomial

$$\overline{b}_0X^d + \overline{b}_1X^{d-1} + \cdots + \overline{b}_d$$

of $R/p[X]$. Hence $\operatorname{tr.deg}_{R/p} A/P = 0$, and so $\operatorname{tr.deg}_{k(p)} k(P) = 0$. This is a contradiction. Therefore $J_{[\alpha]} \subset p$.

We will prove that $P = pA$. Since $J_{[\alpha]} \subset p$, we know that pA is a prime ideal of A by [2], Theorem 1.8 (iii). The inclusion $pA \subset P$ is obvious. Hence

$$\operatorname{tr.deg}_{k(p)} k(P) \leq \operatorname{tr.deg}_{k(p)} k(pA).$$

Note that $\operatorname{tr.deg}_{k(p)} k(pA) \leq 1$. Since $\operatorname{tr.deg}_{k(p)} k(P) > 0$, we get $\operatorname{tr.deg}_{k(p)} k(P) = 1$. Let y be an element of A such that $y \pmod{P}$ is algebraically independent over R/p. Assume that $pA \subsetneq P$. Then there exists an element x of P such that x is not in pA. We will prove that x and $y \pmod{pA}$ are algebraically independent over R/p in order to contradict the fact $\operatorname{tr.deg}_{k(p)} k(pA) \leq 1$. Let

$$a_0(y) + a_1(y)x + \cdots + a_nx^n \equiv 0 \,(\operatorname{mod} pA),$$

$$a_0(y), a_1(y), \ldots, a_n(y) \in R[y]$$

be an algebraic relation such that n is minimal. Then since x is in P, we know that

$$\begin{aligned} -a_0(y) &\equiv a_1(y)x + \cdots + a_nx^n && (\operatorname{mod} P) \\ &\equiv 0 && (\operatorname{mod} P) \end{aligned}$$

Hence $a_0(y) \equiv 0 \,(\operatorname{mod} p)$ because $y \,(\operatorname{mod} P)$ is algebraically independent over R/p. Therefore

$$a_1(y) + a_2(y)x + \cdots + a_nx^{n-1} \equiv 0 \,(\operatorname{mod} pA).$$

This contradicts the minimality of n. Thus x and $y \pmod{pA}$ are algebraically independent over R/p.

Q.E.D.

Lemma 2. *Let R be a Noetherian domain and α an anti-integral element of degree d over R. Set $A = R[\alpha]$ and p a prime ideal of R. If pA is in* $\operatorname{Spec} A$*, then $pA \cap R = p$.*

Proof. Since $pA \neq A$, we know that $pA_p \cap R_p = pR_p$. Hence by [2], Lemma 3.1, we get $pA \cap R = p$.

Q.E.D.

Proposition 3. *Let R be a Noetherian domain and α an anti-integral element of degree d over R. Let β be an element of $R[\alpha]$ such that β is an anti-integral element of degree t ($t \leq d$) over R. If $R[\alpha]/R[\beta]$ is an integral extension, then $\sqrt{J_{[\alpha]}} = \sqrt{J_{[\beta]}}$.*

Proof. We will prove that $\sqrt{J_{[\alpha]}} \supset \sqrt{J_{[\beta]}}$. Let p be an element of $\operatorname{Spec} R$ such that $J_{[\alpha]} \subset p$. Set $P = pR[\alpha]$. Then by [2], Theorem 1.8 (iii), we see that P is a prime ideal of $R[\alpha]$ and $R[\alpha]/P \cong R/p[T]$ where T is an indeterminate over R/p. Set $Q = P \cap R[\beta]$. Then $(R[\alpha]/P)/(R[\beta]/Q)$ is an integral extension because $R[\alpha]/R[\beta]$ is an integral extension. Hence $\operatorname{tr.deg}_{k(p)} R[\beta]/Q > 0$. Furthermore, Lemma 2 implies that $p = P \cap R = (P \cap R[\beta]) \cap R = Q \cap R$. Then Lemma 1 shows that $J_{[\beta]} \subset p$. Since a radical ideal is the intersection of all prime ideals containing it, we see that $\sqrt{J_{[\alpha]}} \supset \sqrt{J_{[\beta]}}$.

Conversely, let p be an element of Spec$R$ such that $J_{[\beta]} \subset p$. Set $Q = pR[\beta]$. Then by [2], Theorem 1.8 (iii), we know that Q is a prime ideal of $R[\beta]$ and $R[\beta]/Q \cong R/p[S]$ where S is an indeterminate over R/p. Besides, there exists a prime ideal P of $R[\alpha]$ such that $Q = P \cap R[\beta]$ because $R[\alpha]/R[\beta]$ is an integral extension. Note that $R[\alpha]/P \supset R[\beta]/Q$. Hence tr.deg$_{k(p)}R[\alpha]/P > 0$. Lemmas 1 and 2 assert that $J_{[\alpha]} \subset p$. Hence $\sqrt{J_{[\alpha]}} \subset \sqrt{J_{[\beta]}}$. Q.E.D.

Note that super-primitive elements are anti-integral elements by [2], Theorem 1.12.

Proposition 4. *Let R be a Noetherian domain and α a super-primitive element of degree d over R. Let β be an element of $R[\alpha]$. If $R[\alpha] = R[\beta]$, then β is a super-primitive element of degree d over R and $\sqrt{J_{[\alpha]}} = \sqrt{J_{[\beta]}}$.*

Proof. Set $A = R[\alpha] = R[\beta]$. We will prove that β is a super-primitive element of degree d over R. Since $[Q(R[\beta]) : K] = d$, we see that β is of degree d over R where $Q(R[\beta])$ stands for the quotient field of $R[\beta]$. Suppose that there exists an element p of $\mathrm{Dp}_1(R)$ such that $J_{[\beta]} \subset p$. Then p does not contain $J_{[\alpha]}$ because α is a super-primitive element of degree d over R. Since $p \supset J_{[\beta]}$, by [2], Theorem 1.8 we know that $A_p/pA_p \cong k(p)[T]$ where T is an indeterminate over $k(p)$. On the other hand, [2], Theorem 1.8 shows that A_p/pA_p is not isomorphic to a polynomial ring $k(p)[T]$ because p does not contain $J_{[\alpha]}$. This is a contradiction. Hence β is a super-primitive element of degree d over R.

The latter half of Proposition 4 is immediate from Proposition 3. Q.E.D.

Remark 5. By [2], Theorem 1.8 we have:

$$\{p \in \mathrm{Spec}R; J_{[\alpha]} \text{ is not contained in } p\}$$
$$= \{p \in \mathrm{Spec}R; A_p/R_p \text{ is a flat extension}\}.$$

So the ideal $\sqrt{J_{[\alpha]}}$ is the obstruction of flatness of $R[\alpha]/R$. Hence $\sqrt{J_{[\alpha]}} = \sqrt{J_{[\beta]}}$ because $R[\alpha] = R[\beta]$. This is another proof of the latter half of Proposition 4.

We will list the following for reference sake whose proof is clear from Proposition 4.

Proposition 6. *Let R be a Noetherian domain and α a super-primitive element of degree d over R. If $R[\alpha] = R[\alpha^2]$, then α^2 is also a super-primitive element of degree d over R and $\sqrt{J_{[\alpha]}} = \sqrt{J_{[\alpha^2]}}$.*

Recall that

$$\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_{d-1}X + \eta_d.$$

Example 7. Let d be an even number, say, $d = 2e$. Assume that $\eta_{2i-1} = 0$ for $i = 1, 2, \ldots, e-1$ and there exists an element a of $I_{[\alpha]}$ such that $a\eta_{d-1} = 1$. Set $a_{2i} = a\eta_{2i}$ for $i = 1, 2, \ldots, e$. Then $R[\alpha] = R[\alpha^2]$ because

$$\alpha = -(a\alpha^{2e} + a_2\alpha^{2e-2} + \cdots + a_{d-2}\alpha^2 + a_d).$$

Furthermore,

$$\varphi_\alpha(X) = X^{2e} + (a_2/a)X^{2e-2} + \cdots + (a_{d-2}/a)X^2 + a^{-1}X + (a_d/a)$$

and

$$\varphi_{\alpha^2}(X) = (X^e + (a_2/a)X^{e-1} + \cdots + (a_{d-2}/a)X + (a_d/a))^2 - a^{-2}X.$$

Proposition 8. *Let R be an integral domain and α an anti-integral element of degree $d \geq 2$ over R. Assume that $R[\alpha] = R[\alpha^2]$. Then the following statements hold:*

(1) $I_{[\alpha]}(\eta_{d-1}, \eta_d) = R$.

(2) $I_{[\alpha]}$ is an invertible ideal of R.

(3) $I_{[\alpha]} = (R :_R \eta_{d-1}) \cap (R :_R \eta_d)$.

Proof. (1) Assume that $I_{[\alpha]}(\eta_{d-1},\eta_d) \neq R$. Then there exists an element p of Spec$R$ such that $I_{[\alpha]}(\eta_{d-1},\eta_d) \subset p$. Since α is in $R[\alpha^2]$, we can write

$$\alpha = b_0 + b_1\alpha^2 + \cdots + b_n\alpha^{2n}$$

for some $b_0, b_1, \ldots, b_n \in R$. Set

$$f(X) = b_0 - X + b_1X^2 + \cdots + b_nX^{2n}.$$

Then $f(X)$ is in $\operatorname{Ker}\pi = I_{[\alpha]}\varphi_\alpha(X)R[X]$. Hence there exist elements a of $I_{[\alpha]}$ and $g(X)$ of $R[X]$ such that $f(X) = a\varphi_\alpha(X)g(X)$. Since $a\eta_{d-1}$ and $a\eta_d$ are in p, we see that

$$b_0 - X + b_1X^2 + \cdots + b_nX^{2n} = (a_2X^2 + \cdots + a_dX^d)g(X)$$

in $R/p[X]$ where $a_2 = a\eta_{d-2}, \ldots, a_{d-1} = a\eta_1$ and $a_d = a$. Comparing the coefficients of X of the both sides of the equation, we get a contradiction. Hence $I_{[\alpha]}(\eta_{d-1},\eta_d) = R$.

(2) It is immediate from the assertion (1) that $I_{[\alpha]}$ is an invertible ideal of R.

(3) By the definition of $I_{[\alpha]}$ it is clear that $I_{[\alpha]} \subset (R :_R \eta_{d-1}) \cap (R :_R \eta_d)$. The assertion (1) implies that there exist elements c_1 and c_2 of $I_{[\alpha]}$ such that $c_1\eta_{d-1} + c_2\eta_d = 1$. Let c be an element of $(R :_R \eta_{d-1}) \cap (R :_R \eta_d)$. Then $c = c_1(c\eta_{d-1}) + c_2(c\eta_d)$ is in $I_{[\alpha]}$. Hence $(R :_R \eta_{d-1}) \cap (R :_R \eta_d) \subset I_{[\alpha]}$. So we obtain the required result. Q.E.D.

Theorem 9. *Let R be a Noetherian domain and α a super-primitive element of degree d over R. If $R[\alpha] = R[\alpha^2]$, then $R[\alpha]/R$ is a flat extension.*

Proof. We have only to prove that $J_{[\alpha]} = R$ by [2], Proposition 2.6. Assume that $J_{[\alpha]} \neq R$. Then there exists an element p of Spec$R$ such that $J_{[\alpha]} \subset p$. Proposition 6 implies that $J_{[\alpha^2]} \subset p$. Set $A = R[\alpha] = R[\alpha^2]$. Then by [2], Theorem 1.8, we know that $A/pA \cong R/p[T] = R/p[T^2]$ where T is an indeterminate over R/p. This is a contradiction. Hence $J_{[\alpha]} = R$. Q.E.D.

Remark 10. Proposition 8 implies that

$$R = I_{[\alpha]}(\eta_{d-1},\eta_d) \subset I_{[\alpha]}(1,\eta_1,\ldots,\eta_d) = J_{[\alpha]}.$$

Hence $J_{[\alpha]} = R$, and so $R[\alpha]/R$ is a flat extension if $d \geq 2$. Theorem 9 says that flatness of $R[\alpha]/R$ holds for $d \geq 1$ if α is a super-primitive element of degree d over R.

Remark 11. We will find a condition that $R[\alpha]/R[\alpha^2]$ is a birational extension:

(1) If d is odd, then $R[\alpha]/R[\alpha^2]$ is a birational extension.

(2) If d is even and if there exists a non-negative integer i such that $\eta_{2i+1} \neq 0$, then $R[\alpha]/R[\alpha^2]$ is a birational extension.

(3) If d is even, say, $d = 2e$ and if $\eta_{2i+1} = 0$ for $i = 0, 1, \ldots, e-1$, then $R[\alpha]/R[\alpha^2]$ is not a birational extension.

Proof. (1) Set $d = 2e-1$ for a positive integer e. Then we have

$$\alpha(\alpha^{2e-2} + \eta_2\alpha^{2e-4} + \cdots + \eta_{d-1}) = -(\eta_1\alpha^{2e-2} + \cdots + \eta_d).$$

Note that $\alpha^{2e-2} + \eta_2\alpha^{2e-4} + \cdots + \eta_{d-1} \neq 0$ because $[K(\alpha) : K] = d = 2e$ where K is the quotient field of R. Therefore

$$\alpha = -(\eta_1\alpha^{2e-2} + \cdots + \eta_d)/(\alpha^{2e-2} + \eta_2\alpha^{2e-4} + \cdots + \eta_{d-1}),$$

and α is in $K(\alpha^2)$. Hence $K(\alpha) = K(\alpha^2)$.

(2) Set $d = 2e$ for a positive integer e. Then

$$\alpha(\eta_1\alpha^{2e-2} + \cdots + \eta_{d-1}) = -(\alpha^{2e} + \eta_2\alpha^{2e-2} + \cdots + \eta_d).$$

Since $\eta_{2i+1} \neq 0$, we know that

$$\eta_1\alpha^{2e-2} + \cdots + \eta_{d-1} \neq 0.$$

Hence

$$\alpha = -(\alpha^{2e} + \eta_2\alpha^{2e-2} + \cdots + \eta_d)/(\eta_1\alpha^{2e-2} + \cdots + \eta_{d-1})$$

and α is in $K(\alpha^2)$. This shows that $K(\alpha) = K(\alpha^2)$.

(3) By the assumptions, we see that

$$\alpha^{2e} + \eta_2\alpha^{2e-2} + \cdots + \eta_d = 0.$$

Therefore $[K(\alpha^2) : K] \leq e = d/2$. Hence $[K(\alpha) : K(\alpha^2)] = 2$ because $[K(\alpha) : K(\alpha^2)] \leq 2$. This means that $R[\alpha]/R[\alpha^2]$ is not a birational extension. Q.E.D.

References

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