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Periodic Critical Orbits of Newton and Halley Iterations

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1. Introduction

In the previous paper [3] I have observed and conjectured that, for the 1-dimensional dynamics arising from iterations of certain typical 1-parameter families of smooth functions, parameter values at which periodic orbits of periods $2^m(2n+1)$, $n = 1, 2, 3, \cdots$, exist with multiplier -1, constitute geometrically convergent sequences.

In the present article we make a computer experiment for dynamical systems generated by Newton and Halley methods from polynomials of odd degree q, $q \geqq 3$. Unlike the previous case we have found parameter sequences at which periodic critical orbits of periods m, $m+1$, $m+2$, $\cdots$(which forms a segment of the natural numbers) appear and, furthermore, that the corresponding asymptotic rates are *rational numbers* depending upon q.

In the case of these iterations I have observed that, on the left of main period-doubling cascade, two-way geometrically convergent parameter values admitting periodic critical orbits of odd periods and, in the right, reverse-ordered geometrically convergent parameters admitting super-stable periodic orbits of odd periods. This phenomenon seem to be similar to anti-monotonicity.

2. Fine Structures of Periodic Orbits

Let $F(x)$ be a real-valued smooth function with a parameter λ and we consider one-dimensional dynamical system arising from iterating F. We shall denote generically by $\lambda(p)$ a parameter value λ at which a super-stable periodic orbit of period p appears. In other words $F(x)$ with $\lambda = \lambda(p)$ admits a p-periodic point whose orbit contains a critical point of $F(x)$. If ,near to $\lambda(p)$, there is the parameter value λ at which a p-periodic orbit with multiplier -1 appears, then we denote it by $\omega(p)$. Set $\delta(p) = \omega(p) - \lambda(p)$. We also consider the *diameter* of the periodic orbit of period p corresponding to $\lambda = \lambda(p)$ and denote it by δ orb(p), namely,

δ orb(p) = the maximum of p-periodic points $-$ the minimum of p-periodic points

It occurs often that, if one takes p-periodic orbits for $p = m, m+1, m+2, \cdots$ in a suitable family, then the corresponding sequence $\lambda(m), \lambda(m+1), \lambda(m+2), \cdots$ becomes monotone. In such cases we denote the successive ratios of its first difference by $\rho(p)$. Namely

$$\rho(p) = [\lambda(p+2) - \lambda(p+1)] \,/\, [\lambda(p+1) - \lambda(p)].$$

Consider the case where the sequence $\rho(p)$ has a limit $\rho(\infty)$, called the *asymptotic ratio*, as p tends to infinity. If $\rho(\infty)$ is smaller than 1 then $\lambda(p)$ converges to a limit $\lambda(\infty)$: if $\rho(\infty)$ is larger than 1 then $\lambda(p)$ diverges, in which we call $\{\lambda(n)\}$ *geometrically divergent*.

Incidentally, it will often observed that $\{\delta$ orb(p)$\}$ for $\lambda = \lambda(p)$ converges geometrically as p tends to infinity in which we denote the asymptotic ratio by ρ_δ. Further it is often observed that the sequence $\{\rho(p)\}$ is also geometrically convergent with the same asymptotic ratio as in $\lambda(p)$; in such a case we denote the ratio by ρ_ρ

If there is a geometrically convergent sequence $\lambda(2n+1)$, $n \geqq m$, thus $\omega(0, n) = \lambda(2n+1)$ in the notation of [3], then we have similar ratios $\rho(0, n)$ with asymptotic ratio $\rho(0, \infty)$.

In order to describe and classify parameter sequences appearing in 1-dimensional dynamics, we shall use the following notations. For a monotone parameter sequence

$$\lambda(m) < \lambda(m+1) < \lambda(m+2) < \cdots \quad \text{or} \quad \lambda(m) > \lambda(m+1) > \lambda(m+2) > \cdots$$

converging to $\lambda(\infty)$ with asymptotic ratio ρ, we denote it by NS($m, \lambda(m), \rho$; $\lambda(\infty)$).

If there is a sequence

$$\lambda(m) < \lambda(m+2) < \lambda(m+4) < \cdots$$

with asymptotic ratio ρ and limit $\lambda(\infty)$ then we denote it by RS($m, \lambda(m), \rho$; $\lambda(\infty)$). Similarly, LS($m, \lambda(m), \rho$; $\lambda(\infty)$) denotes the sequence

$$\cdots \lambda(m+4) < \lambda(m+2) < \lambda(m)$$

with asymptotic ratio ρ and limit $\lambda(\infty)$. The two-way sequence

$$\cdots < \lambda'(m+4) < \lambda'(m+2) < \lambda(m) < \lambda(m+2) < \lambda(m+4) < \cdots$$

with left and right asymptotic ratios ρ' and ρ will be denoted by TS($m, \lambda(m)$; ρ', ρ ; $\lambda'(\infty), \lambda(\infty)$).

3. Newton Method

Applying Newton's method to a given smooth function $f(x)$ we have

$$F(x) = Nf(x) = x - f(x) / f'(x) \tag{3.1}$$

In the frame-work of complex dynamics, iterations of $F(x)$ have been studied extensively by many people (see [1, 2, 4, 5]).

The derivative of $Nf(x)$ is given by

$$(Nf)'(x) = [\, f(x)\, f''(x)\,] \,/\, (f'(x))^2 \tag{3.2}$$

Hence any root of $f''(x) = 0$ which is not a root of $f(x) = 0$ gives a *free* critical point. In the light of theorems of Fatou or Singer, we usually start with such free critical points in case we search for (attractive) p-periodic points of $Nf(x)$.

Take for $f(x)$ polynomials of *odd* degree q :

$$f(x) = x^q + (\lambda - 1)x - \lambda$$

Then $x = 0$ is the free critical point. We have made computer-search of periodic points and corresponding $\lambda(p)$ using a program similar to the one stated in [3]. We have found the following parameter-sequences:

for $q = 3$ RS(7, 0.254338438687, 0.25141; 0.254619273771199) with $\rho_\delta = 0.251413$

TS(3, 0.26744849; 0.251879, 0.259259; 0.256078731774102, 0.275682203651) with $\rho_\delta' = 0.25188$, $\rho_\delta = 0.25926$

NS(5, 0.398756368, 2/3; 0.4459873529729)

NS(3, 0.510650459, 2/3; 1) with $\rho_\delta = 1.5$

for $q = 5$ NS(5, 0.3406616321, 0.8; 0.34777632228) with $\rho_\delta = 0.85$

NS(3, 0.3367402080, 0.8; 0.373062) with $\rho_\delta = 0.8$

NS(3, 0.463469485, 0.8; 1) with $\rho_\delta = 1.25$

for $q = 7$ TS(7, 0.36607039; 0.250885, 0.2518235; 0.3658058158, 0.3667288329)

TS(5, 0.36847724; 0.2524546, 0.254468; 0.3673330067, 0.3691757001)

NS(3, 0.37218037, 6/7; 0.39910233) with $\rho_\delta = 0.85$

NS(5, 0.423125779, 6/7; 0.448767458756) with $\rho_\delta = 0.855$

NS(3, 0.45820702, 6/7; 1) with $\rho_\delta = 7/6$

For even q we have never found periodic points except fixed points.

From these numerical evidences, it is reasonable to state :

Conjecture N : *For odd q there are geometrically convergent sequences*

$$\lambda(m),\ \lambda(m+1),\ \lambda(m+2),\ \cdots$$

with asymptotic rate $(q-1)/q$ *whose orbits have diameters such that* ρ_δ *are equal to* $(q-1)/q$ *or* $q/(q-1)$

4. Halley's method

Given a smooth function $f(x)$, the *Halley iteration function* $Hf(x)$ is a special case of the König iteration function of $f(x)$ (see Vrscay-Gilbert [5]) and defined as follows :

$$Hf(x) = x - (3-1)\,[\,1/f(x)\,]' \,/\, [\,1/f(x)\,]''$$

$$= x - f(x) / [f'(x) - 0.5 f(x) f''(x) / f'(x)]$$
$$= x - 2 f(x) f'(x) / [2 (f'(x)^2 - f(x) f''(x)] \quad (4.1)$$

whose derivative is given by

$$(Hf)'(x) = [f(x) / \{2 (f'(x))^2 - f(x) f''(x)\}]^2 [3 f''(x)^2 - 2 f'(x) f'''(x)]$$
$$= -2 [f(x) / \{2 (f'(x))^2 - f(x) f''(x)]^2 SD(f(x)) \quad (4.2)$$

where $SD(f(x))$ denotes the Schwarzian derivative of $f(x)$ (see Yao [6]).

Take for $f(x)$ polynomials of degree q:

$$f(x) = x^q + (\lambda - 1)x - \lambda$$

Then a simple calculation shows that

$$3f''(x)^2 - 2 f'(x) f'''(x) = q (q-1) x^{q-3} \{ q (q+1) x^{q-1} - 2(q-2)(\lambda - 1) \}$$

Thus *free* critical points of $Hf(x)$ are 0 and

$e [2 (q-2)(\lambda - 1) q^{-1} (q+1)^{-1}]^{(1/(q-1))}$, where $e = \pm 1$ for odd q, $\lambda \geqq 1$

$-[2(q-2)(1-\lambda) q^{-1} (q+1)^{-1}]^{(1/(q-1))}$ for even q, $\lambda < 0$

We have made numerical computation for $q = 3, 5, 7, 9, 11$ using a program similar to the one in [3], and obtained the following parameter-sequences:

for $q = 3$ NS(5, 1.5865404026, 0.5; 1.6417742853)
NS(3, 1.92678793818, 0.5; 2.5) with $\rho_\delta = 0.5$

for $q = 5$ RS(3, 1.058514888, 0.174847384; 1.078447789469) with $\rho_\delta = 0.1748474$
NS(5, 1.69885664, 2/3; 1.8764003026) with $\rho_\delta = 2/3$
NS(3, 2.61686822, 2/3; 16.64114978425844) with $\rho_\delta = 2/3$

for $q = 7$ NS(5, 2.05839513, 0.75; 2.5782312276669) with $\rho_\delta = 4/3$
NS(3, 5.85146057, 0.75; 220649.3075219) with $\rho_\delta = 4/3$
TS(3, 257120.242413; 0.044540715, 0.0432574; 255141.185382, 259189.938888)

for $q = 9$ NS(3, 1.060902495, 0.8; 1.002669187889) with $\rho_\delta = 0.8$
RS(5,1.0815720369, 0.1810494915; 1.086393470229) with $\rho_\delta = 0.181$
NS(5, 2.828392423, 0.8; 4.822331861043367) with $\rho_\delta = 0.8$
NS(7, 182006.8688802, 0.8; 177342.0166683) with $\rho_\delta = 1.25$
NS(5, 641681.236982, 0.8; 602001.8463) with $\rho_\delta = 1.25$
NS(9 ,2005524.47254, 0.8; 95564064.938) with $\rho_\delta = 0.8$
NS(9, 485762030372458522287635945.3, 0.8; 22269277604254934850600000) with $\rho_\delta = 0.8$

for $q = 11$ RS(5,1.00136823439, 0.192062126; 1.002195290795) with $\rho_\delta = 0.1743$
LS(5, 1.06040025182, 0.178954; 1.05992846044719) with $\rho_\delta = 0.017895$
RS(5, 1.0195165052, 0.1742719; 1.2213098025)
NS(5, 5.023110591, 5/6; 24.870979) with $\rho_\delta = 1.2$
NS(6, 169128.315, 5/6; 2116878.842979) with $\rho_\delta = 5/6$

NS(5, 396189822567385.2, 5/6; 1123168.90337) with $\rho_\delta = 5/6$

NS(9, 780523919991.023, 5/6; 76218176274791.62945) with $\rho_\delta = 5/6$

NS(9, 21570955, 5/6; 259305220704360970) with $\rho_\delta = 1.2$

$q=6$ NS(5, −12.780262, 5/7; −412.001234) with $\rho_\delta = 7/5$

NS(5, −2.2974041, 5/7; −0.477889078259) with $\rho_\delta = 7/5$

We could not find periodic points of period 3.

$q=8$ NS(5, −709.13261, 5.808; $-\infty$) with $\rho_\delta = 1.653$

NS(5, −0.334453434, 7/9; 0) with $\rho_\delta = 9/7$

We could not find periodic points of period 3.

$q=10$ NS(9, −0.17716996425, 9/11; −0.02106166)

NS(9, −0.620995042, 9/11; −0.5302298)

NS(9, −9110.600525, 9/11; −21236.960613) with $\rho_\delta = 9/11$

NS(9, −29708498.34, 9/11; −153703407.949836) with $\rho_\delta = 9/11$

We could not find stable periodic points of period 7.

$q=12$ NS(7, −0.2792913671, 11/13; −0.1789) with $\rho_\delta = 11/13$

NS(7, −23823237.5, 11/13; −1194427400) with $\rho_\delta = 11/13$

NS(9, −31111814756509686, 11/13; -31793×10^{10}) with $\rho_\delta = 11/13$

We could not find stable periodic points of period 5.

Thus it is very likely that these results support

Conjecture H : *For $q \neq 1, 2, 4$ there are geometrically convergent sequences*

$$\lambda(m),\ \lambda(m+1),\ \lambda(m+2),\ \cdots$$

with asymptotic ratio $(q-1)/(q+1)$, whose orbits have diameters such that ρ_δ are equal to $(q-1)/(q+1)$ or $(q+1)/(q-1)$.

References

1. K. Barański: From Newton's method to exotic basins Part I: The parameter space, Fund. Math. 158 (1998), 249—288

2. J. H. Curry, L. Garnett and D. Sullivan: On the Iteration of a Rational Function: Computer Experiments with Newton's Method, Commun. Math.. Phys. 91, 267—277 (1983)

3. Y. Nomura: Stable Cycles of odd Periods in Chaotic Dynamics, The Bull. Okayama Univ. Sci. 35, 25—34 (1999)

4. L.Tan: Branched coverings and cubic Newton maps, Fund. Math. 154, 207—260

5. E.R. Vrscay and W. J. Gilbert: Extraneous fixed points, basin boundaries and chaotic dynamics for

Schröder and König rational iteration functions, Numer. Math. 52, 1—16 (1988)

6. Q. Yao: On Halley iteration, Numer. Math. 81, 647—677 (1999)

Table 1 $\lambda(n)$ and $\rho(n)$ for Newton iteration of $x^q + (\lambda - 1)x - \lambda$

$q = 3$

n	$\lambda(n)$	$\delta(n)$	$\rho(n)$
2	0.36110 30805 2	-3.289742×10^{-2}	
3	0.51065 04599 9	-3.50883972×10^{-3}	
4	0.64180 42757 2	-2.42792157×10^{-3}	0.8770051083129
5	0.74542 65254 3	-1.40180586×10^{-3}	0.7900818526189
6	0.82270 63415 2	-7.3382092×10^{-4}	0.7457840020485
7	0.87824 54652 5	-3.6231445×10^{-4}	0.7186756716051
…			
33	0.99999 65839 25232 985	-3.08945×10^{-13}	0.6666680287358
34	0.99999 77226 15581 208	-1.37309×10^{-13}	0.6666675747132
35	0.99999 84817 43169 34703	-6.102634×10^{-14}	0.66666727203200
36	0.99999 89878 28534 471	-2.7123×10^{-14}	0.66666670702413
3	0.26744 849	-9.018×10^{-5}	
4	0.35689 6844	-9.02511×10^{-4}	
5	0.39875 6368	-1.31541×10^{-4}	0.467974223427297
6	0.41724 9849	-4.4089×10^{-5}	0.441798645310098
7	0.42813 61148	-1.56378×10^{-5}	0.588654229022648
…			
35	0.44598 71767 52265 33855	-1.07137×10^{-16}	0.6666647553720277458
36	0.44598 72354 92575 02550 9	-4.7616×10^{-17}	0.666665389243033 4713
37	0.44598 72746 52731 45919 1	-1.49643×10^{-17}	0.6666658150491499 5173
38	0.44598 73007 59480 18195 5	-6.66506×10^{-18}	0.66666609892048386

$q = 5$

n	$\lambda(n)$	$\delta(n)$	$\rho(n)$
3	0.46346 9485	-3.719969×10^{-3}	
4	0.53746 9858	-4.112064×10^{-3}	
5	0.60508 4424	-4.06828×10^{-3}	0.91370574578
6	0.66588 47865	-3.7769341×10^{-3}	0.899219888507
7	0.71972 72770	-3.3528685×10^{-3}	0.88556199808
8	0.76669 77085	-2.8760334×10^{-3}	0.87236736383
…			

59	0.99999 68241 07403	-1.471601×10^{-9}	0. 8000009864352259
60	0.99999 74592 84819	-1.0932×10^{-10}	0. 8000007728241934
61	0.99999 79674 27149	-8.12083×10^{-10}	0. 8000006253370947
62	0.99999 83739 41267	-6.03244×10^{-10}	0. 8000004998599506
$q=7$			
5	0.42312 5779	-3.03754×10^{-4}	
6	0. 43122 22136	-1.801304×10^{-4}	
7	0. 43623 54287	-1.107137×10^{-4}	0. 619187994
8	0. 43948 89649	-7.04182×10^{-5}	0. 648991941
9	0. 44169 22537	-4.6579×10^{-5}	0. 677198182
…			
52	0.44876 20004 6141	-5.69932×10^{-9}	0. 8856968078787
53	0.44876 27803 91479	-4.743642×10^{-9}	0. 8569930614147
54	0.44876 34488 02827	-3.948136×10^{-9}	0. 8570144712719
55	0.44876 40216 53287	-3.285965×10^{-9}	0. 8570328162651
3	0. 45820 702	-2.34891×10^{-3}	
4	0.50885 7184	-2.749562×10^{-3}	
5	0.55631 61443	-2.9436375×10^{-3}	0. 9369951951192
6	0.60077 45068	-3.0076572×10^{-3}	0. 93677489390765
7	0.64230 45144	-2.976515×10^{-3}	0. 934132641524977
…			
80	0.99999 35879 69	-1.4176×10^{-8}	0. 85714383927049130
81	0.99999 45039 69	-1.1792×10^{-8}	0. 85714491960852568
82	0.99999 52891 139	-9.81×10^{-9}	0. 85714339285489198
83	0.99999 59620 956	-8.1606×10^{-9}	0. 85714508733624454

Table 2 λ (n) and ρ (n) for Halley iteration of $x^q+(\lambda-1)x-\lambda$

$q=3$

n	$\lambda(n)$	$\delta(n)$	$\rho(n)$
3	1.92678 79381 8	-4.06252683×10^{-3}	
4	2.19168 17474 7	-1.35657533×10^{-3}	
5	2.33984 12999 2	-3.9016401×10^{-4}	0.5593167799848
6	2.41833 95056 9	-1.0453124×10^{-4}	0.5298221037519
7	2.45876 36191 50	-2.7048023×10	0.5149686297091
…			
40	2.49999 99999 95151 29856 08105 72576	-2.52910×10^{-25}	0.500000000001750327
41	2.49999 99999 97575 64928 71820 50648	-6.322739×10^{-26}	0.500000001398812097
42	2.49999 99999 98787 82464 02015 82293 309	-1.58069×10^{-26}	0.499999995806626788
43	2.49999 99999 99393 91232 01007 02742 41	-3.95171×10^{-27}	0.500000002796092675
5	1.58654 04026 18	-8.035835×10^{-4}	
6	1.61576 23172 48	-1.5485105×10^{-5}	
7	1.62913 60179 39	-3.440801×10^{-6}	0.457659974041
8	1.63554 49315 05	-8.11174×10^{-7}	0.4792176611454
…			
33	1.64177 42807 94355 59509 04108 20	-4.5272×10^{-22}	0.4999999993562528
34	1.64177 42808 85856 66822 22906 22	-1.13179×10^{-22}	0.4999999996781264
35	1.64177 42809 31607 20477 35046 346	-2.82947×10^{-23}	0.4999999998390632
36	1.64177 42809 54482 47304 54301 686	-7.0737×10^{-24}	0.4999999999195316

$q=5$

n	$\lambda(n)$	$\delta(n)$	$\rho(n)$
3	2.61686 822	-2.36051×10^{-2}	
4	4.28192 897	-2.26933×10^{-2}	
5	6.38844 1112	-1.7253962×10^{-2}	1.265126296
6	8.61083 7053	-1.100877×10^{-2}	1.05501216859
7	10.64030 59509	-6.2167065×10^{-3}	0.91318961646
…			
47	16.64114 90921 32836 07734 477	-5.417471×10^{-17}	0.6666666856857448
48	16.64114 93228 35276 96763 782	-2.407764×10^{-17}	0.66666667934605204
49	16.64114 94766 36906 17794 329	-1.070117×10^{-17}	0.66666667511959024
50	16.64114 95791 71326 51819 588	-4.75607×10^{-17}	0.66666667230194903

51	16.64114 96475 27607 13023 788	-2.11381×10^{-17}	0.66666667042352157

$q=7$

5	2.05839 513	-6.9965×10^{-4}	
6	2.23228 5997	-2.16289×10^{-4}	
7	2.34058 0864	-8.0135×10^{-5}	0.6227749233
8	2.41109 6182	-3.3293×10^{-5}	0.6511418311
9	2.45864 51793	-1.50151×10^{-5}	0.67430734695
...			
47	2.57822 93579 37855 47716	-1.71307×10^{-15}	0.74999 86614 578
48	2.57822 98253 70780 73004	-9.6361×10^{-16}	0.74999 89960 937
49	2.57823 01759 45122 72559	-5.4203×10^{-16}	0.74999 92459 1075
50	2.57823 04388 75681 25392 6	-3.0489×10^{-16}	0.74999 94353 0284
51	2.57823 06360 73488 79308 54	-1.715003×10^{-16}	0.74999 95764 7719

3	5.85146 057	-0.174595	
4	47.29110 1	-1.3426689	
5	361.53335 62	-5.7101412	7.581302321
6	1696.707076	-14.127813	4.248867546
7	5435.72649 42	-24.3720025	2.800399201
8	13176.37436 53	-32.4094343	2.0702347341
9	25917.58168 89	-35.387315	1.6460130386
10	43469.59230 84	-33.175574	1.3775782917
11	64505.60010 947	-27.622842	1.1984956172
...			
82	220649.30705 45211 89604 96855 4	-1.1493×10^{-16}	0.7500000005283192
83	220649.30714 89563 69876 62533 6	-6.4648×10^{-17}	0.7500000003962393
84	220649.30721 97827 55108 43212 0	-3.6364×10^{-17}	0.7500000002971797
85	220649.30727 29025 44048 07332 05	-2.0455×10^{-17}	0.7500000002228839