THE BULLETIN OF OKAYAMA UNIVERSITY OF SCIENCE No. 36 A pp. 19-28 (2000)

Discrete and Continuous Analysis to Derive Surface Area and Volume of n-Dimensional Sphere

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(Received November 1, 2000)

Abstract

The surface area and volume of n-dimensional sphere is elementarily derived by introducing several combinatorial functions combined with trigonometric functions.

1. Introduction

The volume of n-dimensional sphere is involved in the theory of fractal geometry, related to the Hausdorff measure[1]. In the theory, the formula is cited in an erroneous form. Another incorrect citation of the formula is found in an enlightening book of mathematics in Japanese. Incorrectness is obvious by substituting some integers for n in the cited formulae. The correct formula is found in 3).

Here, the author dare take a long journey to attain the formula. The long journey passing through combinatorial consideration and Euler's formula might be full of delight.

2. Combinatrics

Several combinatorial functions are introduced. Their properties are discussed in detail.

2.1 Function $a(i,n)$

Definition 2.1 For $n \in \mathrm{N}$,

$$a(i,n) \overset{\text{def}}{=} \sum_{i \le k \le [\frac{n}{2}]} \binom{k}{i}\binom{n}{2k} \quad \left(i \in \mathrm{N}\left(0,[\tfrac{n}{2}]\right)\right),$$

with Gauss' symbol $[n]$ and $\mathrm{N}(m,n) = \{m, m+1, \cdots, n\}$. Furthermore, $\mathrm{N} = \{0,1,2,\cdots\}$, $\mathrm{N}(m) = \{m, m+1, \cdots\}$.

Proposition 2.2 *For* $n \in \mathrm{N}$,

(1) $a(n,2n) = 1$.

(2) $a(n,2n+1) = 2n+1$.

Proof:

(1) $a(n,2n) = \binom{n}{n}\binom{2n}{2n} = 1$.

(2) $a(n,2n+1) = \binom{n}{n}\binom{2n+1}{2n} = \binom{2n+1}{1} = 2n+1$. □

Proposition 2.3

(1) $a(0,n) = 2^{n-1} \quad (n \in \mathrm{N}(1))$.

(2) $a(0,n) = 2a(0,n-1) \quad (n \in \mathrm{N}(1))$.

(3) $2^m a(0,n) = a(0,m+n) \quad (m,n \in \mathrm{N})$.

Proof:

(1) $a(0,n) = \sum_{0 \le k \le [\frac{n}{2}]} \binom{n}{2k} = \frac{2^n}{2} = 2^{n-1}$.

(2), (3) are readily derived by (1). □

Proposition 2.4

$$a(1,n) = n\,a(0,n-2) \quad (n \in \mathrm{N}(3)).$$

Proof:

$$a(1,n) = \sum_{1 \le k \le [\frac{n}{2}]} \binom{k}{1}\binom{n}{2k} = \sum_{1 \le k \le [\frac{n}{2}]} k\binom{n}{2k} \quad (n \in \mathrm{N}(2)).$$

Differentiation by x of

$$(x+1)^n = \sum_{0 \le k \le n} \binom{n}{k} x^k$$

yields

$$n(x+1)^{n-1} = \sum_{1\le k\le n} k\binom{n}{k}x^{k-1} \quad (n\in\mathbf{N}(1)) .$$

By substitution of 1 for x,

$$n\cdot 2^{n-1} = \sum_{1\le k\le n} k\binom{n}{k} .$$

$$\sum_{1\le k\le n} k\binom{n}{k} = \sum_{1\le k\le[\frac{n}{2}]} 2k\binom{n}{2k} + \sum_{1\le k\le[\frac{n+1}{2}]}(2k-1)\binom{n}{2k-1}$$

$$= \left[\sum_{1\le k\le[\frac{n}{2}]} 2k\left\{\binom{n}{2k}+\binom{n}{2k-1}\right\} + \sum_{[\frac{n}{2}]<k\le[\frac{n+1}{2}]} 2k\binom{n}{2k-1}\right] - \sum_{1\le k\le[\frac{n+1}{2}]}\binom{n}{2k-1}$$

$$= 2\left[\sum_{1\le k\le[\frac{n}{2}]} k\binom{n+1}{2k} + \delta_n\frac{n+1}{2}\right] - 2^{n-1} \quad \left(\delta_n \overset{\text{def}}{=} \begin{cases}0\ (n:\text{even})\\ 1\ (n:\text{odd})\end{cases}\right)$$

$$= 2a(1,n+1) - 2^{n-1} .$$

Thus,

$$a(1,n+1) = (n+1)2^{n-2}$$
$$= (n+1)a(0,n-1) \quad (n\in\mathbf{N}(2)).$$

Hence,

$$a(1,n) = n\,a(0,n-2) \quad (n\in\mathbf{N}(3)) . \quad \square$$

Theorem 2.5

$$a(i,n) = a(i-1,n-2) + 2\,a(i,n-1)$$
$$\left(i\in\mathbf{N}\left(1,[\tfrac{n}{2}]\right), n\in\mathbf{N}(2)\right) . \qquad (1)$$

Before proving the theorem, a definition and a lemma are given.

Definition 2.6 For $n\in\mathbf{N}(2)$,

$$b_{i,n}(k) \overset{\text{def}}{=} \binom{k}{i}\binom{n-2}{2(k-1)} + \binom{k}{i-1}\binom{n-2}{2k} + 2\binom{k}{i}\binom{n-1}{2k} \quad \left(k\in\mathbf{N}\left(i,[\tfrac{n}{2}]-1\right)\right).$$

Lemma 2.7

$$b_{i,n}(k) = \binom{k}{i}\binom{n}{2k} + \binom{k+1}{i}\binom{n-2}{2k} .$$

Proof.

$$b_{i,n}(k) = \binom{k}{i}\left\{\binom{n-1}{2k-1} - \binom{n-2}{2k-1}\right\} + \binom{k}{i-1}\binom{n-2}{2k} + 2\binom{k}{i}\binom{n-1}{2k}$$

$$= \binom{k}{i}\left[\left\{\binom{n-1}{2k-1}+\binom{n-1}{2k}\right\} + \left\{\binom{n-1}{2k} - \binom{n-2}{2k-1}\right\}\right] + \binom{k}{i-1}\binom{n-2}{2k}$$

$$= \binom{k}{i}\binom{n}{2k} + \left\{\binom{k}{i}+\binom{k}{i-1}\right\}\binom{n-2}{2k}$$

$$= \binom{k}{i}\binom{n}{2k} + \binom{k+1}{i}\binom{n-2}{2k} . \qquad \square$$

Proof of Theorem 2.5:
By Lemma 2.7,

$$a(i,n) = \sum_{i\le k\le[\frac{n}{2}]-1}\left\{b_{i,n}(k) - \binom{k+1}{i}\binom{n-2}{2k}\right\} + \binom{[\frac{n}{2}]}{i}\binom{n}{2[\frac{n}{2}]} .$$

The righthand side of eqn. (1) is written as

$$a(i-1,n-2) + 2\,a(i,n-1)$$

$$= \binom{i-1}{i-1}\binom{n-2}{2(i-1)} + \sum_{i\le k\le[\frac{n}{2}]-1}\left\{\binom{k}{i-1}\binom{n-2}{2k} + 2\binom{k}{i}\binom{n-1}{2k}\right\} + 2\sum_{[\frac{n}{2}]-1<k\le[\frac{n-1}{2}]}\binom{k}{i}\binom{n-1}{2k} .$$

By Definition 2.6,

$$= \binom{n-2}{2(i-1)}$$

$$+\sum_{i\le k\le[\frac{n}{2}]-1}\left\{b_{i,n}(k)-\binom{k}{i}\binom{n-2}{2(k-1)}\right\}$$
$$+2\sum_{[\frac{n}{2}]-1<k\le[\frac{n-1}{2}]}\binom{k}{i}\binom{n-1}{2k}$$
$$=\sum_{i\le k\le[\frac{n}{2}]-1}\left\{b_{i,n}(k)-\binom{k+1}{i}\binom{n-2}{2k}\right\}$$

$$+\binom{[\frac{n}{2}]}{i}\binom{[\frac{n}{2}]-2}{2([\frac{n}{2}]-1)}+2\sum_{[\frac{n}{2}]-1<k\le[\frac{n-1}{2}]}\binom{k}{i}\binom{n-1}{2k}$$
$$=\sum_{i\le k\le[\frac{n}{2}]-1}\binom{k}{i}\binom{n}{2k}+\binom{[\frac{n}{2}]}{i}\binom{n-2}{2([\frac{n}{2}]-1)}$$
$$+2\sum_{[\frac{n}{2}]-1<k\le[\frac{n-1}{2}]}\binom{k}{i}\binom{n-1}{2k}. \qquad (2)$$

Since

$$\binom{n-2}{2([\frac{n}{2}]-1)}=\begin{cases}1 & (n:\text{even})\\ \binom{n-2}{n-3}=n-2 & (n:\text{odd})\end{cases},$$

$$\sum_{[\frac{n}{2}]-1<k\le[\frac{n-1}{2}]}\binom{k}{i}\binom{n-1}{2k}=\begin{cases}0 & (n:\text{even})\\ \binom{[\frac{n}{2}]}{i} & (n:\text{odd})\end{cases}.$$

The sum of the last two terms of eqn. (2) is, then, expressed as

$$\begin{cases}\binom{[\frac{n}{2}]}{i} & (n:\text{even})\\ \binom{[\frac{n}{2}]}{i}n & (n:\text{odd})\end{cases}=\binom{[\frac{n}{2}]}{i}\binom{n}{2[\frac{n}{2}]}.$$

Thus, the theorem is proved. □

Corollary 2.8

$$a(2,n)=n\,a(1,n-3)\quad (n\in\mathbf{N}(5)).$$

Proof:

For n=5,

$$a(2,5)=\binom{2}{2}\binom{5}{4}=5,$$
$$5a(1,2)=5\binom{1}{1}\binom{2}{2}=5.$$

Then, the corollary holds for n=5.
By Theorem 2.5,

$$a(2,n)=a(1,n-2)+2a(2,n-1).$$

By the supposition of mathematical induction on $n\in\mathbf{N}(6)$,

$$\begin{aligned}a(2,n)&=\{a(0,n-4)+2a(1,n-3)\}\\&\quad+2(n-1)\cdot a(1,n-4)\\&=\{a(0,n-4)+2a(1,n-3)\}\\&\quad+(n-1)\{a(1,n-3)-a(0,n-5)\}\\&=a(0,n-4)-(n-1)a(0,n-5)\\&\quad+a(1,n-3)+n\,a(1,n-3)\\&=2a(0,n-5)-(n-1)a(0,n-5)\\&\quad+(n-3)a(0,n-5)+n\,a(1,n-3)\\&=n\,a(1,n-3).\end{aligned}$$
□

Function $a(i,n)$ is represented in one term as

Theorem 2.9

For $n\ge1$,

$$a(i,n)=2^{n-2i-1}\cdot\frac{n}{n-i}\binom{n-i}{i}\quad \left(i\in\mathbf{N}(0,[\tfrac{n}{2}])\right)$$
$$. \qquad (3)$$

Proof:

For i=0, the theorem holds for $\forall n\ge1$ by Prop. 2.3 (1). Proposition 2.2 (1) asserts $a(i,2i)=1$. For $\forall i\ge1, n=2i$, the righthand side of eqn.(3) is $2^{-1}\cdot\frac{2i}{i}\binom{i}{i}=1$. Then, the theorem holds for $n=2i\ (\forall i\ge1)$. In the case of $n>2i$,

$$a(i,n)=a(i-1,n-2)+2a(i,n-1)$$
$$=2^{(n-2)-2(i-1)-1}\cdot\frac{n-2}{(n-2)-(i-1)}$$
$$\cdot\binom{(n-2)-(i-1)}{i-1}+2\cdot2^{(n-1)-2i-1}\cdot\frac{n-1}{(n-1)-i}$$

$$\cdot\binom{(n-1)-i}{i}$$
$$=2^{n-2i-1}\cdot\frac{n-2}{n-i-1}\binom{n-i-1}{i-1}$$
$$+2^{n-2i-1}\cdot\frac{n-1}{n-i-1}\binom{n-i-1}{i}$$
$$=2^{n-2i-1}\left\{\frac{n-2}{n-i-1}\binom{n-i}{i}+\frac{1}{n-i-1}\right.$$
$$\left.\cdot\binom{n-i-1}{i}\right\}$$
$$=2^{n-2i-1}\cdot\frac{n}{n-i}\binom{n-i}{i}\cdot\frac{1}{n(n-i-1)}$$
$$\cdot\left\{(n-i)(n-2)+(n-i)\frac{\binom{n-i-1}{i}}{\binom{n-i}{i}}\right\}$$
$$=2^{n-2i-1}\cdot\frac{n}{n-i}\binom{n-i}{i}\ . \qquad \square$$

Since

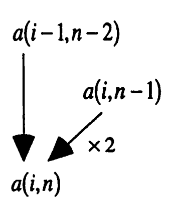
$a(0,n)=2^{n-1}\ \left(n\in\mathbf{N}(1)\right)$,

$a(i,n)=a(i-1,n-2)+2a(i,n-1)$

$\left(i\in\mathbf{N}\left(1,\left[\frac{n}{2}\right]\right), n\in\mathbf{N}(2)\right)$

with $a(0,0)=1$, $a\left(\left[\frac{n}{2}\right]+1,n\right)=0\ (n\in\mathbf{N})$, Table 1 is obtained.

Table 1 Table of $a(i,n)$.



n								i
								0
0							1	1
1						1	0	
2						2	1	2
3					4	3	0	
4					8	8	1	3
5				16	20	5	0	
6				32	48	18	1	4
7			64	112	56	7	0	
8			128	256	160	32	1	5
9		256	576	432	120	9	0	
10		512	1280	1120	400	50	1	6
11	1024	2816	2816	1232	220	11	0	
12	2048	6144	6912	3584	840	72	1	

2.2 Function $b(k,m)$

Definition 2.10

$$b(k,m)\overset{\text{def}}{=}\frac{(-1)^k}{m+k}\binom{m}{k}\binom{m+k}{m}$$
$$\left(k\in\mathbf{N}(0,m), m\in\mathbf{N}\right).$$

Proposition 2.11

(1) $b(0,m)=\dfrac{1}{m}$.

(2) $b(1,m)=-m$.

(3) $b(k,m)=\dfrac{(-1)^k m}{(k!)^2}\left(m^2-1^2\right)\left(m^2-2^2\right)\cdots\left(m^2-(k-1)^2\right)\quad\left(k\in\mathbf{N}(2,m)\right)$.

(4) $b(k,m)=-\dfrac{1}{k^2}\left(m^2-(k-1)^2\right)b(k-1,m)\quad\left(k\in\mathbf{N}(1,m)\right)$.

(5)
$$b(m,m)=-\frac{2m-1}{m^2}b(m-1,m)\quad\left(m\in\mathbf{N}(1)\right).$$

(6) $b(k,m+1)=\dfrac{m+k}{m-k+1}b(k,m)$.

Proof: (1), (2), (3) and (6) follow from Def.2.10, and (4), (5) from (3). □

Proposition 2.12

$$\sum_{0\le k\le l}b(k,m)=\frac{(-1)^l}{(l!)^2 m}\left(m^2-1^2\right)\left(m^2-2^2\right)\cdots\left(m^2-l^2\right)\quad\left(l\in\mathbf{N}(1,m)\right).$$

Especially,

$$\sum_{0\le k\le m}b(k,m)=0\ .$$

Proof:

For $l=1$,

$$b(0,m)+b(1,m)=\frac{1}{m}-m$$
$$=\frac{-1}{(1!)^2 m}\left(m^2-1^2\right).$$

For $l\ge 2$,

$$\sum_{0\le k\le l}b(k,m)=\sum_{0\le k\le l-1}b(k,m)+b(l,m)$$
$$=\left\{\frac{(-1)^{l-1}}{((l-1)!)^2 m}+\frac{(-1)^l m}{(l!)^2}\right\}$$

$$\cdot\left(m^2-1^2\right)\left(m^2-2^2\right)\cdots\left(m^2-(l-1)^2\right)$$

$$=\frac{(-1)^l}{(l!)^2 m}\left(m^2-1^2\right)\left(m^2-2^2\right)\cdots\left(m^2-l^2\right).$$

□

2.3 Function $c(k,m)$

Definition 2.13

$$c(k,m)\overset{\text{def}}{=}\frac{(-1)^k 2^{4k}}{(2k+1)^2}\frac{\binom{m+k}{2k}}{\binom{2k}{k}}$$

$$\left(k\in\mathbf{N}(0,m),\, m\in\mathbf{N}\right).$$

Moreover, the following definitions are given,

$(2m+1)!!\overset{\text{def}}{=}1\cdot3\cdot\cdots\cdot(2m+1)$,

$(2m)!!\overset{\text{def}}{=}2\cdot4\cdot\cdots\cdot2m.\;\; (-1)!!=0!!\overset{\text{def}}{=}1.$

Proposition 2.14

(1) $c(0,m)=1$.

(2) $c(m,m)=(-1)^m\dfrac{2^{2m}\cdot(2m)!}{((2m+1)!!)^2}=(-1)^m\dfrac{(4m)!!}{((2m+1)!!)^2}.$

(3) $c(k,m)=(-1)^k\left(\dfrac{2^k}{(2k+1)!!}\right)^2\cdot(m-k+1)(m-k+2)\cdots(m+k).$

(4) $c(k,m+1)=-\dfrac{m+k+1}{m-k+1}c(k,m)$.

(5) $c(k+1,m)=-\left(\dfrac{2}{2k+3}\right)^2\cdot(m-k)(m+k+1)c(k,m).$

Proof: (1) and (3) are immediately derived by the definition of $c(k,m)$.

$$\text{(2)}\;\; c(k,m)=\frac{(-1)^m 2^{4m}}{(2m+1)^2}\cdot\frac{(m!)^2}{(2m)!}$$

$$=\frac{(-1)^m 2^{2m}}{(2m+1)^2}\cdot\frac{((2m)!!)^2}{(2m)!}=\frac{(-1)^m 2^{2m}((2m)!)^2}{((2m+1)!!)^2(2m)!}$$

$$=\frac{(-1)^m 2^{2m}(2m)!}{((2m+1)!!)^2}\;.$$

□

Proposition 2.15

$$F_m(k)\overset{\text{def}}{=}\sum_{0\le j\le k}c(m-j,m)=\left(\frac{2(m-k)+1}{2m+1}\right)^2 c(m-k,m).$$

Especially,

$$F_m(m)=\frac{1}{(2m+1)^2}\;.$$

Proof:

$F_m(0)=c(m,m)$.

$F_m(k)=F_m(k-1)+c(m-k,m)$.

By the supposition of mathematical induction,

$$F_m(k)=\left(\frac{2(m-(k-1))+1}{2m+1}\right)^2\cdot c(m-(k-1),m)+c(m-k,m).$$

By Prop. 2.14 (5),

$$F_m(k)=\left[-\left(\frac{2(m-(k-1))+1}{2m+1}\right)^2\left(\frac{2}{2(m-k)+3}\right)^2\cdot\{m-(m-k)\}\{m+(m-k)+1\}+1\right]$$

$$\cdot c(m-k,m)=\left(\frac{2(m-k)+1}{2m+1}\right)^2 c(m-k,m)\;.$$

□

3. Integration of Trigonometric Function

Let i be the imaginary unit.

$$e^{in\theta}=(\cos\theta+i\sin\theta)^n\;.$$

Then,

$$\cos n\theta+i\sin n\theta$$

$$=\sum_{k\equiv0(4)}\binom{n}{k}\cos^{n-k}\theta\sin^k\theta-\sum_{k\equiv2(4)}\binom{n}{k}\cos^{n-k}\theta\sin^k\theta$$

$$+i\left[\sum_{k\equiv1(4)}\binom{n}{k}\cos^{n-k}\theta\sin^k\theta\right.$$

$$-\sum_{k\equiv 3(4)}\binom{n}{k}\cos^{n-k}\theta\sin^{k}\theta\Bigg] .$$

For $n\equiv 0(2)$, i.e., an even n,

$$\cos n\theta=\sum_{k\equiv 0(4)}\binom{n}{k}\cos^{n-k}\theta\sin^{k}\theta-\sum_{k\equiv 2(4)}\binom{n}{k}\cos^{n-k}\theta\sin^{k}\theta \qquad (4)$$

$$=\sum_{\frac{k}{2}\equiv 0(2)}\binom{n}{k}\cos^{n-k}\theta\left(1-\cos^{2}\theta\right)^{\frac{k}{2}}-\sum_{\frac{k}{2}\equiv 1(2)}\binom{n}{k}\cos^{n-k}\theta\left(1-\cos^{2}\theta\right)^{\frac{k}{2}}$$

$$=\binom{n}{0}\cos^{n}\theta+\binom{n}{4}\cos^{n-4}\theta\left(1-\cos^{2}\theta\right)^{2}+\binom{n}{8}\cos^{n-8}\theta\left(1-\cos^{2}\theta\right)^{4}+\cdots$$

$$-\Bigg[\binom{n}{2}\cos^{n-2}\theta\left(1-\cos^{2}\theta\right)+\binom{n}{6}\cos^{n-6}\theta\left(1-\cos^{2}\theta\right)^{3}+\binom{n}{10}\cos^{n-10}\theta\left(1-\cos^{2}\theta\right)^{5}+\cdots\Bigg]$$

$$=\sum_{0\le k\le\frac{n}{2}}\binom{n}{2k}\cos^{n}\theta-\sum_{1\le k\le\frac{n}{2}}\binom{k}{1}\binom{n}{2k}\cos^{n-2}\theta+\sum_{2\le k\le\frac{n}{2}}\binom{k}{2}\binom{n}{2k}\cos^{n-4}\theta-\cdots$$

$$+(-1)^{j}\sum_{j\le k\le\frac{n}{2}}\binom{k}{j}\binom{n}{2k}\cos^{n-2j}\theta+\cdots+(-1)^{\frac{n}{2}}$$

$$=\sum_{0\le j\le\frac{n}{2}}(-1)^{j}\sum_{j\le k\le\frac{n}{2}}\binom{k}{j}\binom{n}{2k}\cos^{n-2j}\theta$$

$$=\sum_{0\le j\le\frac{n}{2}}(-1)^{j}a(j,n)\cos^{n-2j}\theta .$$

Here, $a(j,n)$ is defined in Def. 2.1.

In an alternative way, $\cos n\theta$ is represented in terms of $\sin\theta$. By taking k in place of $n-k$, eqn. (4) becomes

$$\cos n\theta=(-1)^{\frac{n}{2}}\Bigg[\sum_{k\equiv 0(4)}\binom{n}{k}\cos^{k}\theta\sin^{n-k}\theta-\sum_{k\equiv 2(4)}\binom{n}{k}\cos^{k}\theta\sin^{n-k}\theta\Bigg]$$

$$=(-1)^{\frac{n}{2}}\sum_{0\le j\le\frac{n}{2}}(-1)^{j}a(j,n)\sin^{n-2j}\theta .$$

Thus, the following proposition is obtained.

Proposition 3.1

For even n,

$$\cos n\theta=\sum_{0\le j\le\frac{n}{2}}(-1)^{j}a(j,n)\cos^{n-2j}\theta=(-1)^{\frac{n}{2}}\sum_{0\le j\le\frac{n}{2}}(-1)^{j}a(j,n)\sin^{n-2j}\theta .$$

For odd n, the following equations are derived quite similarly to the case of even n.

$$\cos n\theta=\sum_{0\le j\le[\frac{n}{2}]}(-1)^{j}a(j,n)\cos^{n-2j}\theta ,$$

$$\sin n\theta=(-1)^{[\frac{n}{2}]}\sum_{0\le j\le[\frac{n}{2}]}(-1)^{j}a(j,n)\sin^{n-2j}\theta .$$

Then, Prop. 3.1 is summarized as

Theorem 3.2

(1) $\displaystyle\sum_{0\le j\le[\frac{n}{2}]}(-1)^{j}a(j,n)\cos^{n-2j}\theta=\cos n\theta \qquad (n\in\mathbf{N})$.

(2) $\displaystyle(-1)^{[\frac{n}{2}]}\sum_{0\le j\le[\frac{n}{2}]}(-1)^{j}a(j,n)\sin^{n-2j}\theta=\begin{cases}\cos n\theta & (n:\text{even})\\ \sin n\theta & (n:\text{odd})\end{cases}$.

Proposition 3.3

$$s(n)\overset{\text{def}}{=}\int_{0}^{\pi}\sin^{n}\theta\,d\theta=\begin{cases}\dfrac{(n-1)!!}{n!!}\cdot\pi & (n:\text{even})\\[2ex] \dfrac{(n-1)!!}{n!!}\cdot 2 & (n:\text{odd})\end{cases} .$$

Proof: The proof is carried out according to even and odd n.

1) n: even.

$$s(0)=\int_0^\pi d\theta=\pi=\frac{(-1)!!}{0!!}\cdot\pi \ .$$

By Theorem 3.2(2),

$$\sin^n\theta=\frac{1}{a(0,n)}\left[(-1)^{\frac{n}{2}}\cos n\theta + \sum_{1\le j\le\frac{n}{2}}(-1)^{j+1}a(j,n)\sin^{n-2j}\theta\right].$$

Since $\int_0^\pi \cos n\theta\, d\theta=0$,

$$s(n)=\frac{1}{2^{n-1}}\sum_{1\le j\le\frac{n}{2}}(-1)^{j+1}a(j,n)s(n-2j)$$

$$=\frac{1}{2^{n-1}}\sum_{1\le j\le\frac{n}{2}}\left\{(-1)^{j+1}\cdot 2^{n-2j-1}\frac{n}{n-j}\binom{n-j}{j}\cdot s(n-2j)\right\}$$

$$=\sum_{1\le j\le\frac{n}{2}}\frac{(-1)^{j+1}}{2^{2j}}\frac{n}{n-j}\binom{n-j}{j}s(n-2j) \ .$$

For $j\ge 1$, the supposition of mathematical induction asserts

$$s(n-2j)=\frac{(n-2j-1)!!}{(n-2j)!!}\cdot\pi \ .$$

Then, it suffices to say

$$\sum_{1\le j\le\frac{n}{2}}\frac{(-1)^{j+1}}{2^{2j}}\frac{n}{n-j}\binom{n-j}{j}\frac{(n-2j-1)!!}{(n-2j)!!}=\frac{(n-1)!!}{n!!}\quad(5)$$

for even $n(\ge 2)$.

Lemma 3.4

For even n,

$$\frac{(n-1)!!}{n!!}=\frac{1}{2^n}\binom{n}{\frac{n}{2}} \ .$$

Proof:

Let $n=2m$.

$$\frac{(2m-1)!!}{(2m)!!}=\frac{(2m)!}{\{(2\cdot 1)(2\cdot 2)\cdots(2m)\}^2}=\frac{(2m)!}{2^{2m}(m!)^2}=\frac{1}{2^{2m}}\binom{2m}{m}. \quad \square$$

Return to the proof of Prop. 3.3. Equation (5) becomes with $n=2m$

$$\sum_{1\le j\le m}\left\{\frac{(-1)^{j+1}}{2^{2j}}\frac{2m}{2m-j}\binom{2m-j}{j}\cdot\frac{1}{2^{2(m-j)}}\binom{2(m-j)}{m-j}\right\}=\frac{1}{2^{2m}}\binom{2m}{m} \ .$$

Thus,

$$\sum_{1\le j\le m}(-1)^{j+1}\frac{2m}{2m-j}\binom{2(m-j)}{m-j}\binom{2m-j}{2(m-j)}=\binom{2m}{m} \ .$$

Let $k=m-j$.

$$\sum_{0\le k\le m-1}(-1)^{m-k+1}\frac{1}{m+k}\binom{2k}{k}\binom{m+k}{2k}=\frac{1}{2m}\binom{2m}{m}.(6)$$

Lemma 3.5

$$\binom{m}{k}\binom{m+k}{m}=\binom{m}{k}\binom{m+k}{k}=\binom{2k}{k}\binom{m+k}{2k}.$$

Proof:

The proof of the first equality is trivial.

$$\binom{2k}{k}\binom{m+k}{2k}=\frac{(2k)!}{(k!)^2}\cdot\frac{(m+k)(m+k-1)\cdots(m-k+1)}{(2k)!}$$

$$=\frac{(m+k)(m+k-1)\cdots(m+1)}{k!}\cdot\frac{m(m-1)\cdots(m-k+1)}{k!}$$

$$=\binom{m+k}{k}\binom{m}{k} . \qquad \square$$

Return to the proof of Prop. 3.3. Equation (6) is written as

$$\sum_{0\le k\le m}\frac{(-1)^k}{m+k}\binom{m}{k}\binom{m+k}{k}=0 ,$$

i.e., $\sum_{0\le k\le m} b(k,m)=0$. (7)

This equation immediately follows from Prop. 2.12. The proof of Prop. 3.3 is completed for even n.

2) n: odd.

$$s(1)=\int_0^{\pi}\sin\theta\, d\theta=2 .$$

By Theorem 3.2(2),

$$\sin^n\theta=\frac{1}{a(0,n)}\left[(-1)^{\frac{n-1}{2}}\sin n\theta + \sum_{1\le j\le \frac{n-1}{2}}(-1)^{j+1}a(j,n)\sin^{n-2j}\theta\right].$$

Since $\int_0^{\pi}\sin n\theta\, d\theta=\frac{2}{n}$,

$$s(n)=\frac{1}{2^{n-1}}\left[(-1)^{\frac{n-1}{2}}\frac{2}{n} + \sum_{1\le j\le \frac{n-1}{2}}(-1)^{j+1}a(j,n)s(n-2j)\right]$$

$$(n=3,5,7,\cdots)$$

$$=\frac{1}{2^{n-1}}\left[(-1)^{\frac{n-1}{2}}\frac{2}{n} + \sum_{1\le j\le \frac{n-1}{2}}(-1)^{j+1}2^{n-1-2j}\frac{n}{n-j}\binom{n-j}{j}s(n-2j)\right].$$

For $j\ge 3$, the supposition of mathematical induction yields

$$s(n-2j)=\frac{(n-2j-1)!!}{(n-2j)!!}\cdot 2 .$$

Then, it suffices to say, with $n=2m+1$ $(m\ge 1)$,

$$\frac{1}{2^{2m}}(-1)^m\frac{1}{2m+1}+\sum_{1\le j\le m}\left\{(-1)^{j+1}\frac{1}{2^{2j}}\frac{2m+1}{2m+1-j}\cdot\binom{2m+1-j}{j}\frac{(2(m-j))!!}{(2(m-j)+1)!!}\right\}=\frac{(2m)!!}{(2m+1)!!} ,$$

i.e.,

$$\sum_{0\le j\le m}\left\{\frac{(-1)^j}{2^{2j}}\frac{2m+1}{2m-j+1}\binom{2m-j+1}{j}\cdot\frac{(2(m-j))!!}{(2(m-j)+1)!!}\right\}=\frac{(-1)^m}{2^{2m}}\frac{1}{2m+1} . \qquad (8)$$

Since

$$\frac{(2k)!!}{(2k+1)!!}=\frac{\{(2\cdot 1)(2\cdot 2)\cdots 2k\}^2}{(2k+1)!}=\frac{2^{2k}}{(2k+1)}\cdot\frac{1}{\binom{2k}{k}} ,$$

$$\sum_{0\le j\le m}\left\{\frac{(-1)^j}{2^{2j}}\frac{1}{2m-j+1}\binom{2m-j+1}{j}\cdot\frac{2^{2(m-j)}}{(2(m-j)+1)\cdot\binom{2(m-j)}{m-j}}\right\}=\frac{(-1)^m}{2^{2m}}\cdot\frac{1}{(2m+1)^2} .$$

$$\sum_{0\le j\le m}\left\{\frac{(-1)^j}{2^{4j}}\frac{1}{(2m-j+1)(2(m-j)+1)}\cdot\frac{(j+1)((m-j)!)^2}{(2(m-j)+1)!}\binom{2m-j+1}{j+1}\right\}$$

$$= \frac{(-1)^m}{2^{4m}} \cdot \frac{1}{(2m+1)^2}.$$

The lefthand side of this equation is written as

$$\sum_{0 \le j \le m} \frac{(-1)^j}{2^{4j}} \frac{1}{2(m-j)+1} \cdot \frac{((m-j)!)^2}{(2(m-j)+1)!} \cdot \binom{2m-j}{j}$$

$$= \sum_{0 \le j \le m} \frac{(-1)^j}{2^{4j}} \frac{1}{(2(m-j)+1)^2} \cdot \frac{\binom{2m-j}{j}}{\binom{2(m-j)}{m-j}} .$$

By $k = m - j$, the above becomes

$$\sum_{0 \le k \le m} \frac{(-1)^{m-k}}{2^{4(m-k)}} \frac{1}{(2k+1)^2} \cdot \frac{\binom{m+k}{2k}}{\binom{2k}{k}} .$$

Thus, it suffices to say

$$\sum_{0 \le k \le m} c(k,m) = \frac{1}{(2m+1)^2} .$$

This has already been shown in Prop. 2.14. The proof is completed for odd n as well as even n. □

N.B. A simple proof of Prop. 3.3 is found in Takagi[2].

4. Surface Area and Volume of n-Dimensional Sphere

An n-dimensional cube of segment of length a has a volume of a^n. The surface of the cube consists of $2n$ cubes of dimension $n-1$. Area of the surface is $2na^{n-1}$. Consider the case of sphere. Let $S_n(r)$, $V_n(r)$ be surface area and volume of the n-dimensional sphere of radius r, respectively. Since

$$V_n(r) = \int_0^r S_n(x)dx ,$$

it suffices to derive $S_n(r)$. All spheres of dimension n are similar, so that $S_n(r)$ is expressed as

$$S_n(r) = r^{n-1}\sigma_n , \quad \sigma_n \overset{\text{def}}{=} S_n(1) .$$

Here, $S_n(1)$ is the surface area of the n-dimensional unit sphere. Let $d\sigma_n$ be a surface element of the unit sphere of dimension n. Then,

$$d\sigma_n = d\theta \cdot \sin^{n-2}\theta \cdot d\sigma_{n-1} .$$

Hence,

$$\sigma_n = \int_{r=1} d\sigma_n = \sigma_{n-1}\int_0^{\pi} \sin^{n-2}\theta d\theta .$$

Since

$$s(n) = \int_0^{\pi} \sin^n \theta d\theta ,$$

$$S_n(r) = s(n-2) \cdot rS_{n-1}(r) .$$

By Prop. 3.3,

$$s(n-2) = \begin{cases} \frac{(n-3)!!}{(n-2)!!} \cdot \pi & (n : \text{even}) \\ \frac{(n-3)!!}{(n-2)!!} \cdot 2 & (n : \text{odd}) \end{cases} .$$

Thus,

$$S_{2m}(r) = \frac{(2m-3)!!}{(2m-2)!!} \pi r S_{2m-1}(r) ,$$

$$S_{2m+1}(r) = \frac{(2m-2)!!}{(2m-1)!!} 2rS_{2m}(r) .$$

By $S_2(r) = 2\pi r$, $S_n(r)$ is iteratively calculated as follows.

$$S_3(r) = 2rS_2(r) = 2^2\pi r^2 = 4\pi r^2 ,$$

$$S_4(r) = \frac{1}{2!!}\pi r S_3(r) = \frac{1}{2!!} 2\pi r^2 S_2(r)$$
$$= \frac{1}{2!!} 2^2 \pi^2 r^3 = 2\pi^2 r^3 ,$$

$$S_5(r)=\frac{2!!}{3!!}2rS_4(r)=\frac{1}{3!!}2^2\pi r^3 S_2(r)$$
$$=\frac{1}{3!!}2^3\pi^2 r^4=\frac{8}{3}\pi^2 r^4 ,$$

$$S_6(r)=\frac{3!!}{4!!}\pi r S_5(r)=\frac{1}{4!!}2^2\pi^2 r^4 S_2(r)$$
$$=\frac{1}{4!!}2^3\pi^3 r^5=\pi^3 r^5 ,$$

$$\vdots$$

$$S_n(r)=\frac{1}{(n-2)!!}2^{[\frac{n+1}{2}]}\pi^{[\frac{n}{2}]-1}r^{n-2}\cdot 2\pi r$$
$$=\frac{1}{(n-2)!!}2^{[\frac{n+1}{2}]}\pi^{[\frac{n}{2}]}r^{n-1} .$$

The volume of n-dimensional sphere is readily calculated.

$$V_n(r)=\int_0^r S_n(x)\,dx$$
$$=\frac{1}{(n-2)!!}2^{[\frac{n+1}{2}]}\pi^{[\frac{n}{2}]}\frac{r^n}{n}=\frac{1}{n!!}2^{[\frac{n+1}{2}]}\pi^{[\frac{n}{2}]}r^n .$$

For instance,

$$V_2(r)=\frac{1}{2}\cdot 2\pi r^2=\pi r^2 ,$$

$$V_3(r)=\frac{1}{3}\cdot 2^2\pi r^3=\frac{4}{3}\pi r^3 ,$$

$$V_4(r)=\frac{1}{2\cdot 4}\cdot 2^2\pi^2 r^4=\frac{1}{2}\pi^2 r^4 ,$$

$$V_5(r)=\frac{1}{3\cdot 5}\cdot 2^3\pi^2 r^5=\frac{8}{15}\pi^2 r^5 ,$$

$$V_6(r)=\frac{1}{2\cdot 4\cdot 6}\cdot 2^3\pi^3 r^6=\frac{1}{6}\pi^3 r^6 .$$

Theorem 4.1

(1) $S_n(r)=\frac{1}{(n-2)!!}2^{[\frac{n+1}{2}]}\pi^{[\frac{n}{2}]}r^{n-1}$.

(2) $V_n(r)=\frac{1}{n!!}2^{[\frac{n+1}{2}]}\pi^{[\frac{n}{2}]}r^n$.

Corollary 4.2

$$V_n(r)=\frac{r}{n}S_n(r) .$$

Especially for even n, say $n=2m$, the above formulae are expressed in a simple form.

Corollary 4.3

$$S_{2m}(r)=\frac{2}{(m-1)!}\pi^m r^{2m-1},$$
$$V_{2m}(r)=\frac{1}{m!}\pi^m r^{2m} .$$

References

1. K. Falconer, *Fractal Geometry, Mathematical Foundations and Applications*, John Wiley and Sons, New York, p26, 1990.
2. T. Takagi, *Kaiseki Gairon (Analysis)* (in Japanese), Iwanami, rev. 3rd ed. 17th printing, p116, 1994.
3. *Dictionary of Mathematics*, Iwanami Pub. 3rd ed.