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Graded commutative Q-algebras $\mathbf{Q}[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ of dimension 10 over Q

Dedicated to the memory of Professor Katsuo Kawakubo

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1. Introduction

Let A be a graded commutative $\mathbf{Q}$-algebra with $\dim_{\mathbf{Q}} A < \infty$. Then A is isomorphic to a truncated weighted polynomial ring

$$\mathbf{Q}[x_1, \ldots, x_n]/(f_1, \ldots, f_m),$$

where $m \geq n$ and f_i is a homogeneous element $(i = 1, \ldots, m)$.

Definition. A graded commutative $\mathbf{Q}$-algebra A is said to be *elliptic* if $\dim_{\mathbf{Q}} A < \infty$ and

$$A \cong \mathbf{Q}[x_1, \ldots, x_n]/(f_1, \ldots, f_n),$$

where f_i is a homogeneous element $(i = 1, \ldots, n)$.

In order to state the following theorem we set

$$|f| = \deg f$$

for a homogeneous element f of weighted polynomial ring $\mathbf{Q}[x_1, \ldots, x_n]$.

Theorem. *Let A be an elliptic graded commutative $\mathbf{Q}$-algebra. If $\dim_{\mathbf{Q}} A = 10$, then A is isomorphic to one of the following:*

(1) $$\mathbf{Q}[x]/(x^{10}).$$

(2) $$\mathbf{Q}[x_1, x_2]/({x_1}^2, {x_2}^5).$$

(3) $$\mathbf{Q}[x_1, x_2]/(x_1 x_2, {x_1}^5 + a{x_2}^5),$$

where $|x_1| = |x_2|$ and $a \in \mathbf{Q}^\times/\sim$ $(a \sim b \Leftrightarrow ab \in \mathbf{Q}^{\times 5}$ or $a/b \in \mathbf{Q}^{\times 5})$.

(4) $$\mathbf{Q}[x_1, x_2]/({x_1}^2 - a{x_2}^2, s x_1 {x_2}^4 + t {x_2}^5),$$

where $|x_1| = |x_2|$ *and* $(a,(s,t)) \in ((\mathbf{Q}^\times \setminus \mathbf{Q}^{\times 2}) \times (\mathbf{Q}^2 \setminus \{(0,0)\}))/\sim$ $((a,(s,t)) \sim (b,(u,v)) \Leftrightarrow b = ar^2, 0 < r \in \mathbf{Q}$ *and* $(\mp v + ur\sqrt{a})(-t + s\sqrt{a})/(\pm v + ur\sqrt{a})(t + s\sqrt{a}) \in \mathbf{Q}(\sqrt{a})_1^{\times 5}$, $\mathbf{Q}(\sqrt{a})_1 = \{c + d\sqrt{a} \in \mathbf{Q}(\sqrt{a})^\times \mid c^2 - ad^2 = 1\})$.

$$\mathbf{Q}[x_1, x_2]/({x_1}^3 + {x_2}^2, {x_1}^5), \tag{5}$$

where $|x_2| = (3/2)|x_1|$.

$$\mathbf{Q}[x_1, x_2]/({x_2}^2, {x_1}^5 + {x_1}^3 x_2), \tag{6}$$

where $|x_2| = 2|x_1|$.

$$\mathbf{Q}[x_1, x_2]/({x_1}^4 + a{x_2}^2, {x_1}^5 + {x_1}^3 x_2), \tag{7}$$

where $|x_2| = 2|x_1|$ *and* $a \in \mathbf{Q} \setminus \{0, -1\}$.

$$\mathbf{Q}[x_1, x_2]/({x_1}^4 + a{x_2}^2, {x_1}^5), \tag{8}$$

where $|x_2| = 2|x_1|$ *and* $a \in \mathbf{Q}^\times/\mathbf{Q}^{\times 2}$.

$$\mathbf{Q}[x_1, x_2]/({x_1}^4 + a{x_2}^2, {x_1}^3 x_2), \tag{9}$$

where $|x_2| = 2|x_1|$ *and* $a \in \mathbf{Q}^\times/\mathbf{Q}^{\times 2}$.

$$\mathbf{Q}[x_1, x_2]/({x_1}^2 x_2, {x_1}^6 + a{x_2}^2), \tag{10}$$

where $|x_2| = 3|x_1|$ *and* $a \in \mathbf{Q}^\times/\mathbf{Q}^{\times 2}$.

$$\mathbf{Q}[x_1, x_2]/(x_1 x_2, {x_1}^8 + a{x_2}^2), \tag{11}$$

where $|x_2| = 4|x_1|$ *and* $a \in \mathbf{Q}^\times/\mathbf{Q}^{\times 2}$.

$$\mathbf{Q}[x_1, x_2]/(x_1 x_2, {x_1}^6 + a{x_2}^4), \tag{12}$$

where $|x_2| = (3/2)|x_1|$ *and* $a \in \mathbf{Q}^\times/\mathbf{Q}^{\times 2}$.

$$\mathbf{Q}[x_1, x_2]/(x_1 x_2, {x_1}^7 + {x_2}^3), \tag{13}$$

where $|x_2| = (7/3)|x_1|$.

$$\mathbf{Q}[x_1, x_2]/({x_1}^2 x_2, {x_1}^4 + {x_2}^3), \tag{14}$$

where $|x_2| = (4/3)|x_1|$.

$$\mathbf{Q}[x_1, x_2, x_3]/({x_1}^2, x_2 x_3, {x_2}^3 + {x_3}^2), \tag{15}$$

where $|x_2| = (2/3)|x_3|$.

$$\mathbf{Q}[x_1, x_2, x_3]/({x_1}^2, x_2 x_3, x_1 {x_2}^2 + {x_2}^3 + {x_3}^2), \tag{16}$$

where $|x_1| = |x_2| = (2/3)|x_3|$.

$$\mathbf{Q}[x_1, x_2, x_3]/({x_1}^2 + a{x_2}^2, x_2 x_3, x_1 {x_2}^2 + {x_3}^2), \tag{17}$$

where $|x_1| = |x_2| = (2/3)|x_3|$ *and* $a \in \mathbf{Q}^\times/\mathbf{Q}^{\times 2}$.

$$\mathbf{Q}[x_1, x_2, x_3]/({x_1}^2 + a{x_2}^2, x_2 x_3, x_1 {x_2}^2 + {x_2}^3 + {x_3}^2), \tag{18}$$

where $|x_1| = |x_2| = (2/3)|x_3|$ *and* $a \in \mathbf{Q} \setminus \{0, -1\}$.

$$\mathbf{Q}[x_1, x_2, x_3]/({x_1}^2 + a{x_2}^2, x_2 x_3, {x_2}^3 + {x_3}^2), \tag{19}$$

where $|x_1|=|x_2|=(2/3)|x_3|$ *and* $a\in\mathbf{Q}^\times/\mathbf{Q}^{\times 2}$.

(20) $$\mathbf{Q}[x_1,x_2,x_3]/(x_1x_2, x_1{}^3+x_2x_3, x_2{}^2+ax_3{}^2),$$

where $|x_2|=|x_3|=(3/2)|x_1|$ *and* $a\in\mathbf{Q}^\times/\mathbf{Q}^{\times 2}$.

(21) $$\mathbf{Q}[x_1,x_2,x_3]/(x_1x_2, x_1{}^3+x_2x_3, ax_2{}^2-x_2x_3+x_3{}^2),$$

where $|x_2|=|x_3|=(3/2)|x_1|$ *and* $a\in\mathbf{Q}^\times$.

(22) $$\mathbf{Q}[x_1,x_2,x_3]/(x_1x_2, x_1{}^3+x_2{}^2, x_2{}^2+ax_3{}^2),$$

where $|x_2|=|x_3|=(3/2)|x_1|$ *and* $a\in\mathbf{Q}^\times/\mathbf{Q}^{\times 2}$.

(23) $$\mathbf{Q}[x_1,x_2,x_3]/(x_1x_2, x_2{}^2+x_1x_3, x_1{}^4+ax_3{}^2),$$

where $|x_2|=(3/4)|x_3|=(3/2)|x_1|$ *and* $a\in\mathbf{Q}^\times/\mathbf{Q}^{\times 2}$.

2. Proof of the Theorem

Let $A\cong\mathbf{Q}[x_1,\dots,x_n]/(f_1,\dots,f_n)$ be an elliptic graded commutative $\mathbf{Q}$-algebra. Then according to [1], we have

(2.1) $$\dim_{\mathbf{Q}}A=|f_1|\cdots|f_n|/|x_1|\cdots|x_n|.$$

We assume that each f_i $(i=1,\dots,n)$ has no linear terms and that

(2.2) $$|x_1|\le\cdots\le|x_n|,\ |f_1|\le\cdots\le|f_n|.$$

Following lemma is [3, Lemma 2.1].

Lemma 2.3. $2|x_i|\le|f_i|$ *for* $i=1,\dots,n$.

It follows from (2.1) and Lemma 2.3 that

(2.4) $$\dim_{\mathbf{Q}}A\ge 2^n.$$

Suppose that $\dim_{\mathbf{Q}}A=10$. Then it follows from (2.4) that $n=1,2$ or 3.

(I) Suppose that $n=1$. Then A is isomorphic to a type of (1) of Theorem.

(II) Suppose that $n=2$. Then it follows from (2.1) that

(2.5) $$|f_1|\cdot|f_2|=10|x_1|\cdot|x_2|.$$

There are following types which correspond to that of (2) of Theorem:

(2*a*) $$\mathbf{Q}[x_1,x_2]/(x_1{}^2,x_2{}^5)$$

and

(2*b*) $$\mathbf{Q}[x_1,x_2]/(x_1{}^5,x_2{}^2).$$

(II.1) Suppose that $|x_1|=|x_2|$. It follows from (2.5) that

$$5|f_1|=2|f_2|=10|x_1|.$$

We set $f_1 = x_1{}^2 - ax_2{}^2$ and $f_2 = sx_1x_2{}^4 + tx_2{}^5$ (a, s and $t \in \mathbf{Q}$). Suppose that

$$\varphi: \mathbf{Q}[x_1, x_2]/(f_{1,1}, f_{1,2}) \to \mathbf{Q}[x_1, x_2]/(f_{2,1}, f_{2,2})$$

is an isomorphism, where $f_{i,1} = x_1{}^2 - a_ix_2{}^2$, $f_{i,2} = s_ix_1x_2{}^4 + t_ix_2{}^5$ and $\varphi(x_i) = p_ix_1 + q_ix_2$ (a_i, s_i, t_i, p_i and $q_i \in \mathbf{Q}; i = 1, 2$). Then $a_2 = r^2a_1$ ($0 \neq r \in \mathbf{Q}$).

(II.1.1) Suppose that $a = 0$. Then $t \neq 0$ and A is isomorphic to a type of $(2a)$.

(II.1.2) Suppose that $a_i \in \mathbf{Q}^{\times 2} = \{r^2 \,|\, 0 \neq r \in \mathbf{Q}\}$ ($i = 1, 2$). Set $u_{i,1} = x_1 + \sqrt{a_i}x_2$ and $u_{i,2} = x_1 - \sqrt{a_i}x_2$ ($i = 1, 2$). Then $f_{i,1} = u_{i,1}u_{i,2}$, $(f_{i,2}) = (u_{i,1}{}^5 + b_iu_{i,2}{}^5)$ ($b_i \in \mathbf{Q}^\times = \mathbf{Q} \setminus \{0\}$) and $\varphi(u_{1,i}) = P_iu_{2,1} + Q_iu_{2,2}$ ($i = 1, 2$), where $P_1 = ((p_1 + \sqrt{a_1}p_2)\sqrt{a_2} + q_1 + \sqrt{a_1}q_2)/2\sqrt{a_2}$, $Q_1 = ((p_1 + \sqrt{a_1}p_2)\sqrt{a_2} - q_1 - \sqrt{a_1}q_2)/2\sqrt{a_2}$, $P_2 = ((p_1 - \sqrt{a_1}p_2)\sqrt{a_2} + q_1 - \sqrt{a_1}q_2)/2\sqrt{a_2}$ and $Q_2 = ((p_1 - \sqrt{a_1}p_2)\sqrt{a_2} - q_1 + \sqrt{a_1}q_2)/2\sqrt{a_2}$. This implies that

$$P_1P_2 = Q_1Q_2 = (P_1{}^5 + b_1P_2{}^5)b_2 - Q_1{}^5 - b_1Q_2{}^5 = 0.$$

If $P_2 = Q_1 = 0$, then $P_1Q_2 \neq 0$ and $b_2 = (Q_2/P_1)^5b_1$. If $P_1 = Q_2 = 0$, then $P_2Q_1 \neq 0$ and $b_1b_2 = (Q_1/P_2)^5$. This case corresponds to that of (3) of Theorem.

(II.1.3) Suppose that $a_i \in \mathbf{Q}^\times \setminus \mathbf{Q}^{\times 2}$ ($i = 1, 2$) and $a_2 = r^2a_1$ ($r \in \mathbf{Q}^\times$). Set $u_{i,1} = x_1 + \sqrt{a_i}x_2$ and $u_{i,2} = x_1 - \sqrt{a_i}x_2$ ($i = 1, 2; \sqrt{a_2} = r\sqrt{a_1}$). Then, in $\mathbf{Q}(\sqrt{a_1})[x_1, x_2]$, $f_{i,1} = u_{i,1}u_{i,2}$, $(f_{i,2}) = (u_{i,1}{}^5 + b_iu_{i,2}{}^5)$ and $\Phi(u_{1,i}) = P_iu_{2,1} + Q_iu_{2,2}$ ($i = 1, 2$), where

$$\Phi = \varphi \otimes 1: (\mathbf{Q}[x_1, x_2]/(f_{1,1}, f_{1,2})) \otimes \mathbf{Q}\sqrt{a_1} \to (\mathbf{Q}[x_1, x_2]/(f_{2,1}, f_{2,2})) \otimes \mathbf{Q}\sqrt{a_1},$$

$b_i = (s_i\sqrt{a_i} - t_i)/(s_i\sqrt{a_i} + t_i)$ ($i = 1, 2$), $P_1 = ((p_1 + \sqrt{a_1}p_2)\sqrt{a_2} + q_1 + \sqrt{a_1}q_2)/2\sqrt{a_2}$, $Q_1 = ((p_1 + \sqrt{a_1}p_2)\sqrt{a_2} - q_1 - \sqrt{a_1}q_2)/2\sqrt{a_2}$, $P_2 = ((p_1 - \sqrt{a_1}p_2)\sqrt{a_2} + q_1 - \sqrt{a_1}q_2)/2\sqrt{a_2}$ and $Q_2 = ((p_1 - \sqrt{a_1}p_2)\sqrt{a_2} - q_1 + \sqrt{a_1}q_2)/2\sqrt{a_2}$. This implies that

$$P_1P_2 = Q_1Q_2 = (P_1{}^5 + b_1P_2{}^5)b_2 - Q_1{}^5 - b_1Q_2{}^5 = 0.$$

If $P_2 = Q_1 = 0$, then $P_1 = p_1 + \sqrt{a_1}p_2$, $Q_2 = p_1 - \sqrt{a_1}p_2$ and $b_2 = (Q_2/P_1)^5b_1$. If $P_1 = Q_2 = 0$, then $P_2 = p_1 - \sqrt{a_1}p_2$, $Q_1 = p_1 + \sqrt{a_1}p_2$ and $b_1b_2 = (Q_1/P_2)^5$. This case corresponds to that of (4) of Theorem.

(II.2) Suppose that $|x_1| < |x_2|$.

(II.2.a) Assume that $|f_1|$ is some integer multiple of $|x_1|$: $|f_1| = k|x_1|$, where k is an integer. According to Lemma 2.3 and (2.5), $k \geq 2$ and $(10/k) = |f_2|/|x_2| \geq 2$. This implies that $2 \leq k \leq 5$.

(II.2.a.1) Assume that $k = 2$. Then $(f_1) = (x_1{}^2)$ and A is isomorphic to a type of $(2a)$.

(II.2.a.2) Assume that $k = 3$. Then $|f_2| = (10/3)|x_2|$, and hence $f_2 \in (x_1)$. This implies that $|f_1|$ is some integer multiple of $|x_2|$: $|f_1| = m|x_2|$, where m is an integer with $2 \leq m < 3$. This implies that $m = 2$, and hence

$$(f_1, f_2) = (x_2{}^2 + ax_1{}^3, x_1{}^5) \quad (a \in \mathbf{Q}).$$

If $a = 0$, then A is isomorphic to a type of $(2b)$. If $a \neq 0$, then A is isomorphic to a type of (5) of Theorem.

(II.2.a.3) Assume that $k = 4$. Then $|f_2| = (5/2)|x_2|$, and hence $f_2 \in (x_1)$. This implies that $|f_1|$ is some integer multiple of $|x_2|$: $|f_1| = m|x_2|$, where m is an integer with $2 \leq m \leq (5/2)$. This implies that $m = 2$, and hence

$$(f_1, f_2) = (x_2{}^2 + ax_1{}^4, bx_1{}^5 + cx_1{}^3x_2) \quad (a, b, c \in \mathbf{Q}; ac^2 + b^2 \neq 0).$$

Suppose that φ: $\mathbf{Q}[x_1,x_2]/(f_{1,1},f_{1,2}) \to \mathbf{Q}[x_1,x_2]/(f_{2,1},f_{2,2})$ is an isomorphism, where $f_{i,1}={x_2}^2+a_i{x_1}^4$, $f_{i,2}=b_i{x_1}^5+c_i{x_1}^3x_2$ and $\varphi(x_1)=px_1$, $\varphi(x_2)=p_2{x_1}^2+qx_2$ (a_i,s_i,t_i,p,p_2 and $q\in\mathbf{Q}; i=1,2$). Then

$$p_2=a_1p^4-a_2q^2=b_1c_2p^2-b_2c_1q=0.$$

If $a=c=0$, then A is isomorphic to a type of ($2b$). If $a=0$ and $bc\neq 0$, then A is isomorphic to a type of (6) of Theorem. If $abc\neq 0$, then A is isomorphic to a type of (7) of Theorem. If $c_i=0$ and $a_ib_i\neq 0$ ($i=1,2$), then we can assume that $b_1=b_2=1$. This case corresponds to that of (8) of Theorem. If $b_i=0$ and $a_ic_i\neq 0$ ($i=1,2$), then we can assume that $c_1=c_2=1$. This case corresponds to that of (9) of Theorem.

(II.2.a.4) Assume that $k=5$. Then $|f_2|=2|x_2|$ and $|x_2|\geq(5/2)|x_1|$. Set $|x_2|=m|x_1|$ ($m\geq 5/2$). If $m=3$, then

$$(f_1,f_2)=(a{x_1}^2x_2+b{x_1}^5, c{x_2}^2+d{x_1}^3x_2+e{x_1}^6) \quad (a,b,c,d,e\in\mathbf{Q}).$$

If $a=0$, then A is isomorphic to a type of ($2b$). If $a\neq 0$, we can set

$$(f_1,f_2)=({x_1}^2x_2, {x_2}^2+e{x_1}^6) \quad (e\in\mathbf{Q}^\times).$$

This case corresponds to that of (10) of Theorem. If $m=4$, then

$$(f_1,f_2)=(ax_1x_2+b{x_1}^5, c{x_2}^2+d{x_1}^4x_2+e{x_1}^8) \quad (a,b,c,d,e\in\mathbf{Q}).$$

If $a=0$, then A is isomorphic to a type of ($2b$). If $a\neq 0$, we can set

$$(f_1,f_2)=(x_1x_2, {x_2}^2+e{x_1}^8) \quad (e\in\mathbf{Q}^\times).$$

This case corresponds to that of (11) of Theorem. Otherwise, A is isomorphic to a type of ($2b$).

(II.2.b) Assume that $|f_1|$ is not an integer multiple of $|x_1|$, and it is some integer multiple of $|x_2|$: $|f_1|=k|x_2|$, where k is an integer. Then $(f_1)\in(x_2)$, and hence $|f_2|$ is some integer multiple of $|x_1|$: $|f_2|=m|x_1|$, where m is an integer. Then $2\leq k<m$ and $km=10$. This implies that $(k,m)=(2,5)$, and hence A is isomorphic to a type of ($2b$).

(II.2.c) Assume that $|f_1|$ is not an integer multiple of $|x_1|$ and it is not an integer multiple of $|x_2|$. Then $(f_1)\in(x_1x_2)$, and hence $|f_2|$ is some integer multiple of $|x_1|$ and $|x_2|$: $|f_2|=k|x_1|=m|x_2|$, where k and m are integers with $2\leq m<k$. It follows from (2.5) that $|x_1|=(m/10)|f_1|$ and $|x_2|=(k/10)|f_1|$. Since $|f_1|\geq|x_1|+|x_2|$, we obtain $m+k\leq 10$. Noting that $m\neq 2$, $k\neq 5$ and $k\neq 2m$, we see $(m,k)=(3,4)$, $(3,7)$ or $(4,6)$. If $(m,k)=(4,6)$, then

$$(f_1,f_2)=(x_1x_2, {x_2}^4+a{x_1}^6) \quad (a\in\mathbf{Q}^\times).$$

This case corresponds to that of (12) of Theorem. If $(m,k)=(3,7)$, then

$$(f_1,f_2)=(x_1x_2, {x_2}^3+a{x_1}^7) \quad (a\in\mathbf{Q}^\times),$$

and A is isomorphic to a type of (13) of Theorem. If $(m,k)=(3,4)$, then

$$(f_1,f_2)=({x_1}^2x_2, {x_2}^3+a{x_1}^4) \quad (a\in\mathbf{Q}^\times),$$

and A is isomorphic to a type of (14) of Theorem.

(III) After this, we consider the case of $n=3$. ¿From (2.1),

$$|f_1|\cdot|f_2|\cdot|f_3|=10|x_1|\cdot|x_2|\cdot|x_3|. \tag{2.6}$$

There are following types which correspond to that of (15) of Theorem:

(15a) $$\mathbf{Q}[x_1, x_2, x_3]/(x_1^2, x_2x_3, x_2^3 + x_3^2),$$

(15b) $$\mathbf{Q}[x_1, x_2, x_3]/(x_2^2, x_1x_3, x_1^3 + x_3^2)$$

and

(15c) $$\mathbf{Q}[x_1, x_2, x_3]/(x_3^2, x_1x_2, x_1^3 + x_2^2).$$

If $|x_1| = |x_2| = |x_3|$, then $|f_i|/|x_i| \geq 2$ $(i = 1, 2, 3)$ are integers. This contradicts to (2.6). Hence $|x_1| < |x_3|$ and there are only three possibilities: $|x_1| = |x_2| < |x_3|$, $|x_1| < |x_2| = |x_3|$ and $|x_1| < |x_2| < |x_3|$.

(III.1) First, we consider the case $|x_1| = |x_2| < |x_3|$.

(III.1.a) Assume that $|f_1|$ is an integer multiple of $|x_2|$: $|f_1| = k|x_2|$, where k is an integer. According to Lemma 2.3 and (2.6),

$$2|x_2| \leq |f_1| \leq |f_1| \cdot (|f_2|/2|x_2|) \cdot (|f_3|/2|x_3|) = (5/2)|x_2|.$$

Then $k = 2$ and $|f_2| \cdot |f_3| = 5|x_2| \cdot |x_3|$. Since $|x_2| < |x_3|$, $f_1, f_2 \in (x_1, x_2)$. Then $|f_3|$ is an integer multiple of $|x_3|$: $|f_3| = m|x_3|$, where m is an integer with $5/2 \geq m \geq 2$. So $m = 2$ and $|f_2| = (5/2)|x_2|$. This implies that $|f_2| = |x_2| + |x_3|$, and hence $|x_3| = (3/2)|x_2|$. Hence we may assume that $f_1 = x_1^2 + ax_2^2, f_2 = x_2x_3$ and $f_3 = x_3^2 + bx_1x_2^2 + cx_2^3$ $(a, b$ and $c \in \mathbf{Q}$; $ab^2 + c^2 \neq 0)$. Suppose that

$$\varphi\colon \mathbf{Q}[x_1, x_2, x_3]/(f_{1,1}, f_{1,2}, f_{1,3}) \to \mathbf{Q}[x_1, x_2, x_3]/(f_{2,1}, f_{2,2}, f_{2,3})$$

is an isomorphism, where $f_{i,1} = x_1^2 + a_ix_2^2$, $f_{i,2} = x_2x_3$, $f_{i,3} = x_3^2 + b_ix_1x_2^2 + c_ix_2^3$, $\varphi(x_i) = p_ix_1 + q_ix_2$ and $\varphi(x_3) = rx_3$ $(r(p_1q_2 - q_1p_2) \neq 0$; a_i, b_i, c_i, p_i, q_i and $r \in \mathbf{Q}$ for $i = 1, 2)$. We obtain

$$p_2 = q_1 = a_2p_1^2 - a_1q_2^2 = b_2r^2 - b_1p_1q_2^2 = c_2r^2 - c_1q_2^3 = 0.$$

If $a = b = 0$, then $c \neq 0$ and A is isomorphic to a type of (15a). If $a = 0$ and $bc \neq 0$, then A is isomorphic to a type of (16) of Theorem. If $c_1 = c_2 = 0$, Then $a_1a_2b_1b_2 \neq 0$ and we may assume that $b_1 = b_2 = 1$. Then $r^2 = p_1q_2^2$ and $a_1q_2^6 = a_2r^4$. This case corresponds to that of (17) of Theorem. If $abc \neq 0$, then A is isomorphic to a type of (18) of Theorem. If $a_1a_2c_1c_2 \neq 0$ and $b_1 = b_2 = 0$, then we may assume that $c_1 = c_2 = 1$. Then $r^2 = q_2^3$ and $a_2p_1^2 = a_1q_2^2$. This case corresponds to that of (19) of Theorem.

(III.1.b) Assume that $|f_1|$ is not an integer multiple of $|x_2|$. Then $f_1 \in (x_3)$. So $|f_2|$ is an integer multiple of $|x_2|$: $|f_2| = m|x_2|$, where m is an integer with $m \geq 3$. This implies that $|f_1| > 2|x_2|$, $|f_2| \geq 3|x_2|$ and $|f_3| < (10/6)|x_3|$. This contradicts to the fact $2|x_3| \leq |f_3|$.

(III.2) Secondly, we consider the case $|x_1| < |x_2| = |x_3|$.

(III.2.a) Assume that $|f_1|$ is some integer multiple of $|x_1|$. Then $|f_1| = 2|x_1|$ and $(f_1) = (x_1^2)$. So $|f_i|/|x_i|$ is an integer with $|f_i|/|x_i| \geq 2$ $(i = 1, 2, 3)$. This contradicts to (2,6).

(III.2.b) Assume that $|f_1|$ is not an integer multiple of $|x_1|$, and it is some integer multiple of $|x_2|$. Then $|f_1| = 2|x_2|$, and hence

$$(5/2)|x_1| \geq |f_i| \geq 2|x_2| > 2|x_1| \quad (i = 1, 2, 3).$$

So $(f_1, f_2, f_3) \subset (x_2, x_3)$. This contradicts to the fact $\dim_{\mathbf{Q}} A < \infty$.

(III.2.c) Assume that $|f_1|$ is not an integer multiple of $|x_1|$, and it is not an integer multiple of $|x_2|$. Then $|f_1| = |x_1| + |x_2|$ and $f_1 \in (x_1)$. So $|f_2| = |f_3| = 2|x_2|$. We can set $f_1 = x_1x_2, f_2 =$

$x_3{}^2+ax_2{}^2+bx_1{}^3, f_3=cx_2x_3+dx_2{}^2+ex_1{}^3$, ($a,b,c,d$, and $e\in\mathbf{Q}$; $e\neq 0$, $(c,d)\neq(0,0)$). Suppose that

$$\varphi\colon \mathbf{Q}[x_1,x_2,x_3]/(f_{1,1},f_{1,2},f_{1,3})\to\mathbf{Q}[x_1,x_2,x_3]/(f_{2,1},f_{2,2},f_{2,3})$$

is an isomorphism, where $f_{i,1}=x_1x_2$, $f_{i,2}=x_3{}^2+a_ix_2{}^2+b_ix_1{}^3$ and $f_{i,3}=c_ix_2x_3+d_ix_2{}^2+e_ix_1{}^3$, $\varphi(x_1)=px_1, \varphi(x_2)=q_2x_2+r_2x_3$ and $\varphi(x_3)=q_3x_2+r_3x_3$ $(p(q_2r_3-r_2q_3)\neq 0; a_i,b_i,c_i,d_i,e_i,p,q_2,q_3,r_2$ and $r_3\in\mathbf{Q}$; $i=1,2)$. Then

$$q_3=r_2=a_2r_3{}^2-a_1q_2{}^2=b_2r_3{}^2-b_1p^3=c_1d_2r_3-c_2d_1q_2=c_1e_2q_2r_3-c_2e_1p^3=d_1e_2q_2{}^2-d_2e_1p^3=0.$$

If $c=0$, then $d\neq 0$ and we can assume that $a=d-1=0$. If $b=c=0$, then A is isomorphic to a type of (15c). If $c_1=c_2=0$ and $b_1b_2\neq 0$, then we may assume that $e_1=e_2=1$. Then $p^3=q_2{}^2$ and $b_2r_3{}^2=b_1q_2{}^2$. This case corresponds to that of (22) of Theorem. If $c\neq 0$, then we can assume that $d=c-1=e-1=0$ and $a\neq 0$. If $b\neq 0$, then A is isomorphic to a type of (21) of Theorem. The case $b=0$ corresponds to that of (20) of Theorem.

(III.3) Lastly, we consider the case $|x_1|<|x_2|<|x_3|$.

(III.3.a) Assume that $|f_1|$ is an integer multiple of $|x_1|$. Then $|f_1|=2|x_1|$, $(f_1)=(x_1{}^2)$ and $|f_2|\cdot|f_3|=5|x_2|\cdot|x_3|$.

(III.3.$a.a$) Assume that $|f_2|$ is an integer multiple of $|x_2|$. Then $|f_2|=2|x_2|$, $|f_3|=(5/2)|x_3|$ and $|f_2|=2|x_3|$. So $|x_2|=|x_3|$. This contradicts to the fact $|x_2|<|x_3|$.

(III.3.$a.b$) Assume that $|f_2|$ is not an integer multiple of $|x_2|$. Then $|f_3|$ is an integer multiple of $|x_2|$: $|f_3|=k|x_2|$, where k is an integer with $k\geq 3$. So $|f_2|=(5/k)|x_3|$, $|f_3|=2|x_3|$ and $|f_2|\geq|x_2|+|x_3|=((k+2)/5)|f_2|$. Hence $k=3$ and A is isomorphic to a type of (15a).

(III.3.b) Assume that $|f_1|$ is not an integer multiple of $|x_1|$, and it is an integer multiple of $|x_2|$. Then $|f_1|=2|x_2|$ and $|f_2|\cdot|f_3|=5|x_1|\cdot|x_3|$.

(III.3.$b.a$) Assume that $|f_2|$ is an integer multiple of $|x_1|$. Then $|f_2|=k|x_1|$ $(3\leq k\in\mathbf{Z})$ and $|f_3|=(5/k)|x_3|<2|x_3|$. This contradicts to the fact $2|x_3|\leq|f_3|$.

(III.3.$b.b$) Assume that $|f_2|$ is not an integer multiple of $|x_1|$. Then $|f_3|=k|x_1|$ $(3\leq k\in\mathbf{Z})$ and $|f_2|=(5/k)|x_3|$. So $|f_3|=2|x_3|$ and $|f_2|\geq|x_1|+|x_3|=((k+2)/5)|f_2|$. Hence $k=3$ and A is isomorphic to a type of (15b).

(III.3.c) Assume that $|f_1|$ is not an integer multiple of $|x_1|$, it is not an integer multiple of $|x_2|$ and $f_1\notin(x_1,x_2)$. Then $|f_1|=2|x_3|$ and $|f_2|\cdot|f_3|=5|x_1|\cdot|x_2|$.

(III.3.$c.a$) Assume that $|f_2|$ is an integer multiple of $|x_1|$. Then $|f_2|=k|x_1|$ $(3\leq k\in\mathbf{Z})$ and $|f_3|=(5/k)|x_2|<2|x_3|$. This contradicts to the fact $2|x_3|\leq|f_3|$.

(III.3.$c.b$) Assume that $|f_2|$ is not an integer multiple of $|x_1|$. Then $|f_3|=k|x_1|$ $(3\leq k\in\mathbf{Z})$ and $|f_2|=(5/k)|x_2|<2|x_2|$. This contradicts to the fact $2|x_2|\leq|f_2|$.

(III.3.d) Assume that $|f_1|$ is not an integer multiple of $|x_1|$, it is not an integer multiple of $|x_2|$ and $f_1\in(x_1,x_2)$.

(III.3.d,a) Assume that $|f_2|$ is an integer multiple of $|x_1|$. Then $|f_2|=k|x_1|$ $(3\leq k\in\mathbf{Z})$ and $|f_1|\leq(5/k)|x_2|$. This implies that $k=3$, $(f_1)=(x_1x_2)$, $|f_1|=(5/2)|x_1|$, and hence $|f_2|\cdot|f_3|=4|x_2|\cdot|x_3|$. So $|f_i|/|x_i|=2$ $(i=2,3)$. We can set $f_1=x_1x_2, f_2=x_2{}^2+ax_1{}^3+bx_1x_3, f_3=x_3{}^2+cx_1{}^4$ $((a,c)\neq(0,0)$,

$(a, b) \neq (0, 0)$). If $b = 0$, then A is isomorphic to a type of (15c). If $b \neq 0$, then we may assume that $a = b - 1 = 0$. This case corresponds to that of (23) of Theorem.

(III.3.$d.b$) Assume that $|f_2|$ is not an integer multiple of $|x_1|$, and it is an integer multiple of $|x_2|$. Then $|f_2| = 2|x_2|$, $|f_3| = k|x_1|$ $(3 \leq k \in \mathbf{Z})$ and $|f_1| = (5/k)|x_3|$. This implies that $|f_3| = 2|x_3|$, $|f_1| = (5/2)|x_1|$ and $f_1 \in (x_1)$. So $(f_1) = (x_1 x_3)$ and $k = 3$. A is isomorphic to a type of (15b).

(III.3.$d.c$) Assume that $|f_2|$ is not an integer multiple of $|x_1|$, and it is not an integer multiple of $|x_2|$. Then $|f_3| = k|x_1| = m|x_2|$ $(3 \leq m < k; k, m \in \mathbf{Z})$. So $|f_1| < (5/k)|x_3|$, $|f_1| < (5/2)|x_1|$ and $(f_1) = (x_1 x_2)$. Since $|x_2| + |x_3| > ((m+2)/5)|f_2| \geq |f_2|$, $f_2 \in (x_1) \supset (f_1)$. This contradicts to the fact $\dim_{\mathbf{Q}} A < \infty$.

This completes the proof of Theorem.

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