THE BULLETIN OF OKAYAMA UNIVERSITY OF SCIENCE No. 36 A pp. 7-10 (2000)

An analogue of Hardy's theorem on the Poincaré disk

Nobukazu SHIMENO*
Department of Applied Mathematics,
Faculty of Science,
Okayama University of Science,
Ridai-cho 1-1, Okayama 700-0005, Japan

(Received November 1, 2000)

We generalize a theorem of Hardy for the Fourier transform on the real line to the Fourier transform on the Poincaré disk.

Introduction

Hardy's theorem for the Fourier transform [3] asserts that f and its Fourier transform $\hat{f}$ cannot both be very small. More precisely, let p and q be positive constants and assume that f is a measurable function on the real line satisfying $|f(x)| \le Ce^{-px^2}$ and $|\hat{f}(y)| \le Ce^{-qy^2}$ for some positive constant C. Then (1) $f = 0$ if $pq > 1/4$; (2) $f = Ae^{-px^2}$ for some constant A if $pq = 1/4$; (3) there are infinitely many f if $pq < 1/4$.

Sitaram and Sundari [8] generalized part (1) of Hardy's result to semisimple Lie groups with one conjugacy class of Cartan subgroups and to the K-invariant case for general semisimple groups. There are several works that extend the result of Sitaram and Sundari to more wider classes of transforms.

In this paper, we prove an analogue of Hardy's theorem on the Poincaré disk, which contains all statements (1), (2), and (3). The key idea is to estimate a function f by the heat kernel. In the case of the classical Fourier transform on the real line and the Hankel transform, the heat kernel is a dilation of $e^{-x^2/2}$ and it's image under the transform is of the same form. In the case of semisimple Lie group, it is not the case and it is natural to change e^{-px^2} in the assumption of the theorem by the heat kernel on the Poincaré disk. Using upper bound for the heat kernel, the result of Sitaram and Sundari for the Poincaré disk follows as a corollary of our result.

1 Preliminaries

In this section we review on some results on harmonic analysis on the Poincaré disk. We refer the reader to Helgason [4] and Terras [9] for details.

1.1 The Poincaré disk D

Let D be the open disk $|z| < 1$ in $\mathbf{C}$ with the Riemannian structure[1]

$$d\sigma(z) = 4\frac{dxdy}{(1-x^2-y^2)^2}, \tag{1.1}$$

where $z = x+iy$. Then the Laplace-Beltrami operator is given by

$$L = \frac{1}{4}(1-x^2-y^2)^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right). \tag{1.2}$$

Let B be the boundary of D and let db be the circular measure on B given by

$$\int_B f(b)db = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})d\theta.$$

The group

$$G = SU(1,1) = \left\{g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a}\end{pmatrix} \; : \; |a|^2 - |b|^2 = 1\right\}$$

acts on D by

$$g\cdot z = \frac{az+b}{\bar{b}z+\bar{a}}.$$

The action is transitive and the isotropy subgroup at o is $K = SO(2)$. Thus

$$D \simeq G/K.$$

The Riemannian structure $d\sigma(z)$ and the Laplacian L is invariant by the action of G.

*This research was supported by the Ministry of Education, Science, Sports and Culture, Grant-in-Aid for Encouragement of Young Scientists, No. 12740097 in 2000.

[1] $d\sigma(z)$ is four times the Riemannian structure given in [4].

1.2 The Poisson kernel

A horocycle is a circle in D that is tangential to B. For $z \in D$ and $b \in B$ let $\langle z, b\rangle = \pm d(o, \xi)$, where ξ is a horocycle such that $z \in \xi$ and the sign is $+$ or $-$ according to the fact that o is outside or inside ξ. Then

$$e^{\langle z,b\rangle} = \frac{1-|z|^2}{|z-b|^2}$$

is the classical Poisson kernel. Complex powers of the classical Poisson kernel are eigenfunctions of the Laplace-Beltrami operator. Namely

$$L(e^{(i\mu+\frac{1}{2})\langle z,b\rangle}) = -(\mu^2+\tfrac{1}{4})e^{(i\mu+\frac{1}{2})\langle z,b\rangle}$$

for $\mu \in \mathbf{C}$.

1.3 The spherical function

A function f on D is called radial if

$$f(\tanh \tfrac{r}{2}\, e^{i\theta}) = f(\tanh \tfrac{r}{2}) \quad \text{for all } r \geq 0,\ \theta \in \mathbf{R},$$

i.e. $f(z) = f(|z|)$ for all $z \in D$.

The function

$$\phi_\mu(z) = \int_B e^{(i\mu+\frac{1}{2})\langle z,b\rangle} db \tag{1.3}$$

is called the spherical function. It is a unique radial function on D such that $f(o) = 1$ and

$$Lf = -(\mu^2+\tfrac{1}{4})f. \tag{1.4}$$

Radial solutions of (1.4) satisfy

$$\frac{\partial^2 f}{\partial r^2} + \coth r \frac{\partial f}{\partial r} = -(\mu^2+\tfrac{1}{4})f,$$

hence

$$\phi_\mu(\tanh \tfrac{r}{2}\, e^{i\theta}) = P_{-\frac{1}{2}-i\mu}(\cosh r),$$

where

$$P_\nu(z) = F(-\nu, \nu+1; 1; (1-z)/2)$$

is the Legendre function of the first kind.

The spherical Fourier transform of a radial function f on D is defined by

$$\tilde{f}(\mu) = \int_D f(z)\phi_{-\mu}(z)d\sigma(z). \tag{1.5}$$

In the coordinates (r, θ) we have

$$d\sigma(z) = \sinh r\, dr\, d\theta$$

and

$$\tilde{f}(\mu) = 2\pi \int_0^\infty f(\tanh \tfrac{r}{2})P_{-\frac{1}{2}+i\mu}(\cosh r)\sinh r\, dr.$$

The inversion of the spherical transform is given by

$$f(\tanh(r/2)) = \frac{1}{2\pi}\int_0^\infty \tilde{f}(\mu)P_{-\frac{1}{2}+i\mu}(\cosh r)\mu \tanh \pi\mu\, d\mu.$$

1.4 The Fourier transform on D

We define the Fourier transform of a function f on D by

$$\tilde{f}(\mu, b) = \int_D f(z)e^{(-i\mu+\frac{1}{2})\langle z,b\rangle}d\sigma(z), \tag{1.6}$$

where $\mu \in \mathbf{C}$ and $b \in B$.

If $f \in C_c^\infty(D)$, then

$$f(z) = \frac{1}{4\pi}\int_{\mathbf{R}}\int_B \tilde{f}(\mu, b)e^{(i\mu+\frac{1}{2})\langle z,b\rangle}\mu \tanh \pi\mu\, db\, d\mu. \tag{1.7}$$

The map $f \mapsto \tilde{f}(\mu, b)$ extends to an isometry of $L^2(D, d\sigma(z))$ onto

$$L^2(\mathbf{R}\times B, (4\pi)^{-1}u \tanh \pi\mu\, d\mu\, db).$$

1.5 K-finite functions

Let $m \in \mathbf{Z}$. The eigenfunctions f of L satisfying

$$f(e^{i\theta}z) = e^{im\theta}f(z) \tag{1.8}$$

are the constant multiple of the function

$$\phi_{\mu,m}(z) = \int_B e^{(i\mu+\frac{1}{2})\langle z,b\rangle}\chi_m(b)db, \tag{1.9}$$

where $\mu \in \mathbf{C}$ and $\chi_m(e^{i\phi}) = e^{im\phi}$. In particular, $\phi_{\mu,0} = \phi_\mu$.

If a function f on D satisfies $f(e^{i\theta}z) = e^{im\theta}f(z)$, then

$$\tilde{f}(\mu, b) = \int_D f(z)\phi_{-\mu,-m}(z)d\sigma(z).$$

Thus $\tilde{f}(\mu, b)$ does not depend on b. In particular, $\tilde{f}(\mu, b)$ is the spherical transform $\tilde{f}(\mu)$ of f, if f is a radial function on D.

1.6 The heat kernel

Let $u = u(z,t)$ be a function on $(z,t) \in H \times (0,\infty)$. We consider an initial value problem of the heat equation on H

$$u_t = L_z u$$
$$u(z,0) = f(z),$$

where f is a radial function on D.

For $t > 0$ let $G_t(z)$ be the radial function on D that is the inverse image of $e^{-(\mu^2+1/4)t}$ under the spherical transform. Thus

$$\tilde{G}_t(\mu) = e^{-(\mu^2+\frac{1}{4})t}. \qquad (1.10)$$

Then

$$u(z,t) = (f * G_t)(z) = \int_D f(g\cdot w)G_t(w)d\sigma(w)$$

is a solution to the above initial value problem. Here $g\cdot o = z$.

The heat kernel is given by the formula

$$G_t(\tanh \tfrac{r}{2}) = (4\pi t)^{-3/2}\sqrt{2}e^{-t/4} \times \int_r^\infty \frac{se^{-s^2/(4t)}}{\sqrt{\cosh s - \cosh r}}ds. \qquad (1.11)$$

Davies and Mandouvalos [1] proved that

$$G_t(\tanh \tfrac{r}{2}) \sim t^{-\frac{1}{2}}e^{-\frac{t}{4}-\frac{r^2}{4t}-\frac{r}{2}}(1+r+t)^{-\frac{1}{2}}(1+r), \qquad (1.12)$$

uniformly for $0 \le r < \infty$ and $0 < t < \infty$. Here we write $f \sim g$ when there is $c > 0$ such that

$$c^{-1}f \le g \le cf$$

for all values of variables in the domain.

2 An analogue of Hardy's theorem

We now state and prove an analogue of Hardy's theorem for the Fourier transform on D.

Theorem 2.1 *Let p and q be positive constants. Suppose f is a measurable function on D satisfying*

$$|f(z)| \le CG_{\frac{1}{4p}}(z) \quad \textit{for all } z \in D \qquad (2.1)$$

and

$$|\tilde{f}(\mu,b)| \le Ce^{-q\mu^2} \quad \textit{for all } \lambda \in \mathbf{R},\ b \in B, \qquad (2.2)$$

where C is a positive constant. Then we have following results:

(1) *If $pq > 1/4$, then $f = 0$ almost everywhere.*

(2) *If $pq = 1/4$, then $\tilde{f}(\mu,b) = h(b)e^{-q\mu^2}$, where h is an arbitrary bounded function on B.*

(3) *If $pq < 1/4$, then there are infinitely many such functions f.*

Proof. By (2.1) and (1.10) we have

$$\begin{aligned}|\tilde{f}(\mu,b)| &\le C\int_D G_{\frac{1}{4p}}(z)e^{(\mathrm{Im}\mu+\frac{1}{2})\langle z,b\rangle}d\sigma(z)\\ &= Ce^{(\mathrm{Im}\mu)^2-1/4)/(4p)}\\ &= C'e^{\frac{(\mathrm{Im}\mu)^2}{4p}}\end{aligned} \qquad (2.3)$$

for all $\mu \in \mathbf{C}$. For fixed b, $\tilde{f}(\mu,b)$ is a holomorphic function of $\mu \in \mathbf{C}$.

If we can prove (2), then (1) is self-evident.

We will prove (2). If $pq = 1/4$, then by using the Phragmén-Lindelöf theorem, (2.2) and (2.3) imply

$$\tilde{f}(\mu,b) = h(b)e^{-q\mu^2}, \qquad (2.4)$$

where the function $h(b)$ is bounded. See Dym and McKean [2, Section 3.2] for details. Conversely if $\tilde{f}(\mu,b)$ is given by (2.4) and h is bounded, then we can show (2.1) by using the inversion formula (1.7).

For (3), choose $p < p' < 1/(4q)$ and let $f(z) = G_{\frac{1}{4p'}}(z)$. It is easy to see that $f(z)$ satisfies (2.1) by using (1.11). Moreover $\tilde{f}(\mu,\lambda) = e^{-(\mu^2+\frac{1}{4})/(4p')}$ satisfies (2.2). □

Remark 2.2 If $pq = 1/4$, then

$$f(\tanh \tfrac{r}{2}\, e^{i\theta}) = \chi_m(e^{i\theta})G_q(\tanh \tfrac{r}{2})$$

satisfies conditions (2.1) and (2.2) of the theorem.

We may replace $G_t(z)$ in the right hand side of (2.2) by the right hand side of (1.12). As a corollary we have the following result of Sitaram and Sundari [8] in the case of D.

Corollary 2.3 *Let p and q be positive constants. Suppose f is a measurable function on D satisfying* (2.2) *and*

$$|\tilde{f}(\tanh \tfrac{r}{2}e^{i\theta})| \le Ce^{-pr^2} \quad \textit{for all } r \ge 0 \textit{ and } \theta \in \mathbf{R},$$

where C is a positive constant.

If $pq > 1/4$, then $f = 0$ almost everywhere.

Proof. Choose $p > p'$ such that $p'q > 1/4$. Then the is a constant C' such that

$$e^{-pr^2} \le C'e^{-p'r^2-r/2}(1+r+1/(4p'))^{-1/2}(1+r).$$

By (1.12) we can apply part (3) of Theorem 2.1. □

Concluding remarks

We can generalize our result to the hyperbolic space $\mathbb{H}^n$. We hope our idea is applicable to general semisimple Lie groups. Moreover it is of interest to consider the case of the Jacobi transform. We will come back to these subject elsewhere.

Acknowledgment

The author thanks Professor S. Koizumi, S. Murakami, T. Nomura, T. Oshima, and S. Takenaka, who gave helpful comments on our previous works [6, 7, 5] and motivated this work.

References

[1] E. B. Davies and N. Mandouvalos, *Heat kernel bounds on hyperbolic space and Kleinian groups*, Proc. London Math. Soc. **57** (1988), pp. 182–208.

[2] H. Dym and H. P. McKean, *Fourier Seres and Integrals*, Academic Press, San Diego, 1972.

[3] G. H. Hardy, *A theorem concerning Fourier transforms*, J. London Math. Soc. **8** (1933), pp. 227–231.

[4] S. Helgason, *Groups and Geometric Analysis*, Academic Press, New York, 1984.

[5] H. Sakase, *An analogue of Hardy's theorem for the Jacobi and Hankel transforms* (in Japanese), Master Thesis, Okayama University of Science, 2000.

[6] N. Shimeno, *A note on the uncertainty principle for the Dunkl transform*, preprint.

[7] N. Shimeno, *An analogue of Hardy's theorem for the Heckman-Opdam transform*, preprint.

[8] A. Sitaram and M. Sundari, *An analogue of Hardy's theorem for very rapidly decreasing functions on semisimple Lie groups*, Pacific J. Math. **177** (1997), pp. 187–200.

[9] A. Terras, *Harmonic Analysis on Symmetric Spaces and Applications* I, Springer Verlag, New York Berlin Heiderberg Tokyo, 1985.