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The Picard group of a finitely generated and birational extension of a Noetherian domain

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Let A be a finitely generated and birational extension of a Noetherian domain R whose generators are super-primitive or anti-integral. In this paper we will determine the relationship between the unit groups $\mathrm{U}(R)$ and $\mathrm{U}(A)$ (Theorem 3) and the kernel of the homomorphism of the Picard groups $\mathrm{Pic}(R)$ into $\mathrm{Pic}(A)$ (Theorem 7).

In this paper a ring will mean a commutative ring with the identity element. An integral domain will stand for a ring without non-trivial zero-divisors.

Let R be a Noetherian domain and $R[X]$ a polynomial ring in an indeterminate X over R. Let K be the quotient field of R and L an algebraic extension of K. Let α be an element of L. We denote by π the R-homomorphism of $R[X]$ into $R[\alpha]$ defined by $\pi(X) = \alpha$. Let $\phi_\alpha(X)$ be the monic minimal polynomial of α over K and $d =$ the degree of $\phi_\alpha(X)$. We will write $\phi_\alpha(X)$ of the form $\phi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$ where $\eta_1, \cdots, \eta_d$ are elements of K. Then η_i ($1 \le i \le d$) are uniquely determined by α. We will set $I_{\eta_i} := R :_R \ \eta_i = \{ a \in R \, ; a\eta_i \in R \}$, $I_{[\alpha]} := \cap_{i=1}^d I_{\eta_i}$.

We will call $I_{[\alpha]}$ the denominator ideal of α. It is easily seen that $I_{[\alpha]} = R[X] :_R \phi_\alpha(X)$. Note that $I_{[\alpha]} = I_\alpha$ if α is in K. We say that α is *anti-integral* over R if $\mathrm{Ker}(\pi) = I_{[\alpha]}\phi_\alpha R[X]$. The notion of anti-integral property is introduced in [4] in the case of birational extensions, and subsequently in [3] in the case of higher degree extensions. We know that an element α of K is anti-integral over R if and only if $R = R[\alpha] \cap R[\alpha^{-1}]$.

Let $f(X)$ be an element of $R[X]$. We denote by $c(f(X))$ the ideal of R generated by all coefficients of $f(X)$. Let J be an ideal of $R[X]$ and $c(J)$ the ideal of R generated by all coefficients of elements of J. If α is anti-integral over R, then we know that $c(\mathrm{Ker}\ (\pi)) = c(I_{[\alpha]} \ \phi_\alpha(X) \ R[X]) = I_{[\alpha]}(1, \ \eta_1, \ \cdots, \ \eta_d)$ where $(1, \ \eta_1, \ \cdots, \ \eta_d)$ is an $R-$module generated by $1, \ \eta_1, \ \cdots, \ \eta_d$. We will set $J_{[\alpha]} = I_{[\alpha]}(1, \ \eta_1, \ \cdots, \ \eta_d)$.

Let $\mathrm{Dp}_1(R)$ be the set of all prime ideals p of R such that $\mathrm{depth}(R_p) = 1$. An element α is called *super-primitive* over R if $J_{[\alpha]} \not\subset p$ for every p of $\mathrm{Dp}_1(R)$. We will use J_α instead of $J_{[\alpha]}$ in case α is in K. Sometimes we will use $I_{R,\ \alpha}$ and $J_{R,\ \alpha}$ instead of I_α and J_α respectively. If α is super-primitive over R, then α is anti-integral over R (cf. [3]). By the definition super-primitive property is characterized by the set $\mathrm{Dp}_1(R)$.

Throughout this paper we will use the notation above unless it is specified. Our terminology is a standard one and we can refer to [2].

Let $\mathrm{U}(R)$ be the unit group of a ring R. Our first purpose is the calculation of $\mathrm{U}(A)$ in the case the extension A/R is finitely generated and birational. Our key Proposition is the following (cf. [1, Proposition 1]).

Proposition 1. *Let R be an integral domain with the quotient field K. Let α be an element of K such that α is anti-integral over R. Set $A = R[\alpha]$, $\mathrm{V}(I_\alpha) = \{p \in \mathrm{Spec}(R);\ I_\alpha \subset p\}$ and $\Gamma_J = \{p \in \mathrm{Spec}(R);\ p + J_\alpha = R\}$. Then $\mathrm{V}(I_\alpha) \cap \Gamma_{J_\alpha} = \{p \in \mathrm{Spec}(R);\ pA = A\} \cdots (*)$.*

By Proposition 1 we obtain the following:

Proposition 2. *Let R be a Noetherian domain with the quotient field K. Let α be an element of K such that α is super-primitive over R and set $A = R[\alpha]$. Let $I_\alpha = q_1 \cap \cdots \cap q_n$, $\mathrm{rad}(q_i) = p_i$ $(1 \le i \le n)$ be the irredundant primary decomposition of an ideal I_α. If $p_i + J_\alpha = R$ $(1 \le i \le n)$, then $\mathrm{U}(A) \cap R = \mathrm{U}(R)$.*

Proof. Let p be a prime ideal of R satisfying $\mathrm{depth}(R_p) = 1$. First we will show that $pA \neq A$. Suppose the contrary. Then the equality (*) in Proposition 1 implies that $I_\alpha \subset p$. Since α is super-primitive over R, we have $(I_\alpha)_p = xR_p$ for some element x of p (cf. [3, Theorem 2.11]). Then p is a prime divisor of I_α because $xR_p \subset pR_p$ and $\mathrm{depth}(R_p) = 1$. Hence $p = p_i$ for some i with $1 \le i \le n$. By the assumption we see that $p + J_\alpha \neq R$. This contradicts the fact $p \in \Gamma_{J_\alpha}$.

We will show that $\mathrm{U}(A) \cap R = \mathrm{U}(R)$. The inclusion $\mathrm{U}(A) \cap R \supset \mathrm{U}(R)$ is obvious, so we will prove the converse inclusion. Let a be an element of R such that a is not a unit of R. Then there exists a prime divisor p of aR because $aR \neq R$. Note that $\mathrm{depth}(R_p) = 1$ (cf. [6, Proposition 10]). The former half of the proof shows that $pA \neq A$. This implies that $aA \neq A$, that is, a is not a unit of A. Hence $\mathrm{U}(A) \cap R \subset \mathrm{U}(R)$. $\square$

Theorem 3. *Let R be a Noetherian domain with the quotient field K. Let $\alpha_1, \cdots, \alpha_n$ be elements of K and set $R_0 = R$, $R_i = R[\alpha_1, \cdots, \alpha_i]$ $(1 \le i \le n)$ and $A = R_n$. Assume that α_{i+1} is super-primitive over R_i for $0 \le i \le n-1$. Let $I_{R_i, \alpha_{i+1}}$ be the denominator ideal of α_{i+1} in $R_{i+1} = R_i[\alpha_{i+1}]$ and $J_{R_i,\alpha_{i+1}} = I_{R_i,\alpha_{i+1}}(1, \alpha_{i+1})$. Let $I_{R_i,\alpha_{i+1}} = q_{i,1} \cap \cdots \cap q_{i,t_i}$ be the irredundant primary decomposition of $I_{R_i,\alpha_{i+1}}$. If $\mathrm{rad}(q_{i,j}) + J_{R_i,\alpha_{i+1}} \neq R_i$ for all $0 \le i \le n-1$ and $1 \le j \le t_i$ hold for the ring extension R_{i+1}/R_i, then $\mathrm{U}(A) \cap R = \mathrm{U}(R)$.*

Proof. We may apply Proposition 2 to the ring extension R_{i+1}/R_i $(0 \le i \le n-1)$. $\square$

Remark 4. Let R be a Noetherian domain with the quotient field K. Let α be an element of K which is anti-integral over R and set $A = R[\alpha]$. It is easily seen from Proposition 1 that $aA = A$ if and only if $\mathrm{rad}(aR) \supset I_\alpha$ and $aR + J_\alpha = R$ for an element a of R. Therefore we get $\mathrm{U}(A) \cap R = \mathrm{U}(R)$ if the following condition holds: let a be an element of R satisfying $a \notin \mathrm{U}(R)$, then

(1) $\mathrm{rad}(aR) \not\supset I_\alpha$

or

(2) there exists a prime ideal $p \in \mathrm{Ass}_R(R/aR)$ such that $p + J_\alpha \neq R$.

If $I_\alpha \neq R$, then there exists an element x of R such that $p = I_\alpha : x = I_{x\alpha}$. This means that we can take $I_{x\alpha} = p$ for the extension $R_0 = R \subset R_1 = R[x\alpha] \subset A$.

In this case we don't know whether $x\alpha$ is super-primitive over R or not.

We will consider a sufficient condition for $\mathrm{U}(A) = \mathrm{U}(R)$.

Proposition 5. *Let R be a Noetherian domain with the quotient field K. Let α be an element of K which is super-primitive over R and set $A = R[\alpha]$. If I_α is a prime ideal of R and $J_\alpha \neq R$, then $\mathrm{U}(A) = \mathrm{U}(R)$.*

Proof. Set $I_\alpha = p$. Let β be a unit of A. Then we can write $\beta = f(\alpha)$ and $\beta^{-1} = g(\alpha)$ for $f(X)$, $g(X)$ in $R[X]$. Furthermore write

$f(X) = a_0 + a_1X + \cdots + a_nX^n$ ($a_i \in R$ and $0 \leq i \leq n$) and $g(X) = b_0 + b_1X + \cdots + b_mX^m$ ($b_j \in R$ and $0 \leq j \leq m$). We may assume that m and n are minimal. By Proposition 2 we know that $\mathrm{U}(A) \cap R = \mathrm{U}(R)$. If $n = 0$, then $\beta = a_0$ is in $\mathrm{U}(A) \cap R = \mathrm{U}(R)$. If $m = 0$, then $\beta^{-1} = b_0$ is in $\mathrm{U}(A) \cap R = \mathrm{U}(R)$, hence β is in $\mathrm{U}(R)$. So we may assume that $n > 0$ and $m > 0$. From the fact $1 = f(\alpha)g(\alpha)$ we get $1 - f(X)g(X) \in \mathrm{Ker}(\pi) = I_\alpha(X - \alpha)R[X]$. This implies that $a_nb_m \in I_\alpha = p$. Since p is a prime ideal, we know that a_n is in I_α or b_m is in I_α. So $a_{n-1} + a_n\alpha$ is in R or $b_{m-1} + b_m\alpha$ is in R. Note that

$\beta = a_0 + a_1\alpha + \cdots + (a_{n-1} + a_n\alpha)\alpha^{n-1}$

and

$\beta^{-1} = b_0 + b_1\alpha + \cdots + (b_{m-1} + b_m\alpha)\alpha^{m-1}$.

This contradicts the minimality of n and m. So we get the required result. $\square$

Next we will consider the case I_α is a prime ideal and $J_\alpha = R$.

Proposition 6. *Let R be a Noetherian domain with the quotient field K. Let α be an element of K which is super-primitive over R. Set $A = R[\alpha]$ and assume that I_α is a prime ideal of R, say, p and $J_\alpha = R$. Then:*

(1) *If there exists an element a of R such that $\mathrm{rad}(aR) = p$, then $A = R[1/a]$.*

(2) *If $\mathrm{rad}(aR) \neq p$ for every element a of R, then $\mathrm{U}(A) \cap R = \mathrm{U}(R)$.*

Proof. (1) By the equality (*) in Proposition 1 we have $aA = A$ because rad $(aR) \supset I_\alpha = p$ and $J_\alpha = R$. Hence $R[1/a] \subset A$.

Let β be an element of A. Then by the assumptions, we see that a is in rad $(aR) = p = I_\alpha$. This shows that $a\alpha$ is in R. Hence there exists a natural number r such that $a^r\beta$ is in R. This implies that β is in $R[1/a]$, and so $R[1/a] \subset A$.

(2) If rad $(aR) \supset p$, then rad $(aR) = p$ by $\mathrm{ht}(aR) = 1$. This conradicts the assumption of (2). Therefore $\mathrm{U}(A) \cap R = \mathrm{U}(R)$ by Remark 4. $\square$

Let A/R be the extension of integral domains and Φ the group homomorphism of the Picard groups $\mathrm{Pic}(R)$ into $\mathrm{Pic}(A)$ defined by $\Phi(\overline{I}) = \overline{IA}$ where I is an invertible ideal of R and $^-$ denotes the residue class modulo non-zero principal ideals.

We can now calculate the kernel of Φ. If an element α is in K and $J_\alpha = R$, then $R = J_\alpha = I_\alpha(1, \alpha)$ and so I_α is an invertible ideal of R.

Theorem 7. *Let R be a Noetherian domain with the quotient field K. Let α be an element of K such that α is anti-integral over R. Set $A = R[\alpha]$ and let Φ be the group homomorphism of $\mathrm{Pic}(R)$ into $\mathrm{Pic}(A)$. Assume that I_α is a prime ideal, say, p and $J_\alpha = R$. Then $\mathrm{Ker}(\Phi) = <\overline{p}> = \mathbf{Z}/t\mathbf{Z}$ where $<\overline{p}>$ denotes* *the subgroup of* $\mathrm{Ker}(\Phi)$ *generated by the residue class of an invertible ideal* p, $t = \min\{i; p^i$ *is principal* $\}$ *if there exists a natural number* i *such that* p^i *is principal and* $t = 0$ *if there is no such* i.

Moreover, the homomorphism Φ *is injective if and only if* p *is principal.*

Proof. Let F be an integral invertible ideal of R such that FA is principal and set $FA = f(\alpha)A$ with $f(X)$ in $R[X]$. We will divide into two cases to prove the assertion.

Case (1): $f(\alpha)$ is in R, say, $f(\alpha) = a$.

Set $H = a^{-1}F$. Then $HA = (a^{-1}F)A = a^{-1}aA = A$. Let β be an element of A. Then there exists a natural number r such that $p^r\beta \subset R$ because $I_\alpha = p$ and $A = R[\alpha]$ by the assumptions. Since $HA = A$, we have $H \subset A$. Hence H is a finite R-module. Therefore $p^nH \subset R$ for some natural number n. We will take n as the smallest one such that $p^nH \subset R$. Assume that $p^nH \neq R$. Then $pA = A$ because $R = J_\alpha = I_\alpha(1, \alpha) = p(1, \alpha)$. Hence $(p^nH)A = p^nA = A$ because $HA = A$. By the equality (*) of Proposition 1 we get rad $(p^nH) \supset I_\alpha = p$. Since p and H are invertible of R, so is p^nH. Furthermore $\mathrm{ht}(p^nH) = 1$ because $p^nH \subset R$. Hence rad $(p^nH) = p$. This implies that $p^nH \subset p$. Therefore $p^{n-1}H = p^{-1}p^nH \subset p^{-1}p = R$. This contradicts the minimality of n. Hence $p^nH = R$. This shows that $H = p^{-n}$, and hence $\mathrm{Ker}(\Phi) =< \overline{p} >$.

Case (2): $f(\alpha)$ is not in R.

Set $n = \deg((f(X))$ and we will take n as the smallest one such that $FA = f(\alpha)A$. Note that $n > 0$. Set $H = \{b \in R;\ bf(\alpha) \in F\}$, then H is an ideal of R. We will prove $F = f(\alpha)H$. It is clear from the definition of H that $F \supset f(\alpha)H$. Let a be an arbitrary element of F. Then we can write $a = f(\alpha)g(\alpha)$ for some $g(X)$ in $R[X]$. Set $m = \deg(g(X))$. If $m = 0$, then $g(\alpha)$ is in H and a is in $f(\alpha)H$. So we may only consider the case $m > 0$. We will take m as small as possible such that $a = f(\alpha)g(\alpha)$. Since α is anti-integral over R, we get $f(X)g(X) - a \in \mathrm{Ker}(\pi) = I_\alpha(X - \alpha)R[X]$. Let c be the leading coefficient of $f(X)$ and y the leading coefficient of $g(X)$. Then $cy \in I_\alpha = p$. Since p is a prime ideal, we know that c *is in* p or y is in p. Hence $c\alpha$ is in R or $y\alpha$ is in R. This contradicts the minimality of n and m. So we get $F = f(\alpha)H$. Note that $H \subset R$. Therefore $HA = (f(\alpha)^{-1}F)A = A$. Assume that $H \neq R$. By the equality (*) of Proposition 1 we have rad (H) $\supset p$. Since H is an invertible ideal, we know that $\mathrm{ht}(H) = 1$. Hence rad $(H) = p$, and $H \subset p$. So $p^{-1}H \subset R$ and $p^{-1}HA = A$. Continuing the argument above for $p^{-1}H$, we can reach the result $p^n = H$ for some non-zero integer n. □

Though the following result is known, we will give a proof for the reference sake (cf. [5, Proposition 1]). Note that a birational extension A/R is flat if and only if $A_P = R_p$ for every prime ideal P of A where $p = P \cap R$.

Proposition 8. *Let* R *be an integral domain with the quotient field* K *and* $\alpha_1, \cdots, \alpha_n$ *elements of* K. *Set* $A = R[\alpha_1, \cdots, \alpha_n]$. *Then* A/R *is a flat extension if and only if* $(I_{\alpha_1} \cap \cdots \cap I_{\alpha_n})A = A$.

Proof. We shall prove the " if" part. Let P be a prime ideal of A and set $p = P \cap R$. If $p \supset I_{\alpha_1} \cap \cdots \cap I_{\alpha_n}$, then $P \supset (I_{\alpha_1} \cap \cdots \cap I_{\alpha_n})A = A$. This is absurd. Hence $p \not\supset I_{\alpha_1} \cap \cdots \cap I_{\alpha_n}$. This implies that $\alpha_1, \cdots, \alpha_n$ are in R_p. Therefore $A \subset R_p$. So we get $A_P = R_p$. Hence the extension A/R is flat.

We shall show the " only if" part. Suppose that $(I_{\alpha_1} \cap \cdots \cap I_{\alpha_n})A = A$. Then there exists a prime ideal P of A such that $P \supset (I_{\alpha_1} \cap \cdots \cap I_{\alpha_n})A$. Set $p = P \cap R$. Then $p \supset I_{\alpha_1} \cap \cdots \cap I_{\alpha_n}$. By the assumptions the extension A/R is flat and birational. This shows that $\alpha_1, \cdots, \alpha_n$ are in $A_P = R_p$. This contradicts the fact $p \supset I_{\alpha_1} \cap \cdots \cap I_{\alpha_n}$. □

Let R be a Noetherian domain with the quotient field K and F a fractional ideal of R. We will define $F^{-1} = R :_K F$.

Theorem 9. *Let R be a Noetherian domain with the quotient field K. Let $\alpha_1, \cdots, \alpha_n$ be elements of K which are anti-integral over R. Set $A = R[\alpha_1, \cdots, \alpha_n]$ and assume that the extension A/R is flat.*

Then the following hold:

(1) *Let I be an invertible ideal of A and set $F = I \cap R$. Then F is a divisorial ideal of R and satisfies* rad $(FF^{-1}) \supset I_{\alpha_1} \cap \cdots \cap I_{\alpha_n}, I = FA$.

(2) *Conversely, let F be an ideal of R such that* rad $(FF^{-1}) \supset I_{\alpha_1} \cap \cdots \cap I_{\alpha_n}$. *Then FA is an invertible ideal of A.*

Proof. (1) First we will prove that F is a divisorial ideal of R. Let $I = Q_1 \cap \cdots \cap Q_m$ be the irredundant primary decomposition and set $P_i =$ rad (Q_i) and $p_i = P_i \cap R$ for $1 \leq i \leq m$. Since I is an invertible ideal of A, we know that P_i is in $\mathrm{Dp}_1(A)$ for $1 \leq i \leq m$. Furthermore, p_i is in $\mathrm{Dp}_1(R)$ for $1 \leq i \leq m$ because the extension A/R is flat. Note that $F = (Q_i \cap R) \cap \cdots \cap (Q_m \cap R)$. We have $\mathrm{Ass}_R(R/F) \subset \{p_1, \cdots, p_m\}$. By renumbering prime ideals $p_1, \cdots, p_m$, we may set $\mathrm{Ass}_R(R/F) = \{p_1, \cdots, p_t\}$ $(t \leq m)$. It suffices to prove that $((F)^{-1})^{-1}R_{p_j} = FR_{p_j}$ for $1 \leq j \leq t$ because $(F^{-1})^{-1} \supset F$. Since the extension A/R is flat and birational, we know that $A_{p_j} = R_{p_j}$. So write $IA_{P_j} = aA_{P_j}$ for some a in I. On the other ha! nd, $IA_{P_j} = Q_jA_{P_j} = (Q_j \cap R)R_{p_j} = FR_{p_j}$. Hence $FR_{p_j} = aA_{P_j} = aR_{p_j}$. Then $(F^{-1})^{-1}R_{p_j} = ((aR_{p_j})^{-1})^{-1}R_{p_j} = aR_{p_j} = FR_{p_j}$. Hence $(F^{-1})^{-1} = F$, and F is a divisorial ideal of R.

Next we shall show that $FA = I$. The inclusion $I \supset FA$ is obvious. Since the extension A/R is flat, we see that FA is also a divisorial ideal of A. In fact $F = I_{\beta_1} \cap \cdots \cap I_{\beta_r}$ for some $\beta_1, \cdots, \beta_r$ in K because F is a divisorial ideal of R. Hence $FA = (I_{\beta_1} \cap \cdots \cap I_{\beta_r})A = I_{\beta_1}A \cap \cdots \cap I_{\beta_r}A = I_{A\beta_1} \cap \cdots \cap I_{A\beta_r}$, and FA is also a divisorial ideal of A. Let P be a prime divisor of FA. Then P is in $\mathrm{Dp}_1(A)$. Set $p = P \cap R$, then $A_P = R_p$. Noting that $I \cap R = F$, we obtain $FA = \cap\ (FA)_P = \cap FA_P = \cap FR_p = \cap(IR_p \cap R_p) = \cap(IA_P \cap A_P) = \cap IA_P \supset I$ where the intersection runs through all prime divisors P's of FA. So we get $FA = I$.

We shall prove the assertion rad $(FF^{-1}) \supset I_{\alpha_1} \cap \cdots \cap I_{\alpha_n}$. Since $F^{-1} = R :_K F$ and the extension A/R is flat, we know that $F^{-1}A = A :_K FA = A :_K I = I^{-1}$. Hence $A = II^{-1} = (FA)(F^{-1}A) = FF^{-1}A$. It is known from the property of the flat extensions that the morphism $\mathrm{Spec}(A) \hookrightarrow \mathrm{Spec}(R)$ is an open immersion. Therefore $\mathrm{Spec}(R) - \mathrm{Spec}(A) = \mathrm{V}(I_{\alpha_1} \cap \cdots \cap I_{\alpha_n})$. Let p be a prime ideal of R such that rad$FF^{-1} \subset p$. Then $pA = A$ because $FF^{-1}A = A$. Hence $p \supset I_{\alpha_1} \cap \cdots I_{\alpha_n}$. This implies that rad $(FF^{-1}) \supset I_{\alpha_1} \cap \cdots \cap I_{\alpha_n}$.

(2) Since rad $(FF^{-1}) \supset I_{\alpha_1} \cap \cdots \cap I_{\alpha_n}$, we have $FF^{-1}A = A$ by Proposition 8. Therefore FA is an invertible ideal of A. $\square$

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