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Solving Differential Stress-Strain Equation of the General Linear Solid

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Abstract

Physical consideration of the general linear solid leads to the differential general linear equation (DGLE). The quasi-static case of creep gives a special solution of the DGLE. For any stress $\sigma \in C^{\infty}$, the general solution of the DGLE is derived by using the convolution.

1. Introduction

The anelastic relaxation in solids is defined as a relaxation of the strain after the applied stress. One of the important anelastic relaxations is the point-defect relaxation. If there exists a point defect in a crystal, it may lower the original crystalline symmetry of the perfect crystal. The symmetry around the defect is called the defect symmetry to distinguish it from the lattice symmetry of the perfect crystal. If the set of all the defect symmetries becomes a subgroup of the point group of all the lattice symmetries and smaller than the point group, then an anelastic relaxation occurs.

The elasticity is a property of solid following Hooke's law and represented by the Hookean spring. The linear viscosity is the property that the stress is proportional to the strain rate, whose property is modelled by the Newtonian dashpot.

The anelasticity involves at least a Hookean spring and a Newtonian dashpot. One of the simplest cases is a combination of the two models connected in parallel. This model is called the Voigt solid. More realistic solid is the standard linear solid composed of a Voigt solid and a Hookean spring in series. The Voigt and the standard linear solids are models of a single relaxation. There often appears a multiple relaxation due to several kinds of point defects in the crystal. Such multiple anelastic relaxation is observed as a superposition of creep curves in the quasi-static experiment and as a superposition of peaks of internal friction in the dynamic experiment. The general linear solid, hereafter abbreviated to GLS, is a model representing the multiple anelastic relaxation. The GLS of order n consists of n Voigt solids and a Hookean spring all in series.

The GLS is expressed in the form of differential stress-strain equation, here called a differential general linear equation abbreviated to DGLE. The DGLE has been considered from various viewpoints[1~22]. Physical experiments always require simplified experiments to obtain results without obscurity. In the study of anelastic relaxation in crystalline solids[23], there have so far been dealt with the quasi-static case of creep, elastic aftereffect or stress relaxation and the dynamic case of internal friction under the condition of the oscillatory harmonic stress. This paper considers the quasi-static case of creep from the physical point of view, and the general solution of the DGLE is mathematically derived. The dynamic case is discussed elsewhere[16].

2. Physico-Mathematical Solution in Quasi-Static Case of Creep

Let D_τ be defined as $D_\tau \overset{\text{def}}{=} d_t + \dfrac{I}{\tau}$ where $d_t = \dfrac{d}{dt}$, I the identity operator and τ the relaxation time. Then, $D_\tau : C^\infty(\mathbf{R}) \to C^\infty(\mathbf{R})$ is a linear differential operator associated with τ, here called a diffusion operator[14]. The strain $\varepsilon = \varepsilon(t)$ $(t \in \mathbf{R})$ is a C^∞ function. Let φ_τ be the solution of the initial value problem: $D_\tau \varepsilon = \dfrac{1}{\tau}$, $\varepsilon(0) = 0$. Uniqueness of the solution for the initial value problem asserts well-definedness of φ_τ. For simplicity, $D_{\tau_\sigma^{(i)}}$ and $\varphi_{\tau_\sigma^{(i)}}$ are replaced by D_i and φ_i, respectively. Similarly, ψ_i is introduced as the solution of $D_i \varepsilon = 0$, $\varepsilon(0) = 1$.

In the ith Voigt solid the Hookean spring gives $\varepsilon_i = \delta J_{\mathrm{R}}^{(i)} \sigma_1^{(i)}$ where $\delta J_{\mathrm{R}}^{(i)}$ is the relaxation of the compliance, $\sigma_1^{(i)}$ the stress component for the spring, and the Newtonian dashpot is specified by $\eta_i \dot{\varepsilon}_i = \sigma_2^{(i)}$ where η_i is the viscosity expressed as $\eta_i = \tau_\sigma^{(i)} / \delta J_{\mathrm{R}}^{(i)}$, $\sigma_2^{(i)}$ the stress component for the dashpot. The stress $\sigma = \sigma_1^{(i)} + \sigma_2^{(i)}$, then, leads to the differential stress-strain equation of the ith Voigt solid:

$$D_i \varepsilon_i = \frac{\delta J_{\mathrm{R}}^{(i)}}{\tau_\sigma^{(i)}} \sigma \, . \tag{1}$$

Since $D_i(\varphi_i \sigma) = \sigma D_i \varphi_i + \varphi_i D_i \sigma - \dfrac{\varphi_i \sigma}{\tau_\sigma^{(i)}} = \dfrac{\sigma}{\tau_\sigma^{(i)}} + \varphi_i \left(D_i \sigma - \dfrac{\sigma}{\tau_\sigma^{(i)}} \right)$,

Proposition 1 $D_i(\varphi_i\sigma)=\dfrac{\sigma}{\tau_\sigma^{(i)}} \Leftrightarrow D_i\sigma=\dfrac{\sigma}{\tau_\sigma^{(i)}} \Leftrightarrow d_t\sigma=0 \Leftrightarrow \sigma$: constant.

Then, for a constant σ, i.e., a creep under a constant stress σ, $\varepsilon_i=\delta J_{\mathrm{R}}^{(\mathrm{i})}\varphi_i\sigma$ is a special solution of (1).

Proposition 2 In the creep of the Voigt solid, the general solution of (1) is represented as

$$\varepsilon_i=\delta J_{\mathrm{R}}^{(\mathrm{i})}\varphi_i\sigma+c_i\psi_i \quad (c_i\text{: constant}).$$

Let ε_0 be the elastic strain given by $\varepsilon_0=J_{\mathrm{U}}\sigma$ with the unrelaxed compliance J_{U} for the Hookean spring attached to the n Voigt solids in the GLS. Hence,

Proposition 3 In the creep of the GLS, the total strain ε is expressed as

$$\varepsilon=\sum_{i=0}^{n}\varepsilon_i=\left(J_{\mathrm{U}}+\sum_{i=1}^{n}\delta J_{\mathrm{R}}^{(i)}\varphi_i\right)\sigma+\sum_{i=1}^{n}c_i\psi_i \quad (c_i\in\mathbf{R}) \ . \tag{2}$$

Let D_{ij} be a differential operator composed of D_i and D_j such that $D_{ij}=D_i\circ D_j$, corresponding to the two components of the Voigt solid connected in series with the ith Voigt solid followed by the jth. The inverted system of the two Voigt solids connected in the order of the jth followed by the ith, is completely of the same physical property as the normal system. Then, $D_{ij}=D_{ji}$, i.e., $D_i\circ D_j=D_j\circ D_i$.

In the general linear solid of n Voigt solids, let D be a differential operator specifying the GLS. Physically, the GLS doesnot depend on the order of connection of the Voigt solids. Thus,

$$D=\prod_{i=1}^{n}\circ D_{s(i)} \quad (\forall s\in\mathfrak{S}(n)), \tag{3}$$

where $\mathfrak{S}(n)$ is the symmetric group of order n. D is anticipated to be

$$D=P(d_t)=\sum_{i=0}^{n}a_i d_t^{\,n-i} \quad (a_0=1,\ a_i\in\mathbf{R}),$$

where a_i is a polynomial of $\alpha_1,\alpha_2,\cdots,\alpha_n$ with $\alpha_i=1/\tau_\sigma^{(i)}$ and written as

$$a_i=a_i(\alpha_1,\alpha_2,\cdots,\alpha_n).$$

By defining

$$sa_i\overset{\mathrm{def}}{=}a_i(\alpha_{s(1)},\alpha_{s(2)},\cdots,\alpha_{s(n)}),$$

eqn.(3) yields $sa_i=a_i\ (\forall s\in\mathfrak{S}(n))$. Then, a_i is represented in terms of the elementary symmetric functions of $\alpha_1,\alpha_2,\cdots,\alpha_n$ (the fundamental theorem on symmetric functions).

The simplest case is the case when each a_i is the elementary symmetric function of degree i. Then,

$$P(z)=\prod_{i=1}^{n}(z+\alpha_i)=\prod_{i=1}^{n}\left(z+\frac{1}{\tau_\sigma^{(i)}}\right), \text{ or } D=\prod_{i=1}^{n}\circ D_i=P(d_t)=\prod_{i=1}^{n}D_i \ .$$

Equation (2) is, thus, the solution of the DGLE of order n in the creep:

$$D\varepsilon=\prod_{i=1}^{n}\frac{1}{\tau_\sigma^{(i)}}\left(J_{\mathrm{U}}+\sum_{i=1}^{n}\delta J_{\mathrm{R}}^{(i)}\right)\sigma \ . \tag{4}$$

The strain ε of eqn.(2) represents a creep behaviour of the GLS under the constant applied stress σ.

3. Convolution

Definitions and propositions related to the convolution to solve the single high-order inhomogeneous constant-coefficient linear ordinary differential equation are summarized based on Ref. 24). New description is given with proof.

Definition 4 The convolution of f and g for $f,g\in C^0(\mathbf{R})$ is defined as

$$f*g(t)\overset{\text{def}}{=}\int_0^t f(t-s)g(s)ds \ .$$

Proposition 5

(1) $f*g=g*f$.

(2) $(f*g)*h=f*(g*h)$.

(3) $(cf)*g=f*(cg)=c(f*g) \quad (c\in\mathbf{R})$.

Definition 6

$$\delta_\lambda\overset{\text{def}}{=}d_t-\lambda I \ .$$

Definition 7 The convolution associated with λ $(\lambda\in\mathbf{R})$ is an operator, γ_λ, defined as

$$\gamma_\lambda(f)\overset{\text{def}}{=}e_\lambda*f \quad \left(f\in C^0(\mathbf{R})\right) .$$

Here, $e_\lambda\overset{\text{def}}{=}e^{\lambda t}$.

Proposition 8

(1) $\delta_\lambda\circ\gamma_\lambda=I$.

(2) $\gamma_\lambda\circ\delta_\lambda(f)=f-e_\lambda f(0) \quad \left(f\in C^0(\mathbf{R})\right)$.

Proposition 9 For $D=\prod_{i=1}^{n}\circ\delta_{\lambda_i}=\delta_{\lambda_1}\circ\delta_{\lambda_2}\circ\cdots\circ\delta_{\lambda_n}$,

$$D\circ\left(\prod_{i=1}^{n}\circ\gamma_{\lambda_i}\right)=I \ .$$

Proof. By Prop. 8 (1), and the commutativity of δ_{λ_i}'s or γ_{λ_i}'s,

$$\left(\prod_{i=1}^{n}\circ\delta_{\lambda_i}\right)\circ\left(\prod_{i=1}^{n}\circ\gamma_{\lambda_i}\right)=\prod_{i=1}^{n}\circ\left(\delta_{\lambda_i}\circ\gamma_{\lambda_i}\right)=I\ . \qquad \square$$

Proposition 10 Let $f\in C^0(\mathbf{R})$ and $D=\prod_{i=1}^{n}\circ\delta_{\lambda_i}$.

$$\varepsilon=\prod_{i=1}^{n}\circ\gamma_{\lambda_i}(f)=e_{\lambda_1}*e_{\lambda_2}*\cdots*e_{\lambda_n}*f \quad\Rightarrow\quad D\varepsilon=f.$$

Definition 11

$$e_{\lambda,m}\overset{\text{def}}{=}\frac{t^m}{m!}e^{\lambda t}\quad (m\geq 0)\ .$$

Especially, $e_{\lambda,0}=e_\lambda$.

Proposition 12 For $i,m\geq 0$,

$$\gamma_\lambda{}^i\left(e_{\lambda,m}\right)=e_{\lambda,m+i}\ \text{, i.e., } \overbrace{e_\lambda*e_\lambda*\cdots*e_\lambda}^{i}*e_{\lambda,m}=e_{\lambda,m+i}.$$

Especially, for $m=0$

$$\gamma_\lambda{}^i\left(e_\lambda\right)=e_{\lambda,i}\ \text{, i.e., } \overbrace{e_\lambda*e_\lambda*\cdots*e_\lambda}^{i}*e_\lambda=e_{\lambda,i}.$$

Proposition 13 For $\lambda\neq\mu$ and $i\geq 1$,

$$\gamma_\lambda{}^i\left(e_\mu\right)=\begin{cases}\dfrac{e_\lambda-e_\mu}{\lambda-\mu} & (i=1).\\[2ex] \dfrac{(-1)^{i-1}}{(\lambda-\mu)^i}\left(e_\lambda-e_\mu\right)+\displaystyle\sum_{j=1}^{i-1}\frac{1}{(\lambda-\mu)^j}e_{\lambda,i-j} & (i\geq 2).\end{cases}$$

Proof. Easy by mathematical induction on *i*. $\square$

Theorem 14 Let $P(z)=\prod_{i=1}^{n}(z-\lambda_i)$ with $\lambda_i\neq\lambda_j\ (i\neq j)$. A special solution of $P(d_t)\varepsilon=f$, denoted by ε^*, is

$$\varepsilon^*=\prod_{i=1}^{n}\circ\gamma_{\lambda_i}(f)=e_{\lambda_1}*e_{\lambda_2}*\cdots*e_{\lambda_n}*f\ ,$$

or

$$\varepsilon^*=\sum_{i=1}^{n}\frac{1}{P'(\lambda_i)}e_{\lambda_i}*f\ ,\quad P'(\lambda_i)=\prod_{\substack{j=1\\ j\neq i}}^{n}\left(\lambda_i-\lambda_j\right).$$

Proof. The first half follows from Prop. 10. The rest is derived from

$$\frac{1}{P(z)}=\sum_{i=1}^{n}\frac{1}{P'(\lambda_i)}\frac{1}{z-\lambda_i}\ . \qquad \square$$

Theorem 15 Let $P(z)=\prod_{i=1}^{r}(z-\lambda_i)^{m_i}$ with $\lambda_i\neq\lambda_j\ (i\neq j)$ and $\sum_{i=1}^{r}m_i=n$. Then,

$$\frac{1}{P(z)}=\sum_{i=1}^{r}\sum_{m=0}^{m_i-1}\frac{a_{i,m_i-m}}{(z-\lambda_i)^{m_i-m}} \ , \quad a_{i,m_i-m}=\frac{1}{m!}\left(\frac{\partial}{\partial\lambda_i}\right)^m\frac{1}{\prod_{\substack{j=1\\ j\neq i}}^{r}(\lambda_i-\lambda_j)^{m_j}} \ ,$$

and a special solution of $P(d_t)\varepsilon=f$, ε^*, is expressed as

$$\varepsilon^*=\sum_{i=1}^{r}\sum_{m=0}^{m_i-1}a_{i,m_i-m}e_{\lambda_i,m_i-m-1}*f=\sum_{i=1}^{r}\sum_{m=0}^{m_i-1}a_{i,m+1}e_{\lambda_i,m}*f \ .$$

The general solution ε is given by

$$\varepsilon=\varepsilon^*+\sum_{i=1}^{r}\sum_{j=0}^{m_i-1}c_{i,j}e_{\lambda_i,j} \quad (c_{i,j}\in\mathbf{R}) \ .$$

Proof. For $D=\delta_\lambda{}^m$, $D\circ\gamma_\lambda{}^m(f)=f$. Since $\gamma_\lambda{}^m(f)=\gamma_\lambda{}^{m-1}(e_\lambda)*f=e_{\lambda,m-1}*f$, $\varepsilon^*=e_{\lambda,m-1}*f$ is a special solution of $D\varepsilon=f$. Then, the theorem is readily verified. □

4. General Solution of DGLE

Let

$$Q(z)=\sum_{i=0}^{n}b_{n-i}z^i \quad (b_i\in\mathbf{R}, b_n\neq 0) \ .$$

Derive a special solution of the DGLE: $P(d_t)\varepsilon=Q(d_t)\sigma$ for $\sigma\in C^\infty(\mathbf{R})$.

Proposition 16

(1) $\quad e_\lambda*d_t\sigma=d_te_\lambda*\sigma+(\sigma-e_\lambda\sigma(0))=\lambda e_\lambda*\sigma+(\sigma-e_\lambda\sigma(0))$.

(2) $\quad d_t(e_\lambda*\sigma)=\lambda e_\lambda*\sigma+\sigma=d_te_\lambda*\sigma+\sigma=e_\lambda*d_t\sigma+e_\lambda\sigma(0)$.

Proposition 16 is proved by integration by parts, and the following corollary is immediately obtained from Prop. 16 (1).

Corollary 17

$$e_\lambda*d_t\sigma=d_te_\lambda*\sigma \quad\Leftrightarrow\quad \sigma=\sigma(0)e_\lambda \ .$$

Mathematical induction verifies Prop.18 by using $e_\lambda*d_t{}^j\sigma=e_\lambda*d_t(d_t{}^{j-1}\sigma)$ and Prop.16(1).

Proposition 18 For $j\geq 1$,

$$e_\lambda*d_t{}^j\sigma=\lambda^je_\lambda*\sigma+\sum_{l=0}^{j-1}\lambda^{j-1-l}d_t{}^l\sigma-e_\lambda\sum_{l=0}^{j-1}\lambda^{j-1-l}d_t{}^l\sigma(0)$$

$$=\lambda^je_\lambda*\sigma+\sum_{l=0}^{j-1}\lambda^{j-1-l}e^{\lambda(t-s)}d_s{}^l\sigma\Big|_{s=0}^{s=t} \ .$$

Theorem 19 In case $P(z)=\prod_{i=1}^{n}(z-\lambda_i)$ $\left(\lambda_i \neq \lambda_j\ (i \neq j)\right)$, a special solution of the DGLE is

$$\varepsilon^* = \sum_{i=1}^{n} \frac{1}{P'(\lambda_i)}\left[b_n e_{\lambda_i} * \sigma + \sum_{j=1}^{n} b_{n-j}\left\{ \lambda_i^{\,j} e_{\lambda_i} * \sigma + \sum_{l=0}^{j-1} \lambda_i^{\,j-1-l} e^{\lambda_i (t-s)} d_s^{\,l} \sigma \Big|_{s=0}^{s=t} \right\}\right],$$

and the general solution is given by

$$\varepsilon = \varepsilon^* + \sum_{i=1}^{n} c_i e_{\lambda_i} \quad (c_i \in \mathbf{R}).$$

Proof is obvious by Prop. 18.

Lemma 20

$$\gamma_\lambda(-\lambda) = \varphi_\tau \quad \left(\tau = -\frac{1}{\lambda}\right).$$

Proof.

$$e_\lambda * (1) = \int_0^t e^{\lambda s} ds = -\frac{1}{\lambda}\left(1 - e^{\lambda t}\right) = -\frac{1}{\lambda}\varphi_\tau .$$ □

In particular, for a constant σ the following theorem is deduced.

Theorem 21 In case $P(z)=\prod_{i=1}^{n}(z-\lambda_i)$ $\left(\lambda_i \neq \lambda_j\ (i \neq j)\right)$ and σ is constant, the general solution of the DGLE is expressed in terms of φ_i and ψ_i as

$$\varepsilon = \left(J_{\mathrm{U}} + \sum_{i=1}^{n} \delta J_{\mathrm{R}}^{(i)} \varphi_i \right)\sigma + \sum_{i=1}^{n} c_i \psi_i \quad (c_i \in \mathbf{R}).$$

Proof. If σ is constant, the DGLE becomes eqn. (4):

$$\left(\prod_{i=1}^{n} \circ \delta_{\lambda_i} \right)\varepsilon = \left(\prod_{i=1}^{n} (-\lambda_i) \right)\left(J_{\mathrm{U}} + \sum_{i=1}^{n} \delta J_{\mathrm{R}}^{(i)} \right)\sigma .$$

with $D = \prod_{i=1}^{n} \circ \delta_{\lambda_i}$. From $\delta_{\lambda_i}(1) = -\lambda_i$, $D(1) = \prod_{i=1}^{n}(-\lambda_i)$. By Lemma 20 and Prop. 8 (1),

$$D\varphi_i = D\left(\gamma_{\lambda_i}(-\lambda_i)\right) = (-\lambda_i) \prod_{j=1, j \neq i}^{n} \circ \delta_{\lambda_j}(1) = \prod_{i=1}^{n}(-\lambda_i) .$$

Thus,

$$\varepsilon^* = J_{\mathrm{U}}\sigma + \sum_{i=1}^{n} \delta J_{\mathrm{R}}^{(i)} \sigma \varphi_i$$

is a special solution of the DGLE for a constant σ. □

Now, the discussion is proceeded to the general case of $P(z)=\prod_{i=1}^{r}(z-\lambda_i)^{m_i}$ $(m_i \geq 1)$ with $\sum_{i=1}^{r} m_i = n$.

Proposition 22 For $j \geq 1$,

$$e_{\lambda,j} * d_t\sigma = \left(\lambda e_{\lambda,j} + e_{\lambda,j-1}\right) * \sigma - e_{\lambda,j}\sigma(0) .$$

Proof.

$$\left(e_{\lambda,j} * d_t\sigma\right)(t) = \int_0^t \frac{(t-s)^j}{j!} e^{\lambda(t-s)} \left(d_s\sigma(s)\right) ds$$

$$= \frac{(t-s)^j}{j!} e^{\lambda(t-s)} \sigma(s) \Bigg|_{s=0}^{s=t} + \int_0^t \left\{ \frac{(t-s)^{j-1}}{(j-1)!} e^{\lambda(t-s)} + \frac{(t-s)^j}{j!} \lambda e^{\lambda(t-s)} \right\} \sigma(s) ds$$

$$= -e_{\lambda,j}\sigma(0) + \left(e_{\lambda,j-1} + \lambda e_{\lambda,j}\right) * \sigma . \qquad \square$$

Mathematical induction on k for $e_{\lambda,j} * d_t{}^k\sigma$ leads to the following theorem with care of the largeness of k relative to j.

Theorem 23 For $k \geq 1$,

$$e_{\lambda,j} * d_t{}^k\sigma = \begin{cases} \sum_{l=0}^{k} \binom{k}{l} \lambda^{k-l} e_{\lambda,j-l} * \sigma - \sum_{l=0}^{k-1} e_{\lambda,j-l} \sum_{s=0}^{k-1-l} \binom{k-1-s}{l} \lambda^{k-1-l-s} d_t{}^s\sigma(0) & (1 \leq k \leq j) \\ \sum_{l=0}^{j} \binom{k}{l} \lambda^{k-l} e_{\lambda,j-l} * \sigma - \sum_{l=0}^{j} e_{\lambda,j-l} \sum_{s=0}^{k-1-l} \binom{k-1-s}{l} \lambda^{k-1-l-s} d_t{}^s\sigma(0) + \sum_{l=0}^{k-1-j} \binom{k-1-l}{j} \lambda^{k-1-j-l} d_t{}^l\sigma & (j < k) \end{cases}$$

$$= \sum_{l=0}^{\min\{j,k\}} \binom{k}{l} \lambda^{k-l} e_{\lambda,j-l} * \sigma - \sum_{l=0}^{\min\{j,k-1\}} e_{\lambda,j-l} \sum_{s=0}^{k-1-l} \binom{k-1-s}{l} \lambda^{k-1-l-s} d_t{}^s\sigma(0) + \sum_{l=0}^{k-1-j} \binom{k-1-l}{j} \lambda^{k-1-j-l} d_t{}^l\sigma ,$$

where the last summation $\sum_{l=0}^{k-1-j} \binom{k-1-l}{j} \lambda^{k-1-j-l} d_t{}^l\sigma$ is defined to be 0 for $k \leq j$, or $k-1<j$.

The general solution of the DGLE is, therefore, readily attained.

Theorem 24 In case $P(z)=\prod_{i=1}^{r}(z-\lambda_i)^{m_i}$,

$$\varepsilon = \sum_{i=1}^{r} \sum_{j=0}^{m_i-1} \sum_{k=0}^{n} a_{i,j+1} b_{n-k} e_{\lambda_i,j} * d_t{}^k\sigma + \sum_{i=1}^{r} \sum_{j=0}^{m_i-1} c_{i,j} e_{\lambda_i,j} \quad \left(c_{i,j} \in \mathbf{R}\right) .$$

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