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Boundary values of the Eisenstein series on $SL(2,\mathbf{Z})\backslash SL(2,\mathbf{R})$

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We study the boundary values of the real analytic Eisenstein series on the upper half plane and give an alternative proof of the functional equation for the Eisenstein series.

Introduction

It is known that a real analytic automorphic form on a semisimple Lie group is determined by its boundary values, which are distribution valued sections of principle series representations.

In this paper we consider the real analytic Eisenstein series given by

$$E_{s,\ell}(z)=\frac{1}{2}\sum_{\substack{a,b\in\mathbf{Z}\\(a,b)=1}}\frac{i^{\ell}(\operatorname{Im}z)^{s}}{|az+b|^{2s-\ell}(az+b)^{\ell}},$$

where s is a complex parameter, $\ell\in 2\mathbf{Z}$, and $\operatorname{Im}z>0$. Since $E_{s,\ell}(z)$ is an eigenfunction of the Laplace-Beltrami operator on a homogeneous line bundle on the upper half plane, we can consider its boundary values on the real line. By a result of Oshima [7], $E_{s,\ell}(z)$ is completely determined by its boundary values, which are distributions on $\mathbf{R}\cup\{\infty\}$. The boundary value with respect to the characteristic exponent $s-1$ is a constant multiple of the distribution

$$\sum_{\substack{(a,b)=1\\a>0}}\frac{1}{a^{2s}}\delta_{\frac{b}{a}}+\delta_{\infty},$$

where $\delta_{\frac{b}{a}}$ and δ_{∞} denote the Dirac delta function supported at b/a and ∞ respectively.

The boundary values of $E_{s,\ell}(z)$ are distributions with period 1 on $\mathbf{R}$. We compute the Fourier series of the boundary values. As an application, we prove the well-known functional equation for $E_{s,\ell}(z)$:

$$\begin{aligned}&\pi^{-s}\Gamma(s)\zeta(2s)(s)_{|\ell|/2}E_{s,\ell}(z)\\&=\pi^{-1+s}\Gamma(1-s)\zeta(2-2s)(1-s)_{|\ell|/2}E_{1-s,\ell}(z),\end{aligned}$$

where ζ is the Riemann ζ function and $(a)_n$ is the shifted factorial.

1 Helgason's conjecture on the upper half plane

In this section we review on boundary value problems on the upper half plane following Oshima [7]. We also refer the reader to Helgason [1] and Schlichtkrull [10].

Let H be the upper half plane

$$H=\{z=x+iy\in\mathbf{C}\,;\,y>0\}.$$

There is a natural action of

$$G=SL(2,\mathbf{R})=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\,;\,ad-bc=1\right\}$$

on $\mathbf{C}\cup\{\infty\}$ by

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\cdot z=\frac{az+b}{cz+d}. \tag{1.1}$$

The orbits are H, $\mathbf{R}\cup\{\infty\}$, and the lower half plane. The isotropy subgroup of i (and of $-i$) is

$$K=SO(2)=\left\{k_\theta=\begin{pmatrix}\cos\theta&-\sin\theta\\ \sin\theta&\cos\theta\end{pmatrix}\,;\,\theta\in T\right\},$$

where $T=\mathbf{R}/2\pi\mathbf{Z}$. Thus $H\simeq G/K$.

We define subgroups of G by

$$\begin{aligned}A&=\left\{a_y=\begin{pmatrix}\sqrt{y}&0\\0&1/\sqrt{y}\end{pmatrix}\,;\,y>0\right\},\\N&=\left\{n_x=\begin{pmatrix}1&0\\x&1\end{pmatrix}\,;\,x\in\mathbf{R}\right\},\\\bar{N}&=\left\{\bar{n}_x=\begin{pmatrix}1&x\\0&1\end{pmatrix}\,;\,x\in\mathbf{R}\right\},\\M&=\{k_0,\,k_\theta\}.\end{aligned}$$

Then P is a minimal parabolic subgroup of G and $\mathbf{R} \cup \{0\} \simeq G/P$, where $P = MAN$.

Notice that the coordinate $z = x + iy$ corresponds the Iwasawa decomposition $G = \bar{N}AK$ by

$$\bar{n}_x a_y \cdot i = x + iy.$$

We will use

$$G \ni \bar{n}_x a_y k_\theta \mapsto (x + iy, \theta) \in H \times T \qquad (1.2)$$

as the coordinate system on G. According to the Iwasawa decomposition $G = KAN$, we define mappings $\kappa : G \to K$ and $H : G \to \mathbf{R}$ by $g \in \kappa(g) a_{\exp H(g)} N$ for $g \in G$.

Let $\mathcal{B}(G)$ denote the space of the hyperfunctions on G. The group G acts on $\mathcal{B}(G)$ from the left by $f^g(x) = f(g^{-1}x)$ for g, $x \in G$.

For $\ell \in \mathbf{Z}$ define one-dimensional representation of K by $\chi_\ell(k_\theta) = e^{i\ell\theta}$. Define the space of the hyperfunction valued section of the homogeneous line bundle on G/K associated with χ_ℓ,

$$\mathcal{B}(G/K;\chi_\ell) = \{f \in \mathcal{B}(G)\,; \\ f(gk) = \chi_\ell(k)^{-1} f(g),\ g \in G,\ k \in K\}.$$

Obviously $\mathcal{B}(G/K;\chi_\ell)$ is a G-invariant subspace of $\mathcal{B}(G)$. In terms of the coordinate (1.2), $f \in \mathcal{B}(G/K;\chi_\ell)$ satisfies

$$f(z,\theta) e^{i\ell\theta}(\theta) = f(z,0), \quad z \in H,\ \theta \in T.$$

On the other hand, define $f(z,\theta) = h(z) e^{i\ell\theta}$ for a hyperfunction $h(z)$ on H. Then $f \in \mathcal{B}(G/K;\chi_\ell)$. Thus we can identify $\mathcal{B}(G/K;\chi_\ell)$ with $\mathcal{B}(H)$. In this identifications the action of G is given by

$$h^g(z) = \left(\frac{cz+d}{|cz+d|}\right)^{-\ell} h(g^{-1} \cdot z),$$

where g and $g \cdot z$ are as in (1.1).

The Casimir operator on G is given by

$$\Omega = 4y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + 4y \frac{\partial^2}{\partial x \partial \theta}.$$

Thus the action of $\Omega/4$ on $\mathcal{B}(G/K;\chi_\ell)$ is given by

$$L_\ell = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - i\ell y \frac{\partial}{\partial x}.$$

Define the Poisson kernel by

$$P_{s,\ell}(z) = \left(\frac{y}{x^2+y^2}\right)^s \left(\frac{xi+y}{\sqrt{x^2+y^2}}\right)^\ell. \qquad (1.3)$$

It satisfies

$$L_\ell P_{s,\ell} = s(s-1) P_{s,\ell}. \qquad (1.4)$$

Let $\varepsilon = \pm 1$. We define the representations τ_ε of M by $\tau_{\pm 1}(k_0) = 1$ and $\tau_{-1}(k_\pi) = -1$. We define the space of hyperfunction valued sections of principal series representation

$$\mathcal{B}(G/P; L_{s,\varepsilon}) = \{f \in \mathcal{B}(G)\,; \\ f(g m a_y n_x) = f(g)\tau_\varepsilon(m) y^{-s+1},\ y > 0,\ x \in \mathbf{R}\},$$

Let $\ell \in \mathbf{Z}$ with $\varepsilon = (-1)^\ell$. For $f \in \mathcal{B}(G/P; L_{s,\varepsilon})$ we define its Poisson transform by

$$(\mathcal{P}_{s,\ell} f)(g) = \int_K f(gk) \chi_\ell(k) dk, \quad g \in G,$$

where dk denote the normalized Haar measure on K. We can write it as

$$(\mathcal{P}_{s,\ell} f)(g) = \int_K f(k) e^{sH(g^{-1}k)} \chi_\ell(\kappa(g^{-1}k)) dk. \qquad (1.5)$$

We have

$$e^{sH(g^{-1})} \chi_\ell(\kappa(g^{-1})) = P_{s,\ell}(z),$$

where $g \cdot i = z$. By (1.5), $\mathcal{P}_{s,\ell}$ can be considered to be a mapping from

$$\mathcal{B}(K/M, \tau_\varepsilon) = \{f \in \mathcal{B}(K)\,; \\ f(km) = \tau_\varepsilon(m) f(k),\ k \in K,\ m \in M\}$$

to $\mathcal{B}(G/K;\chi_\ell)$. For $f \in \mathcal{B}(K/M,\tau_\varepsilon)$, $\mathcal{P}_{s,\ell} f$ depend holomorphically on $s \in \mathbf{C}$.

Let $\mathcal{A}(G/K;\mathcal{M}_{s,\ell})$ denote the space of the functions $u \in \mathcal{B}(G/K;\chi_\ell)$ satisfying $L_\ell u = s(s-1)u$.

The following theorem is proved by Oshima [7]. (We will explain the boundary value maps later.)

Theorem 1.1 *Let $\ell \in \mathbf{Z}$ and $\varepsilon = (-1)^\ell$. If*

$$s \notin \left\{\frac{|\ell|}{2} - j,\ -\frac{1+(-1)^\ell}{4} - j\,;\ j = 0,1,2,\dots\right\},$$

then the Poisson transform $\mathcal{P}_{s,\ell}$ and the boundary value map $\beta_{1-s,\ell}$ give topological G-isomorphisms of $\mathcal{B}(G/P; L_{s,\varepsilon})$ with $\mathcal{A}(G/K;\mathcal{M}_{s,\ell})$.

Remark 1.2 We give some historical remarks. For $\ell = 0$, Helgason proved Theorem 1.1 and conjectured that it can be generalized to general Riemannian symmetric spaces of the noncompact type. Kashiwara et al. [4] proved Helgason's conjecture by employing the techniques of microlocal analysis to study the boundary values of the joint eigenfunctions of invariant differential operators. For general ℓ, Oshima [7] proved Theorem 1.1 and Shimeno [12] generalize the result to Hermitian symmetric spaces.

We review on the boundary value map following Kashiwara et al. [4] and Oshima [7].

The Laplace-Beltrami operator L_ℓ has regular singularities along $\mathbf{R}$ with the characteristic exponents s and $1-s$. Under condition $2s \notin \mathbf{Z}$ we can define the boundary value map

$$\beta_{1-s,\ell} : \mathcal{A}(G/K; \mathcal{M}_{s,\ell}) \to \mathcal{B}(G/P; L_{1-s,\varepsilon}),$$
$$\beta_{s,\ell} : \mathcal{A}(G/K; \mathcal{M}_{s,\ell}) \to \mathcal{B}(G/P; L_{s,\varepsilon}).$$

We can define $\beta_{1-s,\ell}$ alone under weaker condition $-2s \neq 1, 2, 3, \ldots$. Intuitive interpretation of the boundary values are some "limits" when $y \downarrow 0$. Let $u \in \mathcal{A}(G/K; \mathcal{M}_{s,\ell})$ and assume that $2s \notin \mathbf{Z}$. If $\beta_{1-s,\ell}u$ and $\beta_{s,\ell}u$ are both analytic on $\mathbf{R} \simeq \bar{N}P \subset G/P$, then we have an expansion

$$u(z) = a_0(x)y^{1-s} + a_1(x)y^{2-s} + \ldots + b_0(x)y^s + b_1(x)y^{s+1} + \cdots$$

and the boundary values are given by

$$(\beta_{1-s,\ell}u)(x) = a_0(x), \quad (\beta_{s,\ell}u)(x) = b_0(x).$$

Define the Harish-Chandra c-function,

$$c(s,\ell) = \frac{2^{1-2s}\Gamma(2s)}{\Gamma(s+\frac{\ell}{2})\Gamma(s-\frac{\ell}{2})}.$$

Important properties of the boundary value map are the following:

$$\beta_{1-s,\ell} \circ \mathcal{P}_{s,\ell} = c(s,\ell)\mathrm{id}, \tag{1.6}$$

$$\mathcal{P}_{s,\ell} \circ \beta_{1-s,\ell} = c(s,\ell)\mathrm{id}, \tag{1.7}$$

$$\beta_{1-s,\ell} \circ \mathcal{P}_{s,\ell} = A_{s,\varepsilon}. \tag{1.8}$$

Here $A_{s,\varepsilon}$ is the Knapp-Stein intertwining operator

$$(A_{s,\varepsilon}f)(g) = \frac{1}{\pi}\int_\infty^\infty f(gk_\pi \bar{n}_y)dy.$$

In the realization of the principle series as a function space on the real line $\mathbf{R} = \bar{N} \cdot i$,

$$(A_{s,\varepsilon}f)(x) = \frac{i^\ell}{\pi}\int_\infty^\infty \frac{f(y)}{|y-x|^{2s}}dy.$$

Remark 1.3 Schwartz's distributions constitute a subclass of Sato's hyperfunctions. Oshima [7] gave a characterization of an eigenfunction of L_ℓ to be the Poisson transform of a distribution (see also Oshima and Sekiguchi [9]). It follows that boundary values of an automorphic form are distribution valued sections of principal series representations.

2 Eisenstein series

Define $\Gamma = SL(2,\mathbf{Z})$ and $\Gamma_\infty = \Gamma \cap P$. Let $s \in \mathbf{C}$ and $\ell \in \mathbf{Z}$. Define

$$E_{s,\ell}(g) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} P_{s,\ell}(\gamma g). \tag{2.1}$$

It is called the real analytic Eisenstein series. We refer the reader to Ibukiyama (ed.) [2], Kubota [5] and Williams [13] for theories of the Eisenstein series.

If ℓ is an odd integer, then (2.1) is identically zero because $M \subset \Gamma$. Hereafter we assume $\ell \in 2\mathbf{Z}$. We can write $E_{s,\ell}$ as

$$E_{s,\ell}(z) = \frac{1}{2}\sum_{\substack{a,b \in \mathbf{Z} \\ (a,b)=1}} \frac{i^\ell y^s}{|az+b|^{2s-\ell}(az+b)^\ell}. \tag{2.2}$$

If $\mathrm{Re}\, s > 1$, then the series converges absolutely and uniformly on any compact subset of H. As a function of s, $E_{s,\ell}(z)$ is analytic for $\mathrm{Re}\, s > 1$. We denote its analytic continuation by the same notation. By (1.4), $E_{s,\ell} \in \mathcal{A}(G/K; \mathcal{M}_{s,\ell})$, hence we can consider its boundary values. Since $k_\pi \in SL(2,\mathbf{Z})$ gives the change of the coordinate $z \mapsto -1/z$ and $0 \mapsto \infty$, the boundary values are determined completely by their restriction to $\mathbf{R}$.

The following proposition is given by Oshima and Sekiguchi [8] for $\ell = 0$.

Proposition 2.1 *Assume $s - 1/2 \notin \mathbf{Z}$ and $\ell \in 2\mathbf{Z}$. Then*

$$(\beta_{1-s,\ell}E_{s,\ell})(x) = c(s,\ell)\sum_{\substack{(a,b)=1 \\ a>0}} \frac{1}{a^{2s}}\delta\left(x + \frac{b}{a}\right),$$

$$(\beta_{s,\ell}E_{s,\ell})(x) = \frac{1}{2}\sum_{\substack{a,\, b \in \mathbf{Z} \\ (a,b)=1}} \frac{i^\ell}{|ax+b|^{2s}}.$$

Proof. Notice that the Poisson kernel $P_{s,\ell}$ is the Poisson transform of the Dirac delta function

$$P_{s,\ell}(z) = \pi(\mathcal{P}_{s,\ell}\delta)(z).$$

The factor π appears because the invariant measure on $\mathbf{R} = \bar{N} \cdot 0$ is $\frac{1}{\pi}dx$. Let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{Z}).$$

Assume that $a \neq 0$. We have

$$P_{s,\ell}^{\gamma^{-1}}(z) = \frac{i^\ell y^s}{|az+b|^{2s-\ell}(az+b)^\ell} = \frac{1}{|a|^{2s}}P_{s,\ell}^{\bar{n}_{b/a}^{-1}}(z).$$

Thus

$$P_{s,\ell}^{\gamma^{-1}} = \frac{\pi}{|a|^{2s}} \mathcal{P}_{1-s,\ell} \delta_{-\frac{b}{a}}. \qquad (2.3)$$

By (1.6),

$$(\beta_{1-s,\ell} P_{s,\ell}^{\gamma^{-1}}) = \frac{\pi c(s,\ell)}{|a|^{2s}} \delta_{\frac{b}{a}}.$$

It follows from (2.3) that

$$\begin{aligned}(\beta_{s,\ell} P_{s,\ell}^{\gamma^{-1}})(x) &= \frac{\pi}{|a|^{2s}} \beta_{s,\ell} \circ \mathcal{P}_{s,\ell} \delta_{-\frac{b}{a}}(x) \\ &= \frac{\pi}{|a|^{2s}} A_{s,\ell} \delta_{-\frac{b}{a}}(x) \\ &= \frac{i^\ell}{|ax+b|^{2s}}. \qquad (2.4)\end{aligned}$$

If $a = 0$, then $b = \pm 1$ and

$$P_{s,\ell}^{\gamma^{-1}}(z) = i^\ell y^s, \qquad (2.5)$$

hence $\beta_{1-s,\ell} P_{s,\ell}^{\gamma^{-1}} = 0$ on $\mathbf{R}$ and

$$(\beta_{s,\ell} P_{s,\ell})(x) = 1.$$

Thus the formula for the boundary values follows. The assumption $\operatorname{Re} s > 1$ is removed by Theorem 2.3, where analytic continuation to the complex plane will be proved. □

Remark 2.2 On $\mathbf{R} \cup \{\infty\}$ we have

$$\beta_{1-s,\ell} E_{s,\ell} = c(s,\ell) f_s$$

with

$$f_s = \pi \sum_{\substack{(a,b)=1 \\ a>0}} \frac{1}{a^{2s}} \delta_{\frac{b}{a}} + \pi \delta_\infty,$$

where $\delta_{\frac{b}{a}}$ and δ_∞ denote the Dirac delta function supported at b/a and ∞ respectively.

It is more convenient to consider the following series (see Williams [13, Section 15.2]):

$$\begin{aligned}B_{s,\ell}(z) &= \zeta(2s) E_{s,\ell}(z) \\ &= \frac{1}{2} \sum_{\substack{a,b\in\mathbf{Z} \\ (a,b)\neq(0,0)}} \frac{i^\ell y^s}{|az+b|^{2s-\ell}(az+b)^\ell}, \qquad (2.6)\end{aligned}$$

where ζ is the Riemann zeta function

$$\zeta(w) = \sum_{n=1}^{\infty} \frac{1}{n^w}, \quad \operatorname{Re} w > 1.$$

Now we state the main result of this paper:

Theorem 2.3 *Assume $s - 1/2 \notin \mathbf{Z}$ and $\ell \in 2\mathbf{Z}$. Put*

$$\tilde{B}_{s,\ell}(z) = \pi^{-s} \Gamma(s) \zeta(2s) (s)_{|\ell|/2} E_{s,\ell}(z),$$

where $(s)_n$ denote the shifted factorial defined by

$$(s)_n = s(s+1)\cdots(s+n-1) \quad n > 0, \ (s)_0 = 1.$$

Then we have

$$\begin{aligned}(\beta_{1-s,\ell} \tilde{B}_{s,\ell})(x) = \frac{2^{1-2s}\pi^{1-s}\Gamma(2s)}{\Gamma(s-|\ell|/2)} \Big(\zeta(2s-1) + \\ \sum_{n\in\mathbf{Z}\setminus\{0\}} \sigma_{-2s+1}(n) e^{2\pi i n x} \Big), \qquad (2.7)\end{aligned}$$

where

$$\sigma_\nu(n) = \sum_{d>0,\, d|n} d^\nu$$

for a nonzero integer n and $\nu \in \mathbf{C}$. As a function of s, $\beta_{1-s,\ell} \tilde{B}_{s,\ell}$ remains unchanged under substitution $s \mapsto 1-s$ and continues to a meromorphic function on $\mathbf{C}$.

Proof. We claim that

$$\begin{aligned}(\beta_{1-s,\ell} B_{s,\ell})(x) &= \pi c(s,\ell) \big(\zeta(2s-1) \\ &\quad + \sum_{n\in\mathbf{Z}\setminus\{0\}} \sigma_{-2s+1}(n) e^{2\pi i n x}\big), \qquad (2.8)\\ (\beta_{s,\ell} B_{s,\ell})(x) &= i^\ell \pi^{2s-\frac{1}{2}} \frac{\Gamma\left(-s+\frac{1}{2}\right)}{\Gamma(s)} (\zeta(-2s+1) \\ &\quad + \sum_{n\in\mathbf{Z}\setminus\{0\}} \sigma_{2s-1}(n) e^{2\pi i n x}). \qquad (2.9)\end{aligned}$$

Equation (2.8) follows from the proof of Proposition 2.1 and the Fourier series expansion of the periodic distribution

$$\sum_{b=-\infty}^{\infty} \delta\left(x + \tfrac{b}{a}\right) = a \sum_{b=-\infty}^{\infty} e^{2\pi i a b x}.$$

It follows from the proof of Proposition 2.1,

$$(\beta_{s,\ell} B_{s,\ell})(x) = i^\ell \zeta(2s) + \sum_{\substack{b\in\mathbf{Z} \\ a>0}} \frac{i^\ell}{|ax+b|^{2s}}.$$

Since it is a distribution with period 1, it has a Fourier series expansion

$$(\beta_{s,\ell} B_{s,\ell})(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}. \qquad (2.10)$$

We have $c_0 = i^\ell \zeta(2s)$, since

$$\sum_{\substack{b\in\mathbf{Z}\\ a>0}} \frac{1}{|ax+b|^{2s}}$$

is the derivative of

$$\frac{1}{a(-2s+1)} \sum_{\substack{b\in\mathbf{Z}\\ a>0}} \frac{\operatorname{sgn}(ax+b)}{|ax+b|^{2s-1}}$$

in the sense of distributions. For $n \neq 0$,

$$c_n = i^\ell \sum_{a=1}^{\infty} \sum_{b=-\infty}^{\infty} \int_0^1 \frac{e^{-2\pi i n x}}{|ax+b|^{2s}} dx.$$

The b summation above is

$$\sum_{k=1}^{a} \int_{-\infty}^{\infty} \frac{e^{-2\pi\sqrt{-1}nx}}{|ax+k|^{2s}} dx$$
$$= a^{-2s} \int_{-\infty}^{\infty} \frac{e^{-2\pi\sqrt{-1}nv}}{|v|^{2s}} dv \sum_{k=1}^{a} e^{2\pi\sqrt{-1}nk/a}$$

and the sum over k is a or 0 according as $a|n$ or not. Moreover

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi\sqrt{-1}nv}}{|v|^{2s}} dv = \pi^{2s-\frac{1}{2}} \frac{\Gamma(-s+\frac{1}{2})}{\Gamma(s)} |n|^{2s-1}.$$

By writing $n = ab$ and using the functional equation for the Riemann zeta function

$$\zeta(2s) = \pi^{2s-\frac{1}{2}} \frac{\Gamma(-s+\frac{1}{2})}{\Gamma(s)} \zeta(1-2s), \qquad (2.11)$$

we have (2.9).

It follows from (2.8) and (2.9) that (2.7) holds and $(\beta_{s,\ell}\tilde{B}_{s,\ell})(x)$ is given by substitutions $s \mapsto 1-s$ in (2.7). A theorem of Schwartz [11, II Theorem 14] asserts that $\sum_n c_n e^{2\pi i n x}$ gives a distribution if and only if there exist constant C and α $|c_n| \leq C|n|^\alpha$ for for large $|n|$ (see also Helgason [1, Introduction Lemma 4.21, Lemma 4.23]). Thus the distribution in the large curl bracket of the right hand side of (2.7) defines a distribution for all $s \in \mathbf{C}$. This completes the proof of the theorem. □

Remark 2.4 In the proof of Theorem 2.3, we use the functional equation for the Riemann zeta fuction (2.11). Notice that (2.11) can be proved by using the Poisson summation formula (see Mordell [6]).

Equality of the right hand sides of (2.10) and (2.9) can be obtained formally by the Poisson summation formula.

Corollary 2.5 *As a function of s, $E_{s,\ell}$ continues to a meromorphic function on* $\mathbf{C}$ *and* $\pi^{-s}\Gamma(s)\zeta(2s)(s)_{|\ell|/2}E_{s,\ell}(z)$ *is invariant under* $s \mapsto 1-s$.

Proof. The corollary follows from (1.7), Theorem 1.1, and Theorem 2.3. □

Concluding remarks

1. Oshima and Sekiguchi [9] defined the Eisenstein series on a semisimple symmetric space of type K_ε and proved a functional equation. It may be of interest to write down explicitly the Eisenstein series and the functional equation of [9] for $SL(2,\mathbf{R})/SO_0(1,1)$.
2. In principle, the method of this article is applicable to discrete subgroups of $SL(2,\mathbf{R})$ other than $SL(2,\mathbf{Z})$.
3. Professor Yoshihiro Ishikawa informed the author of the work of Kato [3], where a boundary value of the theta series and its image under the Poisson transform are considered.

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