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On Obstructions of Anti-Integrallity and Super-Primitiveness

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Let R be a Noetherian domain and $R[X]$ a polynomial ring. Let α be an element of an algebraic field extension L of the quotient field K of R and let $\pi : R[X] \to R[\alpha]$ be the R-algebra homomorphism sending X to α. Let $\varphi_\alpha(X)$ be the monic minimal polynomial of α over K with $\deg \varphi_\alpha(X) = d$ and write

$$\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$$

Then η_i $(1 \le i \le d)$ are uniquely determined by α. Let $I_{\eta_i} := R :_R \eta_i = \{ a \in R \mid a\eta_i \in R \}$ and $I_{[\alpha]} := \bigcap_{i=1}^d I_{\eta_i}$. It is easy to see that $I_{[\alpha]} = R[X] :_R \varphi_\alpha(X)$. We say that α is an *anti-integral element* over R if $\mathrm{Ker}(\pi) = I_{[\alpha]}\varphi_\alpha(X)R[X]$. For $f(X) \in R[X]$, let $C(f(X))$ denote the ideal of R generated by the coefficients of $f(X)$. For an ideal J of $R[X]$, let $C(J)$ denote the ideal generated by the coefficients of the elements in J. If α is an anti-integral element, then $C(\mathrm{Ker}(\pi)) = C(I_{[\alpha]}\varphi_\alpha(X)R[X]) = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$. Put $J_{[\alpha]} := I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$. If $J_{[\alpha]} \not\subseteq p$ for all $p \in \mathrm{Dp}_1(R) := \{p \in \mathrm{Spec}(R) \mid \mathrm{depth}\, R_p = 1\}$, then α is called a *super-primitive* element over R. It is known that a super-primitive element is an anti-integral element ([OSY,(1.12)]). It is known that any algebraic element over a Krull domain R is anti-integral over R ([OSY,(1.13)]).

Our objective is to investigate when an element $\alpha \in L$ is anti-integral or super-primitive over R.

In this paper, we fix the following notation in addition to the definitions mentioned above unless otherwise specified :

Let R be a *Noetherian* domain with quotient field K. Let L be an algebraic field extension of K and let α be an element in L which is of degree d over K. Let $\varphi_\alpha(X) := X^d + \eta_1 X^{d-1} + \cdots + \eta_d$ denote the minimal polynomial of α over K (that is, $\eta_i \in K$).

Our general reference for unexplained technical terms is [M].

We begin with the following definitions :

Definition 1. For an ideal I, let $\mathcal{R}(I) := I :_K I$, which is an over-ring of R.

Definition 2. For $\alpha \in L$, let

$$R\langle\alpha\rangle := R[\alpha] \cap R[\alpha^{-1}]$$

and

$$D_{(\alpha)} := R \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1} \qquad \text{(as } R\text{-modules)},$$

where $\zeta_i := \alpha^i + \eta_1\alpha^{i-1} + \cdots + \eta_i$ for i $(1 \le i \le d-1)$.

Remark 1. $R\langle\alpha\rangle$, $\mathcal{R}(R)$ and $D_{(\alpha)}$ are integral over R. (*cf.* [OSY])

The proof of the following lemma is obtained from the proof of [OKY,Lemma 4] and that of [KY,Theorem 1] without the assumption that α is anti-integral over R.

Lemma 3. *Under the same situation as in Definition 2, $D_{(\alpha)}$ is a subring of $R\langle\alpha\rangle$ and they are birational i.e., $D_{(\alpha)}$ and $R\langle\alpha\rangle$ have the same quotient fields $K(\alpha)$.*

Proof. When $d = 1$, $D_{(\alpha)} = R$.

Assume that $d = 2$. Note $\zeta_1 = \alpha + \eta_1$ and hence $\zeta_1^2 = \zeta_1(\alpha+\eta_1) = \alpha\zeta_1 + \eta_1\zeta_1 = \zeta_2 - \eta_2 + \eta_1\zeta_1 = \eta_1\zeta_1 - \eta_2$ (here $\zeta_2 = 0$).

Assume that $d = 3$. Note $\zeta_1 = \alpha + \eta_1$, $\zeta_2 = \alpha\zeta_1 + \eta_2$ and $\zeta_3 = \alpha\zeta_2 + \eta_3$. Thus

$\zeta_1^2 = \zeta_1(\alpha + \eta_1) = \alpha\zeta_1 + \eta_1\zeta_1 = \zeta_2 - \eta_2 + \eta_1\zeta_1$,

$\zeta_2\zeta_1 = \zeta_2(\alpha + \eta_1) = \alpha\zeta_2 + \zeta_2\eta_1 = \zeta_3 - \eta_3 + \eta_1\zeta_2 = -\eta_3 + \eta_1\zeta_2$ and

$\zeta_2^2 = \zeta_2(\alpha\zeta_1 + \eta_2) = (\alpha\zeta_2)\zeta_1 + \zeta_2\eta_2 = (\zeta_3 - \eta_3)\zeta_1 + \zeta_2\eta_2 = -\eta_3\zeta_1 + \eta_2\zeta_2$

ause $\zeta_3 = 0$.

Assume that $d \ge 3$. Note first that $\eta_0 := 1$, $\zeta_0 := 1$, $\zeta_d = 0$ and $\zeta_{i+1} = \alpha\zeta_i + \eta_{i+1}$. Now we compute $\zeta_i\zeta_j$ as follows :

$$\begin{aligned}\zeta_i\zeta_j &= \zeta_i(\alpha\zeta_{j-1}+\eta_j)\\ &= \alpha\zeta_i\zeta_{j-1}+\eta_j\zeta_i\\ &= (\zeta_{i+1}-\eta_{i+1})\zeta_{j-1}+\eta_j\zeta_i\\ &= \zeta_{i+1}\zeta_{j-1}-\eta_{i+1}\zeta_{j-1}+\eta_j\zeta_i\\ &= \zeta_{i+2}\zeta_{i-2}-\eta_{i+2}\zeta_{j-2}+\eta_{j-1}\zeta_{i+1}-\eta_{i+1}\zeta_{j-1}+\eta_j\zeta_i\\ &= \zeta_{i+2}\zeta_{i-2}-(\eta_{i+2}\zeta_{j-2}+\eta_{i+1}\zeta_{j-1})+(\eta_j\zeta_i+\eta_{j-1}\zeta_{i+1})\\ &\cdots\cdots\cdots\\ &\cdots\cdots\cdots.\end{aligned}$$

(i) Repeat the above process, we have

$$\zeta_i\zeta_j = \zeta_{i+j}\zeta_0 - \sum_{t=1}^{j}\eta_{i+t}\zeta_{j-t} + \sum_{s=0}^{j-1}\eta_{j-s}\zeta_{i+s},$$

that is,

$$\zeta_i\zeta_j = \zeta_{i+j} - \sum_{t=1}^{j}\eta_{i+t}\zeta_{j-t} + \sum_{s=0}^{j-1}\eta_{j-s}\zeta_{i+s}.$$

(ii) Put $\ell := i+j-d$. Then $j \le i < d$ yields $\ell < d-1$ and $j-\ell \ge 1$. Thus continuing the above process yields

$$\zeta_i\zeta_j = \zeta_d\zeta_\ell - \sum_{t=1}^{j-\ell}\eta_{i+t}\zeta_{j-t} + \sum_{s=0}^{j-\ell-1}\eta_{j-s}\zeta_{i+s},$$

that is,

$$\zeta_i\zeta_j = -\sum_{t=1}^{d-i}\eta_{i+t}\zeta_{j-t} + \sum_{s=0}^{d-i-1}\eta_{j-s}\zeta_{i+s}.$$

Thus we have $I_{[\alpha]}\zeta_i I_{[\alpha]}\zeta_j = I_{[\alpha]}^2\zeta_i\zeta_j \in D_{(\alpha)}$. Hence for $\beta,\gamma \in D_{(\alpha)}$, we have $\beta+\gamma, \beta\gamma \in D_{(\alpha)}$, which shows that $D_{(\alpha)}$ is a subring of $R[\alpha]$.

Next, it is obvious that $D_{(\alpha)} \subseteq R\langle\alpha\rangle$. Take $\zeta_1 = \alpha+\eta_1 \in D_{(\alpha)}$. Then $\alpha = \zeta_1 - \eta_1 \in K(D_{(\alpha)})$. So we have $K(\alpha) = K(D_{(\alpha)}) \subseteq K(R\langle\alpha\rangle) = K(\alpha)$.

Definition 4. For $\alpha \in L$, definie :

$$\mathrm{Ant}(\alpha) := \{\ c \in R\setminus(0) \mid R\langle\alpha\rangle[1/c] = D_{(\alpha)}[1/c]\ \} \cup \{0\}$$

and

$$\mathrm{Sup}(\alpha) := \{\ c \in R\setminus(0) \mid \mathcal{R}(I_{[\alpha]})[1/c] = R[1/c]\ \} \cup \{0\}.$$

The set $\mathrm{Ant}(\alpha)$ is called an *obstruction of anti-integrality* of α over R and the set $\mathrm{Sup}(\alpha)$ is called an *obstruction of super-primitiveness* of α over R.

Theorem 5. *For $\alpha \in L$,*
(1) $\mathrm{Ant}(\alpha)$ *is an radical ideal of R which contains the ideal $I_{[\alpha]}$;*
(2) *for $p \in \mathrm{Spec}(R)$, $\mathrm{Ant}(\alpha) \not\subseteq p$ if and only if α is an anti-integral element over R_p ;*
(3) *the following conditions are equivalent :*
(3.i) *α is anti-integral over R,*
(3.ii) $\mathrm{Ant}(\alpha) = R$,
(3.iii) $\mathrm{Ant}(\alpha) \not\subseteq p$ *for all* $p \in \mathrm{Spec}(R)$,
(3.iv) $\mathrm{Ant}(\alpha) \not\subseteq p$ *for all* $p \in \mathrm{Dp}_1(R)$.

Proof. (1) First take $a, b \in \mathrm{Ant}(\alpha)$ and put $c := a + b$. Take $x \in R\langle\alpha\rangle$. Then $R\langle\alpha\rangle[1/a] = D_{(\alpha)}[1/a]$ and $R\langle\alpha\rangle[1/b] = D_{(\alpha)}[1/b]$. So there exists $n \in \mathbf{N}$ such that $a^n x \in D_{(\alpha)}$ and $b^n x \in D_{(\alpha)}$. Thus we have $c^{2n}x = (a+b)^{2n} = \sum_{i+j=2n} a^i b^j x = \sum_{i+j=2n, i\geq n} a^i b^j x + \sum_{i+j=2n, j\geq n} a^i b^j x \in D_{(\alpha)} + D_{(\alpha)} = D_{(\alpha)}$, which implies that $c^{2n}x \in D_{(\alpha)}$ and hence $x \in D_{(\alpha)}[1/c]$. Therefore $\mathrm{Ant}(\alpha)$ is an ideal of R. Secondly, since $R\langle\alpha\rangle[1/e] = R\langle\alpha\rangle[1/e^n]$ and $D_{(\alpha)}[1/e] = D_{(\alpha)}[1/e^n]$ for every $n \in \mathbf{N}$. So $\mathrm{Ant}(\alpha)$ is a radical ideal. Thirdly, take a non-zero $a \in I_{[\alpha]}$. Then $I_{[\alpha]}[1/a] = R[1/a]$. Hence $\varphi_\alpha(X) \in R[1/a][X]$, that is, $\eta_i \in R[1/a]$ for all i $(1 \leq i \leq d)$. Since $\deg(\varphi_\alpha(X)) = d$, α is integral over $R[1/a]$. Thus $\alpha \in R[1/a][\eta_1, \ldots, \eta_{d-1}] = D_{(\alpha)}[1/a]$. So we have $R\langle\alpha\rangle[1/a] \subseteq R[\alpha][1/a] \subseteq D_{(\alpha)}[1/a] \subseteq R\langle\alpha\rangle[1/a]$ and hence $R\langle\alpha\rangle[1/a] = D_{(\alpha)}[1/a]$, which shows that $a \in \mathrm{Ant}(\alpha)$. Therefore we have $I_{[\alpha]} \subseteq \mathrm{Ant}(\alpha)$.
(2) Let $p \in \mathrm{Spec}(R)$. Assume that $\mathrm{Ant}(\alpha) \not\subseteq p$. Then there exists $a \in \mathrm{Ant}(\alpha) \backslash p$ such that $R\langle\alpha\rangle[1/a] = D_{(\alpha)}[1/a]$. Then $R\langle\alpha\rangle_p = (D_{(\alpha)})_p$. Take $f(X) \in \mathrm{Ker}(\pi)$. with $n := \deg(f(X))(\geq d)$. Put $f(X) := a_0X^n + a_1X^{n-1} + \cdots + a_n \in R[X]$. Then $a_0\alpha^n + a_1\alpha^{n-1} + \cdots + a_n = 0$ and hence $a_0\alpha^{d-1} + a_1\alpha^{d-2} + \cdots + a_{d-1} = -(a_d(1/\alpha) + \cdots + a_n(1/\alpha)^{n-d}) \in R\langle\alpha\rangle_p = R_p\langle\alpha\rangle = (D_{(\alpha)})_p = R_p \oplus (I_{[\alpha]})_p\zeta_1 \oplus \cdots \oplus (I_{[\alpha]})_p\zeta_{d-1}$. Thus we have $a_0 \in (I_{[\alpha]})_p$. Consider $f_2(X) := f(X) - a_0\varphi_\alpha(X)X^{n-d}$. Then $\deg(f_2(X)) \leq n-1$ and $f_2(X) \in \mathrm{Ker}(\pi)$. By induction on n, we have $(\mathrm{Ker}(\pi))_p \subseteq I_{[\alpha]}\varphi_\alpha(X)R_p[X]$. Hence $\mathrm{Ker}(\pi)_p = I_{[\alpha]}\varphi_\alpha(X)R_p[X]$, which shows that α is anti-integral over R_p. Conversely, assume that α is anti-integral over R_p. Then $R_p\langle\alpha\rangle = (D_{(\alpha)})_p$ by [KY,Theorem 1]. Since $(D_{(\alpha)})_p$ is a finitely generated R_p-module, so is $R_p\langle\alpha\rangle$. Thus there exists $a \in R \setminus (0)$ such that $R\langle\alpha\rangle[1/a] = D_{(\alpha)}[1/a]$.
(3) (3.i) $\Leftrightarrow$ (3.ii) : Since α is anti-integral over R_p for every $p \in \mathrm{Spec}(R)$, we have $\mathrm{Ant}(\alpha)_p = R_p$ by (2). Thus $\mathrm{Ant}(\alpha) = R$. The converse implication is obvious. (3.iii) $\Leftrightarrow$ (3.ii) is obvious. (3.iv) $\Leftrightarrow$ (3.ii) follows from [SOY,Theorem 2]. Q.E.D.

The result (3) of Theorem 5 yields the following corollary, which is seen in [KY].

Corollary 5.1. *An element $\alpha \in L$ is anti-integral over R if and only if $R\langle\alpha\rangle = R \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}(= D_{(\alpha)})$.*

Remark 2. If R is normal then $R\langle\alpha\rangle = R \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}(= D_{(\alpha)})$. (*cf.* [OSY,(1.13)])

Theorem 6. *For $\alpha \in L$,*
(1) *$\mathrm{Sup}(\alpha)$ is an radical ideal of R which contains the ideal $I_{[\alpha]}$;*
(2) *for $p \in \mathrm{Spec}(R)$, $\mathrm{Sup}(\alpha) \not\subseteq p$ if and only if α is a super-primitive element over R_p ;*
(3) *The following conditions are equivalent :*
(3.i) *α is super-primitive over R,*
(3.ii) *$\mathrm{Sup}(\alpha) = R$,*
(3.iii) *$\mathrm{Sup}(\alpha) \not\subseteq p$ for all $p \in \mathrm{Spec}(R)$,*
(3.iv) *$\mathrm{Sup}(\alpha) \not\subseteq p$ for all $p \in \mathrm{Dp}_1(R)$.*

Proof. (1) Take $a \in I_{[\alpha]}$. Then $\mathcal{R}(I_{[\alpha]})[1/a] = \mathcal{R}(I_{[\alpha]}[1/a]) = R[1/a]$. So $a \in \mathrm{Sup}(\alpha)$ and hence $I_{[\alpha]} \subseteq \mathrm{Sup}(\alpha)$. In the same manners as in the proof of Theorem 5(1), we can show that $I_{[\alpha]}$ is a radical ideal of R.
(2) By using [OSY,Theorem 2.11] instead of [KY,Theorem 1], our conclusion is obtained in the same manners as in Theorem 5(2).
(3) The equivalences (3.i) $\Leftrightarrow$ (3.ii) $\Leftrightarrow$ (3.iii) follow from the same argument as in Theorem 5(3), and the equivalence (3.i) $\Leftrightarrow$ (3.iv) follows from [OSY,Theorem 1.12].

From Theorems 5 and 6, we have the following result.

Theorem 7. *Let $\alpha \in L$. Then*

$$\mathrm{Ass}_R(R/\mathrm{Ant}(\alpha)) \subseteq \mathrm{Dp}_1(R)$$

and

$$\mathrm{Ass}_R(R/\mathrm{Sup}(\alpha)) \subseteq \mathrm{Dp}_1(R).$$

Remark 3. $\mathrm{Sup}(\alpha) \subseteq \mathrm{Ant}(\alpha)$. In fact, if α is super-primitive over R then α is anti-integral over R by [OSY,(1.12)].

Put $I_n := \{ a \in R \mid a \in R$ is the leading coefficient of some polynomial $f(X) \in \mathrm{Ker}(\pi)$ of degree $n \}$. Note that $\mathrm{Ker}(\pi)$ contains no polynomial of degree less than $d = [K(\alpha) : K]$. If $f(X) \in \mathrm{Ker}(\pi)$, then $Xf(X) \in \mathrm{Ker}(\pi)$ and $\deg(Xf(X)) = \deg(f(X)) + 1$. Hence we have an ascending chain :

$$I_d \subseteq I_{d+1} \subseteq \cdots \subseteq I_n \subseteq \cdots.$$

Since R is a Noetherian doamin, the above chain stops, that is, $I_n = I_{n+1} = \cdots$ for some large n. Put $I_\infty := I_n$.

Proposition 8. *Under the above notation,*
(1) $I_d = I_{[\alpha]}$ *;*
(2) *for* $p \in \mathrm{Spec}(R)$, α *is anti-integral over* R_p *if and only if* $p \not\supseteq I_d :_R I_\infty$.

Proof. (1) Take $a \in I_{[\alpha]}$. Then there exists a polynomial $f(X) = aX^d + a_1X^{d-1} + \cdots + a_d \in \mathrm{Ker}(\pi)$. Since $F(\alpha) = 0$ and $\deg(f(X)) = d$, we have $f(X) = a\varphi_\alpha(X) \in R[X]$. Thus $a \in I_{[\alpha]}$. Conversely, let $a\varphi_\alpha(X) \in R[X]$. Then $a \in I_d$ and hence $I_{[\alpha]} \subseteq I_d$. Therefore we have $I_d = I_{[\alpha]}$.
(2) ($\Rightarrow$) : Assume that $a \in L$ is anti-integral over R_p. Then the kernel of $\pi_p := \pi \otimes_R R_p : R_p[X] \to R_p[\alpha]$ is generated by some polynomials of degree $d := [K(\alpha] K]$. Hence $(I_{[\alpha]})_p = (I\infty)_p$. Thus $p \not\supseteq I_d :_R I_\infty$.
($\Leftarrow$) : Assume that $I_d :_R I_\infty \not\subseteq p$ with $p \in \mathrm{Spec}(R)$. Since $(I_d)_p = (I_\infty)_p$, $\mathrm{Ker}(\pi_p)$ is generated by some polynomial of degree d. So α is anti-integral over R_p. [Indeed, if $(f_1(X), \ldots, f_n(X))R_p[X] = \mathrm{Ker}(\pi_p)$ with $\deg(f_i(X)) = d$ for all i, then $f_i(X) = a_i\varphi_i(X)$ with $a_i \in R$, where $\varphi_i(X)$ denotes the monic polynomial in $K[X]$. Since $\deg(\varphi_i(X)) = d$ and $\varphi_i(\alpha) = 0$, $\varphi_1(X) = \cdots = \varphi_n(X)$. Thus $\mathrm{Ker}(\pi_p) = (a_1, \ldots, a_n)\varphi_1(X)R_p[X]$. So we have $(a_1, \ldots, a_n)R_p = I_{[\alpha]}R_p$.]

Combining Theorem 5 and Proposition 8, we have the following result.

Corollary 8.1. $\mathrm{Ant}(\alpha) = \sqrt{I_d :_R I_\infty}$.

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