Poisson transforms for some principal series representations of $Sp(n, \mathbb{R})$

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1 Introduction

Let $G = Sp(n, \mathbb{R})$ $(n \geq 2)$, K a maximal compact subgroup of G, and let P_Θ be a parabolic subgroup of G with a Langlands decomposition $P_\Theta = M_\Theta A_\Theta N_\Theta^+$, where $M_\Theta \simeq \{\pm 1\} \times Sp(n-1, \mathbb{R})$. We consider an induced representation of G from P_Θ, which is induced from a holomorphic representation of M_Θ, a character of A_Θ, and the trivial representation of N_Θ^+. We consider the problem of characterizing the image of the Poisson transform from the principal series representation to a homogeneous line bundle over G/K. The main result (Theorem 3. 1) asserts that the Poisson transform is injective under certain conditions on parameter and the image is characterized by second-order differential equations, which are given by a K-covariant differential operator between homogeneous vector bundles over G/K. As a corollary we obtain a characterization of the images of degenerate series representations on G/P_Θ under the Poisson transform (Corollary 3. 2).

For the Furstenberg boundary of a Riemannian symmetric space and the Shilov boundary of Hermitian symmetric space of tube type, there are several studies on the Poisson transform[5,9,11]. We believe that it is of importance to construct differential equations that characterize the image of the Poisson transform explicitly for other boundary components of a symmetric space and this article gives a new example on this problem.

2 Notation and preliminary results

2.1 Notation

Let

$$G = Sp(n, \mathbb{R}) = \{g \in SL(2n, \mathbb{R}); {}^t gJg = J\},$$

where

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

and I_n is $n\times n$ identitiy matrix. The group $K=O(2n)\cap Sp(n,\mathbb{R})$ is a maximal compact subgroup of G, which is isomorphic to $U(n)$ by

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}\in K\mapsto A+\sqrt{-1}B\in U(n).$$

Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K respectively. Let θ denote the coresponding Cartan involution of G and $\mathfrak{g}$. We have a Cartan decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$, where $\mathfrak{p}$ is the -1-eigenspace of θ in $\mathfrak{g}$.

For $l\in\mathbb{Z}$ let τ_l denote the one-dimensional representation of $U(n)$ given by $\tau_l(x)=(\det x)^l\ (x\in U(n))$ and we denote corresponding representation of K and $\mathfrak{k}$ by the same notation.

Let E_{ij} denote the $n\times n$ matrix with (i,j)-entry 1 and all other entries being 0. We choose a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{u}(n)$ to be the set of diagonal matrices. We define $\varepsilon_i\in\sqrt{-1}\mathfrak{t}^*$ by $\varepsilon_i(E_{jj})=\delta_{ij}\ (1\le i,j\le n)$. Let Δ denote the root system of $(\mathfrak{g},\mathfrak{t})$ and Δ^+ be the positive system of Δ given by

$$\Delta^+=\{2\varepsilon_i,\ \varepsilon_j\pm\varepsilon_k\,;1\le i\le n,1\le j<k\le n\}.$$

For $\gamma\in\Delta$ let $\mathfrak{g}_\gamma\subset\mathfrak{g}\ =\mathfrak{g}\otimes\ \mathbb{C}$ denote the root space for γ. Let $\mathfrak{p}^\pm=\sum_{\gamma\in\Delta_n^+}\mathfrak{g}_{\pm\gamma}$, where Δ_n^+ is the set of non-compact positive roots.

We put

$$X_i=\begin{pmatrix} E_{ii} & 0 \\ 0 & -E_{ii} \end{pmatrix}\in\mathfrak{p}\quad(1\le i\le n)$$

and $\mathfrak{a}=\sum_{i=1}^n\mathbb{R}X_i$. Then $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$. We put $X_0=X_1+\cdots\cdots+X_n$. Let $e_i\ (1\le i\le n)$ be the linear form on $\mathfrak{a}$ given by $e_i(X_j)=\delta_{ij}$. Let Σ denote the restricted root system of the pair $(\mathfrak{g},\mathfrak{a})$ and Σ^+ be the positive system of Σ given by

$$\Sigma^+=\{2e_i,\ e_j\pm e_k\,;1\le i\le n,1\le j<k\le n\}.$$

For $\alpha\in\Sigma$ let $\mathfrak{g}^\alpha\subset\mathfrak{g}$ be the root space for α. For $\mathfrak{g}=\mathfrak{sp}(n,\mathbb{R})$ we have $\dim\mathfrak{g}^\alpha=1$ for all $\alpha\in\Sigma$. We put $\rho=\frac{1}{2}\sum_{\alpha\in\Sigma^+}\alpha$. For any $\lambda\in\mathfrak{a}_{\mathbb{C}}^*$ let A_λ be the element of $\mathfrak{a}_{\mathbb{C}}$ determined by $B(H,A_\lambda)=\lambda(H)$ for all $H\in\mathfrak{a}$, where B denotes the Killing form of $\mathfrak{g}_{\mathbb{C}}$. For $\lambda,\mu\in\mathfrak{a}^*$ we put $<\lambda,\mu>=B(A_\lambda,A_\mu)$. Since $\{e_1,\cdots\cdots,e_n\}$ forms a basis of $\mathfrak{a}^*$, any $\lambda\in\mathfrak{a}_{\mathbb{C}}^*$ can be written as $\lambda=\sum_{i=1}^n\lambda_ie_i\ (\lambda_i\in\mathbb{C})$. We identify $\mathfrak{a}_{\mathbb{C}}^*$ with $\mathbb{C}^n$ by $\lambda\mapsto(\lambda_1,\cdots\cdots,\lambda_n)$. In this identification we have $\rho=(n,n-1,\cdots\cdots,1)$.

Let A be the analytic subgroups of G corresponding to $\mathfrak{a}$. Let $\mathfrak{n}^+=\sum_{\alpha\in\Sigma^+}\mathfrak{g}^\alpha$ and $\mathfrak{n}^-=\theta(\mathfrak{n}^+)$. Let N^+ and N^- be the corresponding analytic subgroups of G. Let M be the centralizer of $\mathfrak{a}$ in K. The subgroup $P=MAN^+$ is a minimal parabolic subgroup of G.

Put $\alpha_i=e_i-e_{i+1}\ (1\le i\le n-1)$ and $\alpha_n=2e_n$. Then the set of simple roots is $\Psi=\{\alpha_1,\alpha_2,\cdots\cdots,\alpha_n\}$. Let $\{H_1,\cdots\cdots,H_n\}$ denote the basis of $\mathfrak{a}$ which is dual to Ψ. We consider subset

$$\Theta=\{\alpha_2,\cdots\cdots,\alpha_n\}$$

of Ψ and corresponding standard parabolic subgroup P_Θ of G with the Langlands decomposition $P_\Theta = M_\Theta A_\Theta N_\Theta^+$ such that $A_\Theta \subset A$. Then the Lie algebra $\mathfrak{a}_\Theta$ of A_Θ and its orthogonal complement $\mathfrak{a}(\Theta)$ in $\mathfrak{a}$ are given by

$$\mathfrak{a}_\Theta = \mathbb{R}X_1, \quad \mathfrak{a}(\Theta) = \sum_{i=2}^{n} \mathbb{R}X_i,$$

and $M_\Theta \simeq \{\pm 1\} \times Sp(n-1, \mathbb{R})$. Put $K_\Theta = M_\Theta \cap K$ and define a closed subgroup $B_\Theta = K_\Theta A_\Theta N_\Theta^+$ of G. Notice that the pairs $(K, K_\Theta) \simeq (U(n), \{\pm 1\} \times U(n-1))$ is a Gelfand pair.

2.2 Eigenspaces of invariant differential operators

We reviw the main result of Shimeno[9], which gives a characterization of the image of the Poisson transform.

For a real analytic manifold X we denote by $\mathcal{B}(X)$ the space of all hyperfunctions on X. Let $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and $l \in \mathbb{Z}$. We define

$$\mathcal{B}(G/P, L_{\lambda,l}) = \{f \in \mathcal{B}(G)\,;\, f(gman) = e^{(\lambda-\rho)(\log a)}\tau_l(m)^{-1}f(g) \\ g \in G,\, m \in M,\, a \in A,\, n \in N^+\}$$

and

$$\mathcal{B}(G/K\,;\,\tau_l) = \{u \in \mathcal{B}(G)\,;\, u(gk) = \tau_l(k)^{-1}u(g) \text{ for any } g \in G,\, k \in K\}.$$

For $f \in \mathcal{B}(G/P\,;\,L_{\lambda,l})$, we define the Poisson integral $\mathcal{P}_{\lambda,l}f$ by

$$\mathcal{P}_{\lambda,l}f(g) = \int_K f(gk)\tau_l(k)dk.$$

Here dk denotes the invariant measure on K with total measure 1.

Let $\mathbb{D}_l(G/K)$ denote the algebra of invariant differential operators on $\mathcal{B}(G/K\,;\,\tau_l)$ and $L_l \in \mathbb{D}_l(G/K)$ denote the Laplace-Beltrami operator acting on $\mathcal{B}(G/K\,;\,\tau_l)$. We have the Harish-Chandra isomorphism

$$\gamma_l : \mathbb{D}_l(G/K) \xrightarrow{\sim} S(\mathfrak{a}_\mathbb{C})^W,$$

where $S(\mathfrak{a}_\mathbb{C})^W$ denotes the set of W-invariant elements in the symmetric algebra $S(\mathfrak{a}_\mathbb{C})$. Let $\mathcal{A}(G/K, \mathcal{M}_{\lambda,l})$ denote the space of all real analytic functions in $\mathcal{B}(G/K, \tau_l)$ satisfying the system of differential equations,

$$\mathcal{M}_{\lambda,l} : Du = \gamma_l(D)(\lambda)u, \quad D \in \mathbb{D}_l(G/K). \tag{2.1}$$

We define

$$e_{\lambda,l}^{-1} = \prod_{1\le j<k\le n} \Gamma\Big(\frac{1}{2}(1+\lambda_j+\lambda_k)\Big)\Gamma\Big(\frac{1}{2}(1+\lambda_j-\lambda_k)\Big) \\ \times \prod_{1\le i\le n} \Gamma\Big(\frac{1}{2}(\lambda_i+1+l)\Big)\Gamma\Big(\frac{1}{2}(\lambda_i+1-l)\Big).$$

Theorem 2.1 (Shimeno[9]) *If $\lambda \in \mathfrak{a}_\mathbb{C}^*$ satisfies the condition*

$$-2\frac{<\lambda, \alpha>}{<\alpha, \alpha>} \notin \{1, 2, 3, \cdots\cdots\} \text{ for all } \alpha \in \Sigma^{+} \tag{2.2}$$

$$e_{\lambda,l} \neq 0, \tag{2.3}$$

then the Poisson transform $\mathcal{P}_{\lambda,l}$ *is a G-isomorphism of* $\mathcal{B}(G/P\,;L_{\lambda,l})$ *onto* $\mathcal{A}(G/K,\mathcal{M}_{\lambda,l})$.

Under the condition of the above theorem, the inverse of the Poisson transform is given by the boundary value map up to a non-zero constant multiple.

3 The Poisson transforms and the Hua equations

3.1 Poisson transform for principal series representations

Let $s \in \mathbb{C}$, $l \in \mathbb{Z}$, and $\varepsilon \in \{+1, -1\}$. We put

$$\lambda^{\Theta}_{s,l,\varepsilon} = (s, -\varepsilon l+n-1, -\varepsilon l+n-2, \cdots\cdots, -\varepsilon l+1) \in \mathfrak{a}^{*}_{\mathbf{C}},$$

and $\lambda^{\Theta}_{s} = \lambda^{\Theta}_{s,l,\varepsilon}$. Throughout this section we denote $\lambda^{\Theta}_{s,l,\varepsilon}$ by λ for simplicicty.

We define

$$\mathcal{B}(G/B_{\Theta}\,;L_{s,l}) = \{f \in \mathcal{B}(G)\,; f(gman) = \tau_l(m)^{-1}e^{(s-n)e_1(\log a)}f(g),$$
$$g \in G,\, m \in K_{\Theta},\, a \in A_{\Theta},\, n \in N^{+}_{\Theta}\}.$$

The algebra $\mathbb{D}_l(M_{\Theta}/K_{\Theta})$ of invariant differential operators on $\mathcal{B}(M_{\Theta}/K_{\Theta}, \tau_l)$ acts from the right on $\mathcal{B}(G/B_{\Theta}\,;L_{s,l})$. Let $\mathcal{B}(\Theta\,;s, l, \varepsilon)$ denote the subspace of $\mathcal{B}(G/B_{\Theta}\,;L_{s,l})$ consisting of solutions of the system

$$\mathcal{M}_{\lambda|\mathfrak{a}(\Theta),l} : Df = \chi_{\lambda|\mathfrak{a}(\Theta),l}(D)f, \quad D \in \mathbb{D}_l(M_{\Theta}/K_{\Theta}).$$

Let $\mathfrak{m}_{\Theta} = \mathfrak{k}_{\Theta}+\mathfrak{p}_{\Theta}$ be the Cartan decomposition corresponding to $\theta|\mathfrak{m}_{\Theta}$ and put $\mathfrak{p}^{\pm}_{\Theta} = \mathfrak{p}^{\pm} \cap \mathfrak{p}_{\Theta,\mathbf{C}}$. Each element of $\mathfrak{p}^{\varepsilon}_{\Theta}$ acts on $\mathcal{B}(\Theta\,;s, l, \varepsilon)$ from the right as a differential operator, where $\mathfrak{p}^{\varepsilon}_{\Theta}$ denotes $\mathfrak{p}^{\pm}_{\Theta}$ for $\varepsilon = \pm 1$. We define

$$\mathcal{B}(G/P_{\Theta}\,;s, l, \varepsilon) = \{f \in \mathcal{B}(\Theta\,;s, l, \varepsilon)\,; \mathfrak{p}^{\varepsilon}_{\Theta}f = 0\}.$$

We define

$$\mathcal{B}(G/P_{\Theta}\,;s) = \{f \in \mathcal{B}(G)\,; f(gman) = e^{(s-n)e_1(\log a)}f(g),$$
$$g \in G,\, m \in M_{\Theta},\, a \in A_{\Theta},\, n \in N^{+}_{\Theta}\},$$

which equals $\mathcal{B}(\Theta\,;s, 0, +1) \cap \mathcal{B}(\Theta; s, 0, -1)$.

We recall the definition of partial Poisson transforms after Shimeno[10]. For $f \in \mathcal{B}(G/P\,;L_{\lambda,l})$ we define

$$\mathcal{P}^{\Theta}_{s,l,\varepsilon}f(x) = \int_{K_{\Theta}} f(xk)\tau_l(k)dk. \tag{3.1}$$

The image of $\mathcal{P}^{\Theta}_{s,l,\varepsilon}$ is contained in $\mathcal{B}(\Theta\,;s, l, \varepsilon)$. Let $1_{\lambda,l}$ be the element of $\mathcal{B}(G/P\,;L_{\lambda,l})$ such that $1_{\lambda,l}|_K = \tau_l$ and define $\phi^{\Theta}_{\lambda,l} = \mathcal{P}^{\Theta}_{s,l,\varepsilon}1_{\lambda,l}$ and $P^{\Theta}_{s,l,\varepsilon}(x) = \phi^{\Theta}_{-\lambda,-l}(x^{-1})$, $x \in G$. For $f \in \mathcal{B}(\Theta\,;s, l, \varepsilon)$ we define

$$\mathcal{P}_{\Theta,s,l,\varepsilon}f(x) = \int_{K} f(xk)\tau_l(k)dk. \tag{3.2}$$

Then we have $\mathcal{P}_{\lambda,l} = \mathcal{P}_{\Theta,s,l,\varepsilon} \circ \mathcal{P}^{\Theta}_{s,l,\varepsilon}$. A straightforward calculation shows that

$$(\mathcal{P}_{\Theta,s,l,\varepsilon}f)(x) = \int_K P^{\Theta}_{s,l,\varepsilon}(k^{-1}x)f(k)dk. \tag{3.3}$$

We write $\mathcal{P}_{\Theta,s} = \mathcal{P}_{\Theta,s,0,\varepsilon}$ for simplicity.

If $\varepsilon l \gtrdot n-1$, then it follows from Theorem 5.1 and Theorem 5.10 in Shimeno[10] that $\phi^{\Theta}_{\lambda,l}|M_{\Theta}$ is a vector in the holomorphic discrete series representation of M_{Θ} with lowest K-type τ_l. Thus the function $\phi^{\Theta}_{\lambda,l}$ is contained in $\mathcal{B}(G/P_{\Theta}\,;\,l,\,\varepsilon,\,s)$.

For dominant integral weight $\mu \in \sqrt{-1}\mathfrak{t}^*$, let V_{μ} denote the irreducible representation of K with highest weight μ. Let $\{E_i\}$ be a basis of $\mathfrak{p}^+$ and $\{E_j^*\}$ be the dual basis of $\mathfrak{p}^-$ with respect to B. Let p_+ denote the projection of $\mathfrak{p}^+\otimes\mathfrak{p}^+$ onto the K-irreducible component with highest weight $\gamma_1+\gamma_2$. We define an element of $U(\mathfrak{g}_{\mathbf{C}})\otimes V_{\gamma_1+\gamma_2}$ by

$$\mathcal{H}^{\Theta}_{-} = \sum_{i,j} E_i^* E_j^* \otimes p_+(E_i\otimes E_j), \tag{3.4}$$

which does not depend on the choice of basis. Notice that $\mathcal{H}^{\Theta}_{-}$ defines a homogeneous differential operator from $C^{\infty}(G/K\,;\,\tau_l)$ to $C^{\infty}(G/K\,;\,\tau_l\otimes \mathrm{Ad}_K|\,V_{\gamma_1+\gamma_2})$. We define $\mathcal{H}^{\Theta}_{+}$ similarly. We denote $\mathcal{H}^{\Theta}_{\pm}$ for $\varepsilon = \pm 1$ by $\mathcal{H}^{\Theta}_{\varepsilon}$.

We state the main result of this article:

Theorem 3.1 *If $s \in \mathbb{C}$, $l \in \mathbb{Z}$, and $\varepsilon \in \{+1, -1\}$ satisfies condition,*

$$\{s+\varepsilon l-n+1,\, s,\, s-\varepsilon l,\, \frac{1}{2}(s+1-|l|)\}\cap\{0,\,-1,\,-2,\,\cdots\cdots\} = \emptyset, \tag{3.5}$$

then the partial Poisson transform $\mathcal{P}_{\Theta,s,l,\varepsilon}$ is a G-isomorphism of $\mathcal{B}(G/P_{\Theta}\,;s,\,l,\,\varepsilon)$ onto the space of analytic functions u in $\mathcal{B}(G/K,\,\tau_l)$ that satisfy

$$\mathcal{H}^{\Theta}_{\varepsilon}u = 0, \tag{3.6}$$

$$L_l u = (<\lambda^{\Theta}_{s,l,\varepsilon},\,\lambda^{\Theta}_{s,l,\varepsilon}> - <\rho,\,\rho>)u. \tag{3.7}$$

Corollary 3.2 *If $s \in \mathbb{C}$ satisfies the condition,*

$$s-n+1 \notin \{0,\,-1,\,-2,\,\cdots\cdots\} \tag{3.8}$$

then the partial Poisson transform $\mathcal{P}_{\Theta,s}$ is a G-isomorphism of $\mathcal{B}(G/P_{\Theta}\,;\,s)$ onto the space of analytic functions u on G/K that satisfy

$$\mathcal{H}^{\Theta}_{+}u = 0,$$
$$\mathcal{H}^{\Theta}_{-}u = 0,$$
$$L_0 u = (<\lambda^{\Theta}_{s},\,\lambda^{\Theta}_{s}> - <\rho,\,\rho>)u.$$

The proof of the theorem is divided into four steps;

1. The image of $\mathcal{B}(G/P_{\Theta}\,;\,s,\,l,\,\varepsilon)$ under $\mathcal{P}_{\Theta,s,l,\varepsilon}$ satisfies (3.6) (Proposition 3.3),
2. Solutions of (3.6) and (3.7) satisfy $\mathcal{M}_{\lambda^{\Theta}_{s,l,\varepsilon},l}$ (Proposition 3.5),
3. Under condition (3.5), $\mathcal{P}_{\Theta,s,l,\varepsilon}$ gives an isomorphism of $\mathcal{B}(\Theta\,;\,s,\,l,\,\varepsilon)$ onto the joint eigenspace of $\mathcal{M}_{\lambda^{\Theta}_{s,l,\varepsilon},l}$ (Proposition 3.9),
4. Under condition (3.17) boundary values of solutions of (3.6) and (3.7) are contained in $\mathcal{B}(G/P_{\Theta}\,;\,s,\,l,\,\varepsilon)$ (Proposition 3.11).

Proposition 3.3 *For any f in $\mathcal{B}(G/P_{\Theta}\,;\,s,\,l,\,\varepsilon)$, $u = \mathcal{P}_{\Theta,s,l,\varepsilon}f$ satisfies* (3.6).

Proof. We consider the universal covering group of G and may assume that $l \in \mathbb{C}$. It is sufficient to show that $u = P^{\theta}_{s,l,\epsilon}$ satisfies (3.6). We put $F = \mathcal{H}^{\theta}_{-}P^{\theta}_{s,l,-1}$. If $l < -n$, then the restriction of F to $Sp(n-1, \mathbb{R}) \subset M_{\theta}$ is a vector in the holomorphic discrete series representation of lowest K-type τ_l that is $U(n-1)$-finite of type $(\tau_l \otimes \mathrm{Ad}_K | V_{\gamma_1+\gamma_2})|_{U(n-1)}$. Since the holomorphic discrete series of $Sp(n-1, \mathbb{R})$ with lowest $U(n-1)$-type τ_l equals $S(\mathfrak{p}_{\bar{\theta}}) \otimes \tau_l$ as $U(n-1)$-modules, F must be identically zero. We have $F \equiv 0$ for all l by analytic continuation. We can show in the same way that $\mathcal{H}^{\theta}_{+} P^{\theta}_{s,l,+1} = 0$. □

Remark 3.4 The use of operator $\mathcal{H}^{\theta}_{\epsilon}$ is inspired by Miyazaki, Oda[6] and Iida[4], where they construct differential equations for Whittaker functions or matrix coefficients of principal series representations of $Sp(2, \mathbb{R})$ by using K-covariant differential operators between homogeneous vector bundles over G/K (shift operators in their terminology).

3.2 Radial parts of the Hua equations

Proposition 3.5 *Any solution of* (3.6) *and* (3.7) *satisfies* $\mathcal{M}_{\lambda^{\theta}_{s,l,\epsilon},l}$.

Let $\varphi_{\lambda,l}$ denote the Poisson integral of the function $1_{\lambda,l} \in \mathcal{B}(G/P\,;L_{\lambda,l})$ with $1_{\lambda,l}|_K = \tau_{-l}$, i.e.,

$$\varphi_{\lambda,l}(g) = \int_K \tau_l(k^{-1}\kappa(g^{-1}k)) \exp < -\lambda-\rho, H(g^{-1}k) > dk.$$

We shall prove that $\phi_{\lambda,l}$ is a unique solution of (3.6) and (3.7) such that $u(kx) = \tau_l(k)^{-1}u(x)$ for all $k \in K$ and $x \in G$ (Corollary 3.8). Then we can prove Proposition 3.5 in the same way as the proof of Theorem 3.3 in Shimeno[11]. In the proof of Theorem 3.3 in Shimeno[11] we use a characterization of joint eigenfunctions of $\mathbb{D}(G/K)$ by means of an integral formula (Helgason[3], Ch IV, Proposition 2.4), which can easily be generalized to the case of a homogeneous line bundle.

We define elements $T_i\,(i = 1, \cdots\cdots, n)$, $X_{\pm 2\epsilon_i} \in \mathfrak{g}_{\pm 2\epsilon_i}\,(i = 1, \cdots\cdots, n)$ and $X_{\pm\epsilon_j\pm\epsilon_k} \in \mathfrak{g}_{\pm\epsilon_j\pm\epsilon_k}\,(1 \le j \ne k \le n)$ by

$$T_i = \begin{pmatrix} 0 & E_{ii} \\ -E_{ii} & 0 \end{pmatrix}, \quad (1 \le i \le n),$$

$$X_{2\epsilon_i} = \begin{pmatrix} E_{ii} & \sqrt{-1}E_{ii} \\ \sqrt{-1}E_{ii} & -E_{ii} \end{pmatrix}, \quad (1 \le i \le n),$$

$$X_{\epsilon_j+\epsilon_k} = \begin{pmatrix} E_{jk}+E_{kj} & \sqrt{-1}(E_{jk}+E_{kj}) \\ \sqrt{-1}(E_{jk}+E_{kj}) & -E_{jk}-E_{kj} \end{pmatrix}, \quad (1 \le j < k \le n)$$

$$X_{\epsilon_j-\epsilon_k} = \begin{pmatrix} E_{jk}-E_{kj} & -\sqrt{-1}(E_{jk}+E_{kj}) \\ \sqrt{-1}(E_{jk}+E_{kj}) & E_{jk}-E_{kj} \end{pmatrix}, \quad (1 \le j < k \le n)$$

and $X_{-\beta} = \bar{X}_{\beta}(\beta = 2\epsilon_1, 2\epsilon_2, \epsilon_1+\epsilon_2)$.

For $m, l \in \mathbb{Z}$ let $h(t; m, l)$ be the function on $\mathbb{R}^n$ given by

$$h(t; m, l) = \Big(\prod_{j=1}^{n} \cosh t_j\Big)^{\frac{1}{2}(m+l)} \Big(\prod_{j=1}^{n} \sinh t_j\Big)^{\frac{1}{2}(l-m)}$$

A function u on G is called (τ_{-m}, τ_{-l})-spherical when it satisfies

$$u(k_1gk_2) = \tau_m(k_1)^{-1}u(g)\tau_l(k_2)^{-1} \text{ for all } g \in G,\ k_1,\ k_2 \in K.$$

We will calculate (τ_{-m}, τ_{-l})-radial parts of (3.6). For $Sp(2, \mathbb{R})$ this was done by Iida[4]. The calculation for general n reduces to this case.

Proposition 3.6 *If $u \in C^\infty(G)$ is a (τ_{-m}, τ_{-l})-spherical solution of* (3.6), *then the function*

$$\phi(t) = h(t;\, \varepsilon m,\, \varepsilon l)u\Big(\exp\Big(\sum_{j=1}^{n} t_j X_j\Big)\Big) \tag{3.9}$$

satisfies

$$\begin{aligned}(2\partial_{t_j}\partial_{t_k} &+ (\coth(t_j+t_k) - \coth(t_j-t_k))\partial_{t_j} \\ &+ (\coth(t_j+t_k) + \coth(t_j-t_k)\partial_{t_k})\phi = 0\end{aligned} \tag{3.10}$$

for all $1 \leq j < k \leq n$.

Lemma 3.7 *The highest weight vector of the irreducible K-module $V_{\gamma_1+\gamma_2} \subset \mathfrak{p}^+ \otimes \mathfrak{p}^+$ is up to constant given by*

$$X_{2\varepsilon_1} \otimes X_{2\varepsilon_2} + X_{2\varepsilon_2} \otimes X_{2\varepsilon_1} - \frac{1}{2} X_{\varepsilon_1+\varepsilon_2} \otimes X_{\varepsilon_1+\varepsilon_2}. \tag{3.11}$$

Proof. Since each weight space of $\mathfrak{p}^+ \otimes \mathfrak{p}^+$ is multiplicity one, it is enough to show that (3.11) is a highest weight vector with weight $\gamma_1+\gamma_2$. Since $\alpha+\beta \notin \Delta$ for all $\alpha \in \{\pm\varepsilon_1 \pm\varepsilon_2,\ \pm 2\varepsilon_1,\ \pm 2\varepsilon_2\}$ and

$$\beta \in \{\varepsilon_j - \varepsilon_k\,;\, 1 \leq j < k \leq n\} \backslash \{\varepsilon_1 - \varepsilon_2\}. \tag{3.12}$$

Thus it suffices to show that (3.11) is annihilated by $X_{\varepsilon_1-\varepsilon_2}$. It follows easily from direct computation. So we omit it. □

Proof of Proposition 3.6. We prove the proposition for $\varepsilon = -1$. The case of $\varepsilon = +1$ can be proved in a similar way.

The coefficient of vector (3.11) in $\mathcal{H}^{\mathfrak{e}}_{-}$ is up to constant given by

$$\|2\varepsilon_1\|^2\|2\varepsilon_2\|^2(X_{-2\varepsilon_1}X_{-2\varepsilon_2} + X_{-2\varepsilon_1}X_{-2\varepsilon_2}) - 2\|\varepsilon_1+\varepsilon_2\|^4 X^2_{-\varepsilon_1-\varepsilon_2},$$

which is a non-zero constant multiple of

$$2X_{-2\varepsilon_1}X_{-2\varepsilon_2} - \frac{1}{2} X^2_{-\varepsilon_1-\varepsilon_2}. \tag{3.13}$$

It follows from Proposition 6.6 (ii) and Proposition 7.1 in Iida[4] that (τ_{-m}, τ_{-l})-radial part of element (3.13) gives differential equation (3.10) for $j = 1$, $k = 2$. For each $\sigma \in S_n$ we get equation (3.10) for $j = \sigma(1)$, $k = \sigma(2)$ from the coefficient

$$2X_{-2\varepsilon_{\sigma(1)}}X_{-2\varepsilon_{\sigma(2)}} - \frac{1}{2} X^2_{-\varepsilon_{\sigma(1)}-\varepsilon_{\sigma(2)}} \tag{3.14}$$

in $\mathcal{H}^{\mathfrak{e}}_{-}u = 0$ of the highest weight vector with respect to the ordering $\varepsilon_{\sigma(1)} > \varepsilon_{\sigma(2)} > \cdots\cdots > \varepsilon_{\sigma(n)}$. □

Let u be a (τ_{-m}, τ_{-l})-spherical solution of (3.7). By Proposition 2.6 in Shimeno[10], the function ϕ given by (3.9) is a solution of the differential equation

$$\Big(\sum_{i=1}^{n} \partial_{t_i}^2 + \sum_{1\le j<k\le n} ((\coth(t_j+t_k)+\coth(t_j-t_k))\partial t_j + (\coth(t_j+t_k)-\coth(t_j-t_k))\partial t_j) + \sum_{i=1}^{n} (\varepsilon 2m \coth t_i + 2(1-\varepsilon l-\varepsilon m)\coth 2t_i)\partial t_i\Big)\phi = (s^2-(n-\varepsilon l)^2)\phi. \tag{3.15}$$

Corollary 3.8 *Assume that ϕ is a W-invariant solution of* (3.10) *for* $1 \le j < k \le n$ *and* (3.15) *that is analytic at $t = 0$. Then ϕ is a constant multiple of the hypergeometric function $F(\exp(\sum_{i=1}^{n} t_iX_i)\,;\,\lambda_{s,l}^{\Theta}, k)$ of Heckman and Opdam, where k is given by $k_{\pm e_j \pm e_k} = 1/2\ (1 \le j \neq k \le n)$ and $k_{e_i} = \varepsilon m$, $k_{2e_i} = 1/2(1-\varepsilon l-\varepsilon m)\ (1 \le i \le n)$. In particular, if $l = m$, then $\phi(t)$ is a constant multiple of $h(t; \varepsilon l, \varepsilon l,)\varphi_{\lambda_{s,l,\varepsilon},l}(\exp(\sum_{i=1}^{n} t_iX_i))$.*

Proof. By the change of variables $y_i = -\sinh^2 t_i\ (1 \le i \le n)$, the system of differential equations (3.10) for $1 \le j < k \le n$ and (3.15) become a system of differential equation that was investigated by Debiard and Gaveau[1]. By Theorem 4[1], there is a unique solution up to constant subject to condition that it is W-invariant and analytic at $t = 0$. Moreover, by Corollay 4[1] it is a joint eigenfunction of commutative family of W-invariant differential operators, which turns out to be the hypergeometric function of Heckman and Opdam for the root system of type BC_n. The latter statement follows from Remark 3.8 in Shimeno[10]. □

3.3 Boundary value map

If $s \in \mathbb{C}$ satisfies condition

$$\frac{1}{2} < w\lambda-\lambda, H_1 > \notin \{0, 1, 2, \cdots\cdots\} \text{ for all } \bar{w} \in W_\Theta\backslash W \text{ with } \bar{w} \neq \bar{1}, \tag{3.16}$$

then we can define the boundary value map (cf. Section 4 of Shimeno[10]),

$$\beta_{\Theta,\mathrm{I},s,l,\varepsilon} : \mathcal{A}(G/K,\mathcal{M}_{\lambda,l}) \to \mathcal{B}(\Theta; s, l, \varepsilon).$$

Condition (3.16) is equivalent to

$$\{s+\varepsilon l-n+1, s, s-\varepsilon l\} \cap \{0, -1, -2, \cdots\cdots\} = \emptyset. \tag{3.17}$$

Proposition 3.9 *Assume that s, l, and $\varepsilon \in \{+1, -1\}$ satisfy condition* (3.5). *Then the partial Poisson transform $\mathcal{P}_{\Theta,s,l,\varepsilon}$ is a G-isomorphism of $\mathcal{B}(\Theta\,;\, s, l, \varepsilon)$ onto $\mathcal{A}(G/K\,;\,\mathcal{M}_{\lambda,l})$.*

Proof. We consider the universal covering group of G and may assume that $l \in \mathbb{C}$. First assume conditions (2.2) and (2.3) so that the Poisson transform $\mathcal{P}_{\lambda,l}$ is bijective. It follows from Proposition 4.13 in Shimeno[10] that

$$\beta_{\Theta,\mathrm{I},s,l,\varepsilon} = \boldsymbol{c}^{\Theta}(\lambda, l)\mathcal{P}_{s,l,\varepsilon}^{\Theta}\circ\mathcal{P}_{\lambda,l}^{-1}.$$

Here

$$c^{\Theta}(\lambda, l) = c2^{-s}\frac{\Gamma(s)\Gamma\left(\frac{1}{2}(s-\varepsilon l)\right)\Gamma\left(\frac{1}{2}(s+\varepsilon l-n+2)\right)}{\Gamma\left(\frac{1}{2}(s+1+l)\right)\Gamma\left(\frac{1}{2}(s+1-l)\right)\Gamma\left(\frac{1}{2}(s-\varepsilon l+n-1)\right)\Gamma\left(\frac{1}{2}(s+\varepsilon l+1)\right)},$$

where c is a non-zero constant (cf. Section 4 in Shimeno[10]).

Since $\mathcal{P}_{\lambda,l} = \mathcal{P}_{\Theta,s,l,\varepsilon}\circ\mathcal{P}^{\Theta}_{s,l,\varepsilon}$, we have

$$\beta_{\Theta,\mathrm{I},s,l,\varepsilon}\circ\mathcal{P}_{\Theta,s,l,\varepsilon} = \mathcal{P}_{\Theta,s,l,\varepsilon}\circ\beta_{\Theta,\mathrm{I},s,l,\varepsilon} = c^{\Theta}(\lambda, l)\mathrm{id}. \tag{3.18}$$

Equation (3.18) holds under condition (3.17) by analytic continuation. Therefore the inverse of $\mathcal{P}_{\Theta,s,l,\varepsilon}$ is given by $c^{\Theta}(\lambda, l)^{-1}\beta_{\Theta,\mathrm{I},s,l,\varepsilon}$ under conditions (3.17) and $c^{\Theta}(\lambda, l) \neq 0$, which are equivalent to (3.5). □

From the Iwasawa decompositon $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{n}^{-} = \mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{n}_{\bar{\Theta}}\oplus\mathfrak{n}(\Theta)^{-}$ and the Poincaré-Birkoff-Witt theorem it follows that

$$U(\mathfrak{g}_{\mathrm{C}}) = U(\mathfrak{g}_{\mathrm{C}})\mathfrak{k} \oplus U(\mathfrak{a}_{\mathrm{C}}+\mathfrak{n}_{\bar{\mathrm{C}}})\mathfrak{n}_{\bar{\Theta}\mathrm{C}}\oplus U(\mathfrak{n}(\Theta)_{\bar{\mathrm{C}}}+\mathfrak{a}_{\mathrm{C}}). \tag{3.19}$$

Let π be the projection of $U(\mathfrak{g}_{\mathrm{C}})$ to $U(\mathfrak{n}(\Theta)_{\bar{\mathrm{C}}}+\mathfrak{a}_{\mathrm{C}})$ with respect to this decomposition. Let $\iota_{s,l,\varepsilon}$ be the algebra homomorphism of $U(\mathfrak{n}(\Theta)_{\bar{\mathrm{C}}}+\mathfrak{a}_{\mathrm{C}})$ to $U(\mathfrak{n}(\Theta)_{\bar{\mathrm{C}}}+\mathfrak{a}(\theta)_{\mathrm{C}})$ such that $\iota_{s,l,\varepsilon}(Y) = Y$ if $Y \in \mathfrak{a}(\Theta)$ and $\iota_{s,l,\varepsilon}(Y) = < \lambda-\rho, Y >$ if $Y \in \mathfrak{a}_{\Theta}$. We state the following proposition, which is a special case of Theorem 4.4 in Shimeno[10].

Proposition 3.10 *Assume that* $\lambda = \lambda^{\Theta}_{s,l,\varepsilon}$ *satisfjes condition* (3.16). *Let* $u \in \mathcal{A}(G/K, \mathcal{M}_{\lambda,l})$ *and* $U \in U(\mathfrak{g}_{\mathrm{C}})$. *If* $Uu = 0$ *then* $\iota_{s,l,\varepsilon}\circ\pi(U)\beta_{\Theta,\mathrm{I},s,l,\varepsilon}(u) = 0$.

Proposition 3.11 *Assume condition* (3.17). *Then boundary values* $\beta_{\Theta,\mathrm{I},s,l,\varepsilon}(u)$ *of solutions of* (3.6) *and* (3.7) *satisfy*

$$\mathfrak{p}^{\varepsilon}_{\bar{\Theta}}\beta_{\Theta,\mathrm{I},s,l,\varepsilon}(u) = 0. \tag{3.20}$$

Proof. We prove the proposition for $\varepsilon = -1$. The case of $\varepsilon = +1$ can be proved in a similar way. We apply Proposition 3.10 to operators

$$U_i = 2X_{-2\varepsilon_1}X_{-2\varepsilon_i}-\frac{1}{2}X^2_{-\varepsilon_1-\varepsilon_i} \quad (2 \leq i \leq n) \tag{3.21}$$

and

$$U_{jk} = X_{-2\varepsilon_1}X_{-e_j-e_k}-X_{-e_1-e_j}X_{-e_1-e_k} \quad (2 \leq j < k \leq n). \tag{3.22}$$

Operator (3.21) is a coefficient in $\mathcal{H}^{\Theta}_{-}$ of weight vector of weight $2\varepsilon_1+2\varepsilon_j$ as we see in the proof of Proposition 3.6. We see by direct computations that operator (3.22) is a coefficient in $\mathcal{H}^{\Theta}_{-}$ of weight vector of weight $2\varepsilon_1+\varepsilon_j+\varepsilon_k$.

We define elements $E_{\pm 2e_i} \in \mathfrak{g}_{\pm 2e_i}$ $(i = 1, \cdots\cdots, n)$ and $E_{\pm e_j\pm e_k} \in \mathfrak{g}_{\pm e_j\pm e_k}$ $(1 \leq j \neq k \leq n)$ by

$$E_{2\varepsilon_i} = \begin{pmatrix} 0 & E_{ii} \\ 0 & 0 \end{pmatrix}, \quad (1 \leq i \leq n),$$

$$E_{e_j+e_k} = \begin{pmatrix} 0 & E_{jk}+E_{kj} \\ 0 & 0 \end{pmatrix}, \quad (1 \leq j < k \leq n)$$

$$E_{e_j-e_k}=\begin{pmatrix} E_{jk} & 0 \\ 0 & -E_{kj}\end{pmatrix}, \quad (1\leq j<k\leq n)$$

and $E_{-\alpha}={}^tE_\alpha\,(\alpha\in\Sigma^+)$.

We can show by direct computations that

$$\iota_{s,l,\varepsilon}\circ\pi(U_i)=2(s-n+1-l)(X_i-2\sqrt{-1}E_{-2e_i}+l),$$

which is identical to $2(s-n+1-l)X_{-2\varepsilon_i}$ modulo $\mathfrak{k}_{\Theta,l}=\sum_{X\in\mathfrak{k}_{\Theta,\mathbf{C}}}(X+\tau_l(X))$, and

$$\iota_{s,l,\varepsilon}\circ\pi(U_{jk})=2(s-n+2-l)(E_{e_k-e_j}-\sqrt{-1}E_{-e_j-e_k}),$$

which is identical to $2(s-n+2-l)X_{-\varepsilon_j-\varepsilon_k}$ modulo $\mathfrak{k}_{\Theta,l}$. Since

$$\{X_{-2\varepsilon_i},X_{-\varepsilon_j-e_k};\,2\leq i\leq n,2\leq j<k\leq n\}$$

forms a basis of $\mathfrak{p}_{\Theta}^-$, we have (3.20) □

Remark 3.12 We can consider generalizations of Theorem 3.1 for

1. any Hermitian symmetric space,
2. parabolic subgroups that correspond to

$$\Theta_k=\{\alpha_k,\alpha_{k+1},\cdots\cdots,\alpha_n\}\quad(2\leq k\leq n\}.$$

We will discuss these problems in a forthcoming paper.

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