An Integral Formula related to the Whittaker Functions on Semisimple Lie Groups

Michihiko HASHIZUME and Yoshiyuki MORI
Department of Applied Mathematics, Faculty of Science
Okayama University of Science
Ridai-cho 1-1, Okayama 700-0005, Japan
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1 Introduction

In this paper we consider the certain multiple integral, which appears as the integral representation of the Whittaker functions on semisimple Lie groups of real rank 2 ([H]). We transform our multiple integral into the integral of a single variable, which enables us to clarify the properties of the Whittaker functions such as the analyticity, the behavior at infinity and so on.

2 Results and Proofs

Let $\mathbb{F}$ be a normed algebra over $\mathbb{R}$ of dimension d, namely a d-dimensional algebra over $\mathbb{R}$ with multiplicative unit 1, and equipped with a positive definite inner product $<, >$ whose associated norm $\| \|$ satisfies

$$\|xy\| = \|x\| \, \|y\| \qquad (x, y \in \mathbb{F}). \tag{1}$$

Then it is known that $\mathbb{F}$ is either $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$, and correspondingly d is either 1, 2, 4 or 8. Here $\mathbb{H}$ is the quaternion algebra and $\mathbb{O}$ is the octanion algebra. We denote by $\bar{x}$ the canonical conjugate of $x \in \mathbb{F}$. Define the polynomial functions $P(x, y, z)$ on $\mathbb{F}^3$, $Q(y, z)$ on $\mathbb{F}^2$ and $R(y)$ on $\mathbb{F}$ respectively by

$$\begin{cases} P(x, y, z) = 1+\|x\|^2+\|z\|^2+\|x\|^2\|y\|^2-2 < \bar{y}z, x >, \\ Q(y, z) \quad\; = 1+\|y\|^2+\|z\|^2, \\ R(y) \qquad\;\, = 1+\|y\|^2. \end{cases} \tag{2}$$

Note that

$$P(x, y, z) = R(y)\|x-\bar{y}z/R(y)\|^2+Q(y, z)/R(y),\ Q(y, z) = R(y)+\|z\|^2 \tag{3}$$

and hence $P(x, y, z)$, $Q(y, z)$ and $R(y)$ are positive. Let $\xi, \eta \in \mathbb{F}-(0)$ and $\lambda, \mu \in \mathbb{C}$. We consider the multiple integral

$$I(\xi, \eta; \lambda, \mu) = \int_{\mathbb{F}^3} P(x, y, z)^{-\lambda} Q(y, z)^{-\mu} \exp\,(i < \xi, x > +i < \eta, y >)dx\, dy\, dz \tag{4}$$

where dx means the euclidean measure on $\mathbb{F}$ canonically identified with $\mathbb{R}^d$. Our aim is to show that the integral $I(\xi, \eta; \lambda, \mu)$ can be transformed into the integral of one variable. More precisely, let's define the integral

$$K(t_1, t_2; s_1, s_2) = \int_0^\infty K_{s_1}(t_1(1+r)^{1/2})K_{s_1}(t_2(1+r^{-1})^{1/2})r^{s_2}d^\times r \tag{5}$$

where $t_1, t_2 > 0$, $s_1, s_2 \in \mathbb{C}$, $d^\times r = dr/r$ and $K_s(z)$ means the modified Bessel function of second kind, which is also called Macdonald's function. Then our main result is the identity

$$I(\xi, \eta; \lambda, \mu) = \frac{2^2\pi^{3d/2}(\|\xi\|\,\|\eta\|/4)^{\lambda+\mu-d}}{\Gamma(\lambda)\Gamma(\mu)\Gamma(\lambda+\mu-d/2)}K(\|\xi\|, \|\eta\|; \lambda+\mu-d,(\mu-\lambda)/2) \tag{6}$$

where $\xi, \eta \in \mathbb{F}-(0)$ and $\lambda, \mu \in \mathbb{C}$.

Before establishing the above identity, we mention about the prototype of our consideration.

For $\xi \in \mathbb{R}^d$ and $\lambda \in \mathbb{C}$, consider the integral

$$I(\xi; \lambda) = \int_{\mathbb{R}^d}(a\|x\|^2+b)^{-\lambda}\exp(i<\xi, x>)dx \tag{7}$$

where a, b are positive constants. The integral can be viewed as the Fourier transform of the function $(a\|x\|^2+b)^{-\lambda}$ on $\mathbb{R}^d$. First we note that the integral representation of the Γ-function yields that

$$a^{-\lambda} = \frac{1}{\Gamma(\lambda)}\int_0^\infty \exp(-at)t^\lambda d^\times t \tag{8}$$

where $a > 0$ and $\lambda \in \mathbb{C}$ such that $Re\lambda > 0$. From (8), we can write

$$(a\|x\|^2+b)^{-\lambda} = \frac{1}{\Gamma(\lambda)}\int_0^\infty \exp(-at\|x\|^2)\exp(-bt)t^\lambda d^\times t$$

and consequently

$$I(\xi; \lambda) = \frac{1}{\Gamma(\lambda)}\int_0^\infty \exp(-bt)t^\lambda d^\times t\int_{\mathbb{R}^d}\exp(-at\|x\|^2)\exp(i<\xi, x>)dx$$

Applying the well known integral formula

$$\int_{\mathbb{R}^d}\exp(-c\|x\|^2)\exp(i<\xi, x>)dx = (\pi/c)^{d/2}\exp(-\|\xi\|^2/4c) \tag{9}$$

where c is a positive constant, we obtain, for $Re\lambda > 0$,

$$I(\xi; \lambda) = \frac{(\pi/a)^{d/2}}{\Gamma(\lambda)}\int_0^\infty \exp(-bt-\|\xi\|^2/4at)t^{\lambda-d/2}d^\times t. \tag{10}$$

Therefore the multiple integral $I(\xi; \lambda)$ can be transformed into the integral of a single variable. Assume $\xi = 0$. Then the integral representation of the Γ-function yields that, for $Re\lambda > 0$,

$$I(0; \lambda) = \int_{\mathbb{R}^d}(a\|x\|^2+b)^{-\lambda}dx = \frac{\Gamma(\lambda-d/2)}{\Gamma(\lambda)}(\pi b/a)^{d/2}b^{-\lambda}. \tag{11}$$

On the contrary, if $\xi \neq 0$, the integrand in (10) is well behaved in $(0, +\infty)$ and hence the integral (10) is well-defined for all $\lambda \in \mathbb{C}$. It is known ([L]) that for $p, q > 0$ and $\nu \in \mathbb{C}$

$$\int_0^\infty \exp(-pt-qt^{-1})t^\nu d^\times t = 2(q/p)^{\nu/2}K_\nu(2(pq)^{1/2}). \tag{12}$$

Using (12), we get, for $\xi \neq 0$,

$$I(\xi;\lambda)=\frac{2(\pi/a)^{d/2}}{\Gamma(\lambda)}(\|\xi\|/2(ab)^{1/2})^{\lambda-d/2}K_{\lambda-d/2}(\|\xi\|(b/a)^{1/2}). \tag{13}$$

Now we prove our identity (6). First we consider the integral with respect to $x\in\mathbb{F}$ in $I(\xi,\eta;\lambda,\mu)$.

Lemma 1. Put

$$I_1(y,z)=\int_{\mathbb{F}}P(x,y,z)^{-\lambda}\exp(i<\xi,x>)dx. \tag{14}$$

Then

$$I_1(y,z)=\frac{\pi^{d/2}}{\Gamma(\lambda)}R(y)^{-d/2}\exp(i<y\xi,z>/R(y))J_1(y,z) \tag{15}$$

where

$$J_1(y,z)=\int_0^\infty\exp(-t_1\|\xi\|^2/4R(y)-Q(y,z)/t_1R(y))t_1^{d/2-\lambda}d^\times t_1. \tag{16}$$

Proof. Using the expression (3) of $P(x,y,z)$ and the property $<\xi,\bar{y}z>=<y\xi,z>$ in normed algebras, we have

$$I_1(y,z)=\exp(i<y\xi,z>/R(y))\int_{\mathbb{F}}(R(y)\|x\|^2+Q(y,z)/R(y))^{-\lambda}\exp(i<\xi,x>)dx$$

Since the integral in the right-side is of the from (7), it follows from (10) that

$$I_1(y,z)=\frac{(\pi/R(y))^{d/2}}{\Gamma(\lambda)}\exp(i<y\xi,z>/R(y))$$
$$\times\int_0^\infty\exp\{-t_1Q(y,z)/R(y)-\|\xi\|^2/4t_1R(y)\}t_1^{\lambda-d/2}d^\times t_1.$$

Changing t_1 into t_1^{-1}, we obtain (15). //

Applying (8) for $a=Q(y,z)/R(y)$, we can get

$$Q(y,z)^{-\mu}=\frac{R(y)^{-\mu}}{\Gamma(\mu)}J_2(y,z) \tag{17}$$

where

$$J_2(y,z)=\int_0^\infty\exp(-t_2Q(y,z)/R(y))t_2^{\mu}d^\times t_2. \tag{18}$$

Put $J_3(y,z)=J_1(y,z)J_2(y,z)$. Then from (16) and (18),

$$J_3(y,z)=\int_0^\infty\int_0^\infty\exp\{-t_1\|\xi\|^2/4R(y)-(t_1^{-1}+t_2)Q(y,z)/R(y)\}t_1^{d/2-\lambda}t_2^{\mu}d^\times t_1d^\times t_2. \tag{19}$$

Since

$$I(\xi,\eta;\lambda,\mu)=\int_{\mathbb{F}^2}I_1(y,z)Q(y,z)^{-\mu}\exp(i<\eta,y>)dydz,$$

it follows from (15) and (17) that

$$I(\xi,\eta;\lambda,\mu)=\frac{\pi^{d/2}}{\Gamma(\lambda)\Gamma(\mu)}$$
$$\times\int_{\mathbb{F}^2}\exp(i<y\xi,z>/R(y))J_3(y,z)R(y)^{-(\mu+d/2)}\exp(i<\eta,y>)dy\,dz. \tag{20}$$

Next we consider the integral with respect to $z\in\mathbb{F}$ in (20).

Put

$$I_2(y) = \int_{\mathbb{F}} J_3(y, z) \exp(i < y\xi, z > /R(y))dz. \tag{21}$$

Then we can write $I(\xi, \eta; \lambda, \mu)$ in the form

$$I(\xi, \eta; \lambda, \mu) = \frac{\pi^{d/2}}{\Gamma(\lambda)\Gamma(\mu)} \int_{\mathbb{F}} R(y)^{-(\mu+d/2)} I_2(y) \exp(i < \eta, y >)dy. \tag{22}$$

If we introduce the integral

$$I_3(y, s) = \int_{\mathbb{F}} \exp(-sQ(y, z)/R(y)) \exp(i < y\xi, z > /R(y))dz \tag{23}$$

where s is a positive constant, then in view of (19) and (21) we can write $I_2(y)$ in the form

$$I_2(y) = \int_0^\infty \int_0^\infty \exp(-t_1\|\xi\|^2/4R(y)) I_3(y, t_1^{-1}+t_2) t_1^{d/2-\lambda} t_2^{\mu} d^\times t_1 d^\times t_2. \tag{24}$$

Lemma 2. The value of the integral $I_3(y, s)$ is given by

$$I_3(y, s) = (\pi R(y)s^{-1})^{d/2} \exp(-s) \exp(-\|\xi\|^2\|y\|^2/4sR(y)) \tag{25}$$

and consequently one can write

$$I_2(y) = \pi^{d/2}R(y)^{d/2} \int_0^\infty \int_0^\infty \exp\{-\|\xi\|^2(1+R(y)^{-1}t_1t_2)/4(t_1^{-1}+t_2)\}$$
$$\times \exp\{-(t_1^{-1}+t_2)\}(t_1^{-1}+t_2)^{-d/2} t_1^{d/2-\lambda} t_2^{\mu} d^\times t_1 d^\times t_2. \tag{26}$$

Proof. Since $\exp(-sQ(y, z)/R(y)) = \exp(-s)\exp(-s\|z\|^2/R(y))$, the assertion (25) is a direct consequence of (9). Furthermore since

$$t_1\|\xi\|^2/4R(y) + \|\xi\|^2\|y\|^2/4(t_1^{-1}+t_2)R(y) = \|\xi\|^2(1+\|y\|^2+t_1t_2)/4(t_1^{-1}+t_2)R(y)$$

and $1+\|y\|^2 = R(y)$, we can deduce (26) from (24) and (25) easily. //

Now put

$$r_1 = t_1^{-1}+t_2, \qquad r_2 = R(y)^{-1}t_1t_2.$$

Then we have

$$t_1 = r_1^{-1}(1+R(y)r_2), \qquad t_2 = R(y)r_1r_2(1+R(y)r_2)^{-1}$$

and

$$d^\times t_1 d^\times t_2 = d^\times r_1 d^\times r_2.$$

Consequently we can write $I_2(y)$ in the form

$$I_2(y) = \pi^{d/2}R(y)^{\mu+d/2} \int_0^\infty \int_0^\infty \exp\{-r_1 - \|\xi\|^2(1+r_2)/4r_1\}$$
$$\times (1+R(y)r_2)^{d/2-\lambda-\mu} r_1^{\lambda+\mu-d} r_2^{\mu} d^\times r_1 d^\times r_2. \tag{27}$$

Consider the integral with respect to r_1 in (27). Applying (12), we get

$$\int_0^\infty \exp\{-r_1 - \|\xi\|^2(1+r_2)/4r_1\} r_1^{\lambda+\mu-d} d^\times r_1$$
$$= 2(\|\xi\|/2)^{\lambda+\mu-d}(1+r_2)^{(\lambda+\mu-d)/2} K_{\lambda+\mu-d}(\|\xi\|(1+r_2)^{1/2}).$$

Therefore we have

$$I_2(y) = 2\pi^{d/2}(\|\xi\|/2)^{\lambda+\mu-d}R(y)^{\mu+d/2}\int_0^\infty (1+R(y)r_2)^{-(\lambda+\mu-d/2)} \times(1+r_2)^{(\lambda+\mu-d)/2}K_{\lambda+\mu-d}(\|\xi\|(1+r_2)^{1/2})r_2^\mu d^\times r_2. \tag{28}$$

Applying (28) into (22), we obtain

$$I(\xi, \eta; \lambda, \mu) = \frac{2\pi^d(\|\xi\|/2)^{\lambda+\mu-d}}{\Gamma(\lambda)\Gamma(\mu)} \times \int_0^\infty K_{\lambda+\mu-d}(\|\xi\|(1+r_2)^{1/2})(1+r_2)^{(\lambda+\mu-d)/2}r_2^\mu K(r_2)d^\times r_2 \tag{29}$$

where we put

$$K(r_2) = \int_{\mathbb{F}} (1+R(y)r_2)^{-(\lambda+\mu-d/2)} \exp(i < \eta, y >)dy.$$

Since $1+R(y)r_2 = (1+r_2)+r_2\|y\|^2$, the integral $K(r_2)$ is of the form (7), so by (13) we have

$$K(r_2) = \frac{2\pi^{d/2}(\|\eta\|/2)^{\lambda+\mu-d}}{\Gamma(\lambda+\mu-d/2)}(1+r_2)^{-(\lambda+\mu-d)/2}r_2^{-(\lambda+\mu)/2}K_{\lambda+\mu-d}(\|\eta\|(1+r_2^{-1})^{1/2}).$$

Using this into (29), we finally obtain

$$I(\xi, \eta; \lambda, \mu) = \frac{2^2\pi^{3d/2}(\|\xi\|\,\|\eta\|/4)^{\lambda+\mu-d}}{\Gamma(\lambda)\Gamma(\mu)\Gamma(\lambda+\mu-d/2)} \times\int_0^\infty K_{\lambda+\mu-d}(\|\xi\|(1+r_2)^{1/2})K_{\lambda+\mu-d}(\|\eta\|(1+r_2^{-1})^{1/2})r_2^{(\mu-\lambda)/2}d^\times r_2.$$

We remark that the integral $I(\xi, \eta; \lambda, \mu)$ where either ξ or $\eta = 0$ can be computed in a similar and slightly simple manner.

We summarize our results in the theorem.

Theorem. Let $\xi, \eta \in \mathbb{F}$ and $\lambda, \mu \in \mathbb{C}$. Define the integral $I(\xi, \eta; \lambda, \mu)$ by

$$I(\xi, \eta; \lambda, \mu) = \int_{\mathbb{F}^3} P(x, y, z)^{-\lambda}Q(y, z)^{-\mu} \exp\{i < \xi, x > +i < \eta, y >\}dxdydz.$$

Then (i) if $\xi \neq 0$ and $\eta \neq 0$,

$$I(\xi, \eta; \lambda, \mu) = \frac{2^2\pi^{3d/2}(\|\xi\|\,\|\eta\|/4)^{\lambda+\mu-d}}{\Gamma(\lambda)\Gamma(\mu)\Gamma(\lambda+\mu-d/2)} \times\int_0^\infty K_{\lambda+\mu-d}(\|\xi\|(1+r)^{1/2})K_{\lambda+\mu-d}(\|\eta\|(1+r^{-1})^{1/2})r^{(\mu-\lambda)/2}d^\times r,$$

(ii) if $\xi = 0$ and $\eta \neq 0$,

$$I(0, \eta; \lambda, \mu) = \frac{2\pi^{3d/2}\Gamma(\lambda-d/2)\Gamma(\lambda+\mu-d)}{\Gamma(\lambda)\Gamma(\mu)\Gamma(\lambda+\mu-d/2)}(\|\eta\|/2)^{\mu-d/2}K_{\mu-d/2}(\|\eta\|),$$

(iii) if $\xi \neq 0$ and $\eta = 0$,

$$I(\xi, 0; \lambda, \mu) = \frac{2\pi^{3d/2}\Gamma(\mu-d/2)\Gamma(\lambda+\mu-d)}{\Gamma(\lambda)\Gamma(\mu)\Gamma(\lambda+\mu-d/2)}(\|\xi\|/2)^{\lambda-d/2}K_{\lambda-d/2}(\|\xi\|),$$

(iv) and finally if $\xi = \eta = 0$,

$$I(0, 0; \lambda, \mu) = \frac{\pi^{3d/2}\Gamma(\lambda-d/2)\Gamma(\mu-d/2)\Gamma(\lambda+\mu-d)}{\Gamma(\lambda)\Gamma(\mu)\Gamma(\lambda+\mu-d/2)}.$$

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