Anti-Integral Polynomials, Super-Primitive Polynomials and Flat Polynomials

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Let R be a Noetherian integral domain and $R[X]$ a polynomial ring and let K be the quotient field of R. Let α be an element of an algebraic field extension L of K and let π: $R[X]\rightarrow R[\alpha]$ be the R-algebra homomorphism sending X to α. Let $\varphi_{\alpha}(X)$ be the monic minimal polynomial of α over K with deg $\varphi_{\alpha}(X)=d$ and write $\varphi_{\alpha}(X)=X^{d}+\eta_{1}X^{d-1}+\cdots+\eta_{d}$. Let $I_{[\alpha]}:=\bigcap_{i=1}^{d}(R:_{R}\eta_{i})(=R[X]:_{R}\varphi(x)))$. For $f(X)\in R[X]$, let $C(f(X))$ denote the ideal generated by the coefficients of $f(X)$. Let $J_{[\alpha]}:=I_{[\alpha]}C(\varphi_{\alpha}(X))$, which is an ideal of R and contains $I_{[\alpha]}$. The element α is called an anti-integral element of degree d over R if $Ker\ \pi=I_{[\alpha]}\varphi_{\alpha}(X)R[X]$. When α is an anti-integral element over R, $R[\alpha]$ is called an anti-integral extension of R. In the case $K(\alpha)=K$, an anti-integral element α is the same as an anti-integral element (i.e., $R=R[\alpha]\cap R[1/\alpha]$) defined in [OY]. The element α is called a super-primitive element of degree d over R if $J_{[\alpha]}\not\subset p$ for all primes p of depth one. In [YOS], we have treated an ideal H of a polynomial ring $R[X]$ such that $P\cap R=(0)$ for all $P\in Ass_{R[X]}(R[X]/H)$. We defined the super-primitiveness, anti-integralness and flatness of the ideal H. If necessary, we can consider a simple ring-extension $R[X]/H$.

As was seen in the above, [OSY] concerned a simple extension with certain properties by use of the ideal $Ker\ \pi$. This paper deals with a monic polynomial $\varphi(X)$ in $K[X]$. We define the super-primitiveness, anti-integralness and flatness of a polynomial $\varphi(X)$. If necessary, we can consider a simple ring-extension $R[X]/(\varphi(X)K[X]\cap R[X])$. When $\varphi(X)\in K[X]$ is an irreducible polynomial, we come back to the case in [OSY]. When $\varphi(X)\in K[X]$ is not an irreducible polynomial, we can extend the super-primitiveness, anti-integralness and flatness to a simple extension which is not necessarily an integral domain.

This paper can be considered to be a continuation of [YOS].

We use the following notation throughout this paper unless otherwise specified:

Let R be a Noetherian integral domain and $R[X]$ a polynomial ring and let K is the quotient field of R. Let H be an ideal of $R[X]$ and let $\varphi(X)$ be a monic polynomial in $K[X]$.

Our unexplained technical terms are standard and are seen in [M1] and [M2].

We start with the following definitions.

Definition 1. Let $\varphi(X)$ be a monic polynomial in $K[X]$ and let $I^R_{\varphi(X)} := R[X] :_R \varphi(X)$. The polynomial $\varphi(X)$ is called a *super-primitive polynomial* (resp. an *anti-integral polynomial*) if $\mathrm{grade}(I^R_\varphi C(\varphi(X))) > 1$ (resp. if $\varphi(X)K[X] \cap R[X] = I^R_{\varphi(X)}\varphi(X)R[X]$).

Proposition 2. *Let $\varphi(X) \in K[X]$ be a monic polynomial. If the ideal $I^R_{\varphi(X)}\varphi(X)R[X]$ contains a monic polynomial in $R[X]$, then $I^R_{\varphi(X)} = R$, i.e., $\varphi(X) \in R[X]$.*

Proof. Let $f(X) \in I^R_{\varphi(X)}\varphi(X)R[X]$ be a monic polynomial in $R[X]$. Then $f(X) = \sum a_i\varphi(X)g_i(X) = (\sum a_i g_i(X))\varphi(X)$ with some $a_i \in I^R_{\varphi(X)}$ and some $g_i(X) \in R[X]$. Compare the coefficient of the leading term, We conclude that $I^R_{\varphi(X)} = R$. □

Corollary 2.1. *Let $\varphi(X) \in K[X]$ be a monic polynomial. Assume that $\varphi(X)$ is a factor of a monic polynomial in $R[X]$ and that $\varphi(X)$ is an anti-integral polynomial. Then $\varphi(X) \in R[X]$.*

Proof. By the assumption, there exists a monic polynomial $f(X) \in \varphi(X)k[X] \cap R[X]$. So we have $f(X) \in \varphi(X)K[X] \cap R[X] = I_{\varphi(X)}\varphi(X)R[X]$. Hence $\varphi(X) \in R[X]$ by Proposition 3. □

The following example shows that every polynomial $\varphi(X) \in K[X]$ is not always an anti-integral polynomial.

Example 3. Assume that $R \subsetneq \bar{R}$ and take $\eta \in \bar{R} \backslash R$. Then $\varphi(X) := X - \eta$ is not an anti-integral polynomial. Indeed, Proposition 2 shows that $\varphi(X) = X - \eta$ is an anti-integral polynomial if and only if $\eta \in R$. So if $\varphi(X)$ is an anti-integral polynomial then $\eta \in R$, which is a contradiction.

It is easy to see the following remark by definition.

Proposition 4. *Let $\varphi(X) \in K[X]$ be a monic polynomial. If $\varphi(X)$ is a super-primitive polynomial. then $\varphi(X)$ is an anti-integral polynomial.*

Proof. Since $\varphi(X)$ is super-primitive, $H := \varphi(X)K[X] \cap R[X]$ is a super-primitive ideal of $R[X]$. Note that the ideal H is exclusive i.e., $R \cap H = (0)$. Hence the conclusion follows from [YOS, (1.6)]. □

Let $f(X) = a_0X^n + a_1X^{n-1} + \cdots + a_n$ be a polynomial in $R[X]$. We say that $f(X)$ is a *Sharma polynomial* in $R[X]$ if there does not exist $t \in R$ with $t \notin a_0R$ such that $ta_i \in a_0R$ for $1 \leq i \leq n$.

Lemma 5. *If both $f(X)$ and $g(X) \in R[X]$ are Sharma polynomials, then $f(X)g(X)$ is a Sharma polynomial.*

Proof. Suppose that there exists $p \in \mathrm{Dp}_1(R)$ such that $f(X)g(X) \in pR[X]$. Then $f(X) \in pR[X]$ or $g(x) \in pR[X]$ and hence $C(f(X)) \subseteq p$ or $C(g(X)) \in p$, which is a contradiction by [YOS; (1.4)]. □

Proposition 6. *Let $\varphi_1(X), \cdots, \varphi_n(X) \in K[X]$ be super-primitive polynomials. Then*
(1) $\varphi(X) := \varphi_1(X) \cdots \varphi_n(X) \in K[X]$ is a super-primitive polynomial;
(2) if $\varphi(X) \in R[X]$, then $\varphi_i(X) \in R[X]$ for each $1 \leq i \leq n$.

Proof. (1) Since $\varphi_i(X)$ is super-primitive, we have $I^R_{\varphi_i(X)}C(\varphi_i(X)) \nsubseteq p$ for every $p \in \mathrm{Dp}_1(R)$. Take $p \in \mathrm{Dp}_1(R)$. Then there exists $a_i \in I^R_{\varphi_i(X)}$ such that $C(a_i\varphi_i(X)) \nsubseteq p$. Thus $a_i\varphi_i(X)$ is a Sharma polynomial in $R_p[X]$. Put $a := a_1 \cdots a_n$. Then $a \in I^R_{\varphi(X)}$ and $a\varphi(X) = (a_1\varphi_1(X)) \cdots (a_n\varphi_n(X))$. So by Lemma 5, $a\varphi(X)$ is a Sharma polynomial in $R_p[X]$ and hence $C(a\varphi(X)) \nsubseteq p$. Therefore we have $\mathrm{grade}(I^R_{\varphi(X)}\, C(\varphi(X))) > 1$. So $a\varphi(X)$ is a super-primitive polynomial.
(2) follows Proposition 4 and Corollary 2.1. □

Question. Is the similar statement to Proposition 8 valid for anti-integral polynomials?

The following proposition asserts that a partial answer to the above question is valid.

Proposition 7. *Let $\varphi_1(X), \cdots, \varphi_n(X) \in K[X]$ be anti-integral polynomials. If $\varphi(X) := \varphi_1(X) \cdots \varphi_n(X) \in R[X]$ is a monic polynomial, then $\varphi_i(X) \in R[X]$ for all $1 \leq i \leq n$.*

Proof. Since $\varphi_i(X)$ is an anti-integral polynomial, $\varphi(X) \in \varphi_i(X)K[X] \cap R[X] = I^R_{\varphi_i(X)}\varphi_i(X)R[X]$. So $I^R_{\varphi_i(X)} = R$ by Proposition 2. Thus we conclude that $\varphi_i(X) \in R[X]$. □

Example 8. Let $R = k[t^2, t^3]$, where k is a field and t is an indeterminate. Let $\varphi_1(X) := X - t$, $\varphi_2(X) := X + t \in K[X]$. Then neither of $\varphi_1(X)$ nor $\varphi_2(X)$ is an anti-integral polynomial. But $\varphi(X) := \varphi_1(X)\varphi_2(X) = (X-t)(X+t) = X^2 - t^2$ is an anti-integral polynomial.

Proposition 9. *Let $\varphi(X) \in K[X]$ and let $\varphi(X) = \varphi_1(X)^{e_1} \cdots \varphi_n(X)^{e_n}$ be the irreducible decomposition in $K[X]$. Assume that $\varphi_i(X)$ is a super-primitive polynomial for each $1 \leq i \leq n$. Then $I^R_{\varphi(X)} = (((I^R_{\varphi_1(X)})^{e_1} \cdots (I^R_{\varphi_n(X)})^{e_n})^{-1})^{-1}$.*

Proof. $(I^R_{\varphi_1(X)})^{e_1} \cdots (I^R_{\varphi_n(X)})^{e_n} \subseteq I^R_{\varphi(X)}$ is obvious. Since $I^R_{\varphi(X)}$ is a divisorial ideal, we have $(((I^R_{\varphi_1(X)})^{e_1} \cdots (I^R_{\varphi_n(X)})^{e_n})^{-1})^{-1} \subseteq I^R_{\varphi(X)}$. Note that both sides of this implication are divisorial. Consider the localization at a prime ideal of depth one, we may assume that (R, m) is a local domain with the maximal ideal $m \in \mathrm{Dp}_1(R)$. Since $\varphi_i(X)$ is a super-primitive polynomial, there exists $a_i \in I^R_{\varphi_i(X)}$ such that $a_i\varphi_i(X)$ is a Sharma polynomial by [YOS, (1.4)]. Put $a := a_1 \cdots a_n$, Then we have $(a_1\varphi_1(X))^{e_1} \cdots (a_n\varphi_n(X))^{e_n} = a\varphi_1(X)^{e_1} \cdots \varphi_n(X)^{e_n}$ is a Sharma polynomial by Lemma 5. Thus $I^R_{\varphi(X)}\varphi(X)R[X] = a\varphi(X)R[X]$ and hence $I^R_{\varphi(X)} = aR$. Therefore $I^R_{\varphi(X)} = (((I^R_{\varphi_1(X)})^{e_1} \cdots (I^R_{\varphi_n(X)})^{e_n})^{-1})^{-1}$. □

Definiton 10. Let $\varphi(X) \in K[X]$ be a monic polynomial. We say that $\varphi(X)$ is *a flat polynomial* if $I^R_{\varphi(X)}C(\varphi(X)) = R$.

Poposition 11. *Let $\varphi(X)$, $\Psi(X)$ be flat polynomials. Then*
(1) *$\varphi(X)\Psi(X)$ is also a flat polynomial;*
(2) *$I^R_{\varphi(X)\Psi(X)} = I^R_{\varphi(X)}I^R_{\Psi(X)}$;*
(3) *$\varphi(X)$ is a super-primitive polynomial.*

Proof. Note that $I^R_{\varphi(X)} \cdot I^R_{\Psi(X)} \subseteq I^R_{\varphi(X)} \cap I^R_{\Psi(X)} \subseteq I^R_{\varphi(X)\Psi(X)}$.
Hence $I^R_{\varphi(X)}I^R_{\Psi(X)}(\varphi(X)\Psi(X))R[X] \subseteq I^R_{\varphi(X)\Psi(X)}\varphi(X)\Psi(X)R[X]$.

(1) Hence $R = I^R_{\varphi(X)} I^R_{\Psi(X)} C(\varphi(X)\Psi(X)) \subseteq I^R_{\varphi(X)\Psi(X)} C(\varphi(X)\Psi(X)) \subseteq R$. Thus $\varphi(X)\Psi(X)$ is flat polynomial.

(2) By the argument in the proof or (1). $C(\varphi(X)\Psi(X))$ is an invertible ideal of R. Thus $I^R_{\varphi(X)\Psi(X)} = I^R_{\varphi(X)}I^R_{\Psi(X)}$.

(3) follows from the definition. □

Proposition 12. *Let $\varphi(X) \in K[X]$ be a monic polynomial and let $\varphi(X) = \varphi_1(X)^{e_1} \cdots \varphi_n(X)^{e_n}$ be an irreducible decomposition in $K[X]$. If $\varphi_i(X)$ is a flat polynomial for each $1 \leq i \leq n$, then $I^R_{\varphi(X)} = (I^R_{\varphi_1(X)})^{e_1} \cdots (I^R_{\varphi_n(X)})^{e_n}$ and $I^R_{\varphi(X)}\varphi(X)R[X] = (\prod_{i=1}^n (I^R_{\varphi_i(X)})^{e_i}\varphi_i(X)^{e_i})R[X]$ is an primary decomposition.*

Proof. This follows from Proposition 9 and Proposition 11 (1)(2). □

Definition 13. Let $\varphi(X) \in K[X]$ be a monic polynomial. We say that $\varphi(X)$ is an *ultra-primitive polunomial* if $\mathrm{grade}(I^R_{\varphi(X)} + C(\overline{R}/R)) > 1$.

Proposition 14. *Let $\varphi(X)$, $\Psi(X) \in K[X]$. be monic polynomials. If both $\varphi(X)$ and $\Psi(X)$ are ultra-primitive polynomials, then so is $\varphi(X)\Psi(X)$.*

Proof. Note that $I^R_{\varphi(X)} \cdot I^R_{\psi(X)} \subseteq I^R_{\varphi(X)} \cap I^R_{\psi(X)} \subseteq I^R_{\varphi(X)\psi(X)}$. Since $\mathrm{grade}(I^R_{\varphi(X)} + C(\overline{R}/R)) > 1$ and $\mathrm{grade}(I^R_{\psi(X)} + C(\overline{R}/R)) > 1$, we have $1 < \mathrm{grade}(I^R_{\varphi(X)} I^R_{\psi(X)} + C(\overline{R}/R)) \le \mathrm{grade}(I^R_{\varphi(X)\psi(X)} + C(\overline{R}/R))$. □

Remark 15. (1) We have the following implications:
an ultra-primitive polynomial ⇒ a super-primitive polynomial ⇒ an anti-integral polynomial.

The reverse implications are not valid in general.

(2) Take $\eta \in K$.

(i) $X - \eta$ is a super-primitive polynomial if and only if $\mathrm{grade}(I_\eta(1, \eta) > 1$.

(ii) $X - \eta$ is an ultra-primitive polynomial if and only if $\mathrm{grade}(I_\eta + C(\overline{R}/R)) > 1$.

(iii) $X - \eta$ is a flat polynomial if and only if $I_\eta(1, \eta) = R$.

Definition 16. Let $\eta \in K$.

1) η is called a *super-primitive element* if $\mathrm{grade}(I_\eta(1, \eta)) > 1$.

2) η is called an *ultra-primitive element* if $\mathrm{grade}(I_\eta + C(\overline{R}/R)) > 1$.

3) η is called a *flat element* if and only if $I_\eta(1, \eta) = R$.

Remark 17. The above definition is that same as in [OSY], that is, $\eta \in K$ is super-primitive over R if the extension $R[\eta]$ is a super-primitive extension of R in the sense of [OSY]. An element $\eta \in K$ is a flat element if $R[\eta]$ is aflat extension of R.

Proposition 18. *Let* $\varphi(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d \in K[X]$. *If* η_i *is an ultra-primitive element for each* $1 \le i \le d$, *then* $\varphi(X)$ *is an ultra-primitive polynomial.*

Proof. Note that $I_{\eta_1} \cdots I_{\eta_d} \subseteq \bigcap_{i=1}^d I_{\eta_i}$. Since $\mathrm{grade}(I_{\eta_i} + C(\overline{R}/R)) > 1$ by definition, $\mathrm{grade}(I_{\eta_1} \cdots I_{\eta_d} + C(\overline{R}/R))\ 1$ and hence $\mathrm{grade}(\bigcap_{i=1}^d I_{\eta_i} + C(\overline{R}/R)) > 1$. Therefore $\mathrm{grade}(I^R_{\varphi(X)} + C(\overline{R}/R)) > 1$ because $I^R_{\varphi(X)} = \bigcap_{i=1}^d I_{\eta_i}$, which implies that $\varphi(X)$ is an ultra-primitive polynomial. □

In the notation as in Proposition 18, even if η_i is a super-primitive (resp. flat) element for each $1 \le i \le d$, $\varphi(X)$ is not necessarily a super-primitive (resp. flat) polynomial. We see this in the following example.

Example 19. Note first that a flat element (resp. polynomial) is a super-primitive element (resp. polynomial). Let $R := k[t^2, t^3]$, where k is a field and t be an indeterminate. Put $\eta_1 := 1/t^3$, $\eta_2 := 1/t^4$ and $\varphi(X) := X^2 + \eta_1 X + \eta_2 = X^2 + (1/t^3)X + (1/t^4)$. Then $I_{\eta_1} = t^3 R$, $I_{\eta_2} = t^4 R$. Hence $I_{\eta_1}(1, \eta_1) = I_{\eta_2}(1, \eta_2) = R$. Thus η_1 and η_2 are flat elements and hence they are super-primitive elements. We have $I^R_{\varphi(X)} = I_{\eta_1} \cap I_{\eta_2} = t^3 R \cap t^4 R = ((t^6, t^7)R$. So we have $I^R_{\varphi(X)} C(\varphi(X) = I^R_{\varphi(X)}(1, \eta_1, \eta_2) = (t^6, t^7)(1, 1/t^3, 1/t^4)R = (t^2, t^3)R \subsetneq R$. Thus $\mathrm{grade}(I^R_{\varphi(X)} C(\varphi(X))) = 1$, which means that $\varphi(X)$ is not a super-primitive polynomial.

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