# Solitons on the Surface of the Binary Stars

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**Abstract**

Equations describing the surface waves of the binary stars are studied as an analogy to that of water on the earth. Except the weak gravity wave, such waves can be nonlinear ones. As is expected, the equations of the surface wave in the rotating frame of reference around the center of mass of two components are reduced to the so-called Korteweg-de Vries (KdV) equation and non-linear Schrödinger (NLS) equation acoording to its"depth". As is well known these equations have the solution of soliton. The solitons can cause various kind of activities of binary star systems especially when they are sent to the other component of a close binary system through the inner Lagrangian point.

## 1 Introduction

According to the recent result of astronomical observations, more than half of the stars are thought to be membership of some binary or multiple star systems. Therefore studies of the structure and evolution of not single stars but steller systems have become important for the purpose of understanding the real content of Galaxy and the Universe. From the point of view of stellar photometric observations, we could find various kind of activities that are not yet understood completely. The aim of this paper is to find out the equations which describe the nonlinear behavior of the surface waves like soliton solutions. Before preceeding to the equation itself, we describe the hydrodynamical properties of the suface of stars in section 2 where we show the incompressibility is valid. In section 3, we give basic equation describing the surface wave in the rotating frame of reference. In section 4, we derive an equation describing the propagation of the surface (shallow-water) wave which can be reduced to the so-called Korteweg-de Vries (KdV) equation whose solution is well known as solitons. In the final section we mention about the opposite (deep-water) case. Also some comments are given for further development of the theory.

## 2 Properties of matter on the stellar surface

In order to apply the theory of gravity wave to the stellar surface, we must be sure that the constituent matter is incompressible. It can be proved in the similar way to the case of pre-recombination universe.

The equation of state of the stellar material is a combination of gas and radiation. As the temperature of it is thought to be higher than that of hydrogen recombination (approximately 3000K), the pressure of gas and radiation are written as follows :

$$P_g = \frac{\rho}{\mu} R_g T \tag{1}$$

$$P_r = \frac{1}{3}\frac{4\sigma}{c} T^4 \tag{2}$$

where $\sigma$ is a Thomson scattering cross section, $R_g$ is the gas constant and $T$ and $\rho$ are temperature and density, respectively. As the total pressure of the matter is $P = P_g + P_r$, the sound velocity $c_s$ is

$$c_s = \sqrt{\frac{dp}{d\rho}} \simeq \sqrt{r\frac{P}{\rho}}. \tag{3}$$

This last relation is valid for normal (main sequence) stars like the Sun.

To calculate the compressibility defined as $\frac{1}{\rho}\frac{d\rho}{dp} = \frac{1}{\rho {c_s}^2}$, we insert the solar values into these quantities. As a result we obtain for the compressibility as

$$10^{-15}\mathrm{cm}^2/\mathrm{dyne}$$

This value is much smaller (by approximately $10^{-10}$) than that of water on the earth. Accordingly the validity of the incompressibility on the surface of stars is guaranteed.

## 3 Basic equations

The Euler equation which describes the flow of ionized gas in the binary systems is

$$\frac{D\vec{u}}{Dt} + 2\vec{\Omega}\times\vec{u} = -\frac{\nabla p}{\rho} - \nabla\phi_G - \nabla\phi_\Omega, \tag{4}$$

where $\vec{\Omega}$ is an angular velocity, and $\phi_G$ and $\phi_\Omega$ are gravitational and rotational potentials given as follows, respectively :

$$\nabla^2\phi_G = 4\pi G\rho,\ \nabla^2\phi_\Omega = -\frac{1}{2}(X^2 + Y^2). \tag{5}$$

The equation of continuity is

$$\frac{D\rho}{Dt} + \rho \,\mathrm{div}\,\vec{u} = 0. \tag{6}$$

As we can assume the incompressibe and irrotational flow for $\vec{u} = \mathrm{div}\Phi$, *i.e.*

$$\Delta\Phi = 0, \tag{7}$$

we obtain

$$\frac{\partial\Phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2}|\mathrm{grad}\Phi|^2 + \phi_G + \phi_\Omega + \psi = 0. \tag{8}$$

Here $\psi$ is due to the rotation of the system with respect to the inertial frame of reference given by the following equations :

$$\nabla\psi = 2\vec{\Omega}\times\nabla\Phi,\ \nabla^2\psi = 0. \tag{9}$$

As the surface of the star is the equipotential ones, we adopt a different coordinate system from the case on the earth. The most useful system is cylindrical one $(r,\theta,z)$, where $z$-axis is parallel to that of its rotation. Then we take the following coordinate system whose origin is the center of mass of the system :

$$x = r_0\theta,\ y = r - r_0,\ z = z. \tag{10}$$

Here, we assume the surface of the star is a sphere. [Fig. 1]

## 4 Equations of the surface wave — the case of shallow "water"

Let $\eta(x,z,t)$ describe the surface wave. According to the following kinematic boundary condition

$$\frac{DF}{Dt} = 0\ ;\ F = y - \eta(x,z,t), \tag{11}$$

we get

$$\frac{\partial\Phi}{\partial y} = \frac{\partial\eta}{\partial t} + \frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x}. \tag{12}$$

At $y = -h$ (bottom),

$$(\mathrm{grad}\Phi)_n = 0. \tag{13}$$

As $y = 0$ denotes the equipotential surface, from Euler equation we get

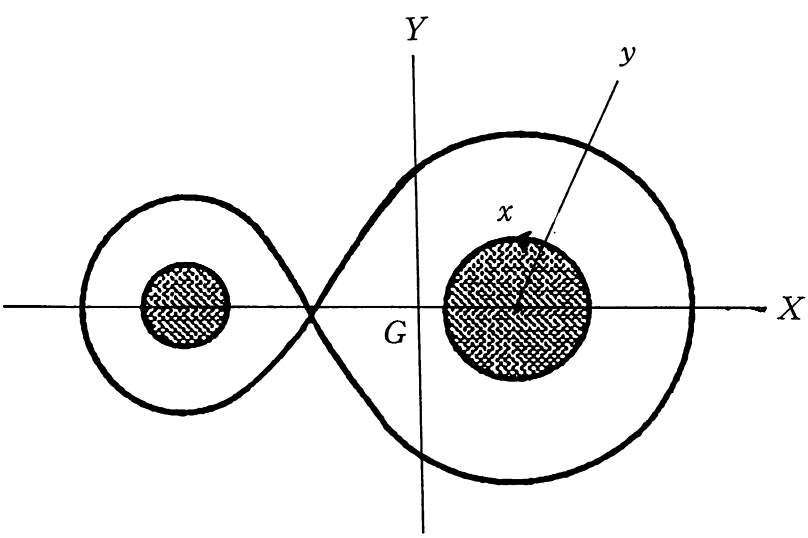


Fig. 1 Coordinate system adopted fo describing the surface waves.

$$\frac{\partial\Phi}{\partial t}+\frac{1}{2}((\frac{\partial\Phi}{\partial x})^2+(\frac{\partial\Phi}{\partial y})^2)+g\eta+\psi = 0, \tag{14}$$

where $g$ is a gravitational accelleration at the surface (radius $r_0$). At first we consider the case shallow-"Water". As usual, we take the so-called Gardner-Morikawa transformation :

$$\xi = \varepsilon^{1/2}(x-c_0t)\ ,\ \tau = \varepsilon^{3/2}t. \tag{15}$$

Here, $\varepsilon$ is expressed by $h$ (depth) and $k$ (wave number) as follows :

$$\varepsilon = h^2k^2. \tag{16}$$

Now we expand $\Phi$ (velocity potential) and $\eta$ as the following series :

$$\Phi(x,y,t) = \varepsilon^{1/2}[\Phi^{(1)}(\xi,y,\tau)+\varepsilon\Phi^{(2)}(\xi,y,\tau)+...], \tag{17}$$

$$\eta(x,t) = \varepsilon[\eta^{(1)}(\xi,\tau)+\varepsilon\eta^{(2)}(\xi,\tau)+...]. \tag{18}$$

By making use of the reductive peturbation method, we obtain for $\eta^{(1)}$

$$\frac{\partial\eta^{(1)}}{\partial\tau}+\frac{3}{2}\frac{g}{c_0}\frac{\partial\eta^{(1)}}{\partial\xi}+\frac{1}{6}c_0h^2\frac{\partial^3\eta^{(1)}}{\partial\xi^2}-\Omega(-\frac{1}{6}h^3\frac{\partial^3\eta^{(1)}}{\partial\xi^3}+h\frac{\partial\eta^{(1)}}{\partial\xi}) = 0, \tag{19}$$

where $\Omega$ is the $z$-component of $\vec{\Omega}$. The last term is additional to KdV equation. However, if we define $\zeta$ as

$$\zeta(\xi,\tau) = \eta^{(1)}-\frac{2c_0h}{3g}\Omega, \tag{20}$$

we get the following equation

$$\frac{\partial\zeta}{\partial\tau}+\frac{3g}{2c_0}\zeta\frac{\partial\zeta}{\partial\zeta}+\frac{1}{6}(c_0h^2+\Omega h^3)\frac{\partial^3\zeta}{\partial\xi^3} = 0. \tag{21}$$

This is exactly the same form as the Korteweg-de Vries (KdV) equation, whose solution is solitary wave (soliton!).

## 5 Discussion — the case of deep "water" and other cases

For majority of the lobe-filling components of binary systems, it is rather realistic to consider the opposite case of "deep-water" wave (no actual bottom). To treat this problem, we make use of the procedure for deriving the envelope soliton. At first, we assume the potential $\psi$ in the basic equation of motion Eq (8) is connected with the Keplerian rotation frequency $\Omega$, in such a way as

$$\psi = g'\eta.$$

Then from Eq (9) and the relation $\eta \sim exp\{i(k_0x-\omega_0t)\}$, we get

$$g' = 2\frac{\omega_0}{k_0}\Omega.$$

We introduce a small parameter $\varepsilon \sim |k-k_0|$, and another G-M transformation given by

$$\xi = \varepsilon(x - c_{g0}t)\ ,\ \tau = \varepsilon^2 t \tag{22}$$

and furthermore the following series expansion :

$$\eta = \varepsilon(\eta^{(1,1)}(\xi,\tau) + c.c) + \varepsilon^2(\eta^{(2,0)} + \eta^{(2,1)}E + \eta^{(2,2)}E^2 + c.c) + \dots, \tag{23}$$

$$\Phi = \varepsilon(\phi^{(1,0)}(\xi,y,\tau) + \phi^{(1,1)}E + c.c) + \varepsilon(\phi^{(2,0)} + \phi^{(2,1)}E + \dots) + \dots, \tag{24}$$

where

$$E = \exp[i(k_0 x - \omega t)]$$

and

$$\omega_0 = \omega(k_0).$$

Substituting these into Eqs (8), (9), (10) and (11), we obtain the following Non Linear Schrödinger (NLS) equation,

$$i\frac{\partial \eta^{(1,1)}}{\partial \tau} + \mu\frac{\partial^2 \eta^{(1,1)}}{\partial \xi^2} + \nu|\eta^{(1,1)}|^2\eta^{(1,1)} = 0 \tag{25}$$

with the dispersion relation

$$\omega_0 = \sqrt{k_0(g+g')\sigma}\ ,\ c_{p0} = \frac{\omega_0}{k_0}\ ,\ \sigma = \tanh k_0 r_0 \tag{26}$$

and formula of the group velocity

$$c_{g0} = \frac{\omega_0}{2k_0\sigma}[\sigma + k_0 r_0(1-\sigma^2)]. \tag{27}$$

The explict forms of the coefficients $\mu$ and $\nu$ are not given here because of their tediousness. NLS equation has a sort of soliton solution such as

$$\eta^{(1,1)} = A\mathrm{sech}\sqrt{\frac{A^2\nu}{2\mu}}\xi\exp(i\frac{A^2\nu}{2}\xi).$$

Finally we mention about the most spectacular case of close binary star systems where one component fills the so-called Roche-lobe (common equipotential suface of the two components). In this case, the solitary waves will surely be transferred to the other component through the inner Lagrangian point. The treatment near the cusp seems to be difficult but not so hopeless. The study by Kakutani[1] on "Tsunami" is useful for our case. He derived the following soliton equation for the shallow water case of the uneven bottom :

$$\frac{\partial \eta^{(1)}}{\partial \tau} + c_1 \eta^{(1)} \frac{\partial \eta^{(1)}}{\partial \xi} + c_2 \frac{\partial^3 \eta^{(1)}}{\partial \xi^3} - c_3 \frac{dB}{d\tau} \eta^{(1)} = 0.$$

The problem is how we interpret the "bottom". Our case seems to be a counterpart of this for that of deep "water". At any rate soliton can be a model of flare-like light curves obtained in some close binary systems detected by the high-speed photometry (See Tanabe et al[2], and Marar et al[3]).

**References**

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3) T. M. K. Marar et al, Astron. Astrophys. Vol. **189**, p119 (1988).