Boundary Value Problems for the Shilov Boundary of the Siegel Upper Half Plane

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1 Introduction

In this article we consider the problem of characterizing the image of the Poisson transforms from the Shilov boundary of the Siegel upper half plane H_n to a homogeneous line bundle over H_n ;

$$H_n = \{Z \in M_n(\mathbb{C}) \ ; \ {}^tZ = Z, \operatorname{Im} Z > 0\} \simeq Sp(n, \mathbb{R})/U(n).$$

The space $\{Z \in M_n(\mathbb{R}) \ ; \ {}^tZ = Z\}$ is a dense subspace of the Shilov boundary of H_n, which is a small component of the boundary of H_n in the maximal Satake compactification.

We define the Poisson integrals of functions on the Shilov boundary, which is a generalization of the Poisson integrals on the complex upper half plane. The main result (Theorem 3. 1) asserts that we can characterize the image by second-order differential equations.

In the case of trivial line bundle over H_n, i. e. the case of functions on H_n, the answer was given by Johnson[5] for harmonic functions and by Sekiguchi[6] for arbitrary eigenfunctions of invariant differential operators on H_n. Our result is a generalization of their results to the case of homogeneous line bundles over H_n.

To prove the theorem we use techniques and results in Shimeno[7)8)9)].

2 Notation and preliminary results

2. 1 Notation

Let

$$G = Sp(n, \mathbb{R}) = \{g \in SL(2n, \mathbb{R}) \ ; \ {}^tgJg = J\},$$

where

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

and I_n is $n \times n$ identity matrix. The group $K = O(2n) \cap Sp(n, \mathbb{R})$ is a maximal compact subgroup of G, which is isomorphic to $U(n)$ by

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in K \rightarrow A+\sqrt{-1}B \in U(n).$$

Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K respectively. Let θ denote corresponding Cartan involution of G and $\mathfrak{g}$. We have a Cartan decomposition $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$, where $\mathfrak{p}$ is the -1-eigenspace of θ in $\mathfrak{g}$.

For $l \in \mathbb{Z}$ let τ_l denote the one-dimensional representation of $U(n)$ given by $\tau_l(x) = (\det x)^l\,(x \in U(n))$ and we denote corresponding representation of K and $\mathfrak{k}$ by the same notation.

Let E_{ij} denote the $n\times n$ matrix with (i, j)-entry 1 and all other entries being 0. We choose a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{u}(n)$ to be the set of diagonal matrices. We define $\varepsilon_i \in \sqrt{-1}\mathfrak{t}^*$ by $\varepsilon_i(E_{jj}) = \delta_{ij}(1 \leq i, j \leq n)$. Let Δ denote the root system of $(\mathfrak{g}, \mathfrak{t})$ and Δ^+ be the positive system of Δ given by

$$\Delta^+ = \{2\varepsilon_i,\ \varepsilon_j\pm\varepsilon_k\ ;\ 1 \leq i \leq n,\ 1 \leq j < k \leq n\}.$$

For $\gamma \in \Delta$ let $\mathfrak{g}_\gamma \subset \mathfrak{g}_{\mathbb{C}} = \mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}$ cenote the root space for γ. Let $\mathfrak{p}^\pm = \sum_{\gamma\in\Delta_n^+}\mathfrak{g}_{\pm\gamma}$, where Δ_n^+ is the set of non-compact positive roots. We have $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}^+ + \mathfrak{p}^-$ and $[\mathfrak{p}^+, \mathfrak{p}^-] = \mathfrak{k}_{\mathbb{C}}$.

We put

$$X_i = \begin{pmatrix} E_{ii} & 0 \\ 0 & -E_{ii} \end{pmatrix} \in \mathfrak{p} \quad (1 \leq i \leq n)$$

and $\mathfrak{a} = \sum_{i=1}^n \mathbb{R} X_i$. Then $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$. We put $X_0 = X_1+\cdots+X_n$. Let $e_i(1 \leq i \leq n)$ be the linear from on $\mathfrak{a}$ given by $e_i(X_j) = \delta_{ij}$. Let Σ denote the restricted root system of the pair $(\mathfrak{g}, \mathfrak{a})$ and Σ^+ be the positive system of Σ given by

$$\Sigma^+ = \{2e_i,\ e_j\pm e_k\ ;\ 1 \leq i \leq n,\ 1 \leq j < k \leq n\}.$$

For $\alpha \in \Sigma$ let $\mathfrak{g}^\alpha \subset \mathfrak{g}$ be the root space for α. For $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{R})$ we have $\dim\mathfrak{g}^\alpha = 1$ for all $\alpha \in \Sigma$. We put $\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+}\alpha$. For any $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ let A_λ be the element of $\mathfrak{a}_{\mathbb{C}}$ determined by $B(H, A_\lambda) = \lambda(H)$ for all $H \in \mathfrak{a}$, where B denotes the Killing form of $\mathfrak{g}_{\mathbb{C}}$. For $\lambda, \mu \in \mathfrak{a}_{\mathbb{C}}^*$ we put $\langle\lambda, \mu\rangle = B(A_\lambda, A_\mu)$. Since $\{e_1,\cdots, e_n\}$ forms a basis of $\mathfrak{a}^*$, any $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ can be written as $\lambda = \sum_{i=1}^n\lambda_i e_i(\lambda_i \in \mathbb{C})$. We identify $\mathfrak{a}_{\mathbb{C}}^*$ with $\mathbb{C}^n$ by $\lambda\rightarrow(\lambda_1,\cdots, \lambda_n)$. In this identification we have $\rho = (n, n-1,\cdots, 1)$.

Let A be the analytic subgroups of G corresponding to $\mathfrak{a}$. Let $\mathfrak{n}^+ = \sum_{\alpha\in\Sigma^+}\mathfrak{g}^\alpha$ and $\mathfrak{n}^- = \theta(\mathfrak{n}^+)$. Let N^+ and N^- be the corresponding analytic subgroups of G. Let M be the centralizer of $\mathfrak{a}$ in K. The subgroup $P = MAN^+$ is a minimal parabolic subgroup of G.

We cosider the subset $\Xi = \{e_i - e_{i+1}\ ;\ 1 \leq i \leq n-1\}$ of simple roots and corresponding standard parabolic subgroup P_Ξ of G with the Langlands decomposition $P_\Xi = M_\Xi A_\Xi N_\Xi^+$ such that $A_\Xi \subset A$. We have

$$\mathfrak{a}_{\Xi} = \mathbb{R} X_0, \quad \mathfrak{a}(\Xi) = \sum_{i=1}^{r-1} \mathbb{R}(X_i - X_{i+1}),$$

2. 2 Eigenspaces of invariant differential operators

We review the main reslt of Shimeno[7], which gives a characterization of the image of the Poisson transform.

For a real analytic manifold X we denote by $\mathcal{B}(X)$ the space of all hyperfunctions on X. Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $l \in \mathbb{Z}$. We define

$$\mathcal{B}(G/P, L_{\lambda,l}) = \{f \in \mathcal{B}(G) \; ; \; f(gman) = e^{(\lambda-\rho)(\log a)} \tau_l(m)^{-1} f(g) \\ g \in G, \; m \in M, \; a \in A, \; n \in N^+\}$$

and

$$\mathcal{B}(G/K \; ; \; \tau_l) = \{u \in \mathcal{B}(G) \; ; \; u(gk) = \tau_l(k)^{-1} u(g) \text{ for any } g \in G, \; k \in K\},$$

For $f \in \mathcal{B}(G/P \; ; \; L_{\lambda,l})$, we define the Poisson integral $\mathcal{P}_{\lambda,l} f$ by

$$\mathcal{P}_{\lambda,l} f(g) = \int_k f(gk) \tau_l(k) dk.$$

Here dk denotes the invariant measure on K with total measure 1.

For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $l \in \mathbb{Z}$ let $\varphi_{\lambda,l}$ denote the Poisson integral of the function $1_{\lambda,l} \in \mathcal{B}(G/P \; ; \; L_{\lambda,l})$ with $1_{\lambda,l}|_K = \tau_{-l}$, i. e.,

$$\varphi_{\lambda,l}(g) = \int_K \tau_l(k^{-1} \kappa(g^{-1}k)) \exp \langle -\lambda - \rho, H(g^{-1}k) \rangle \; dk.$$

Let $\mathbb{D}_l(G/K)$ denote the algebra of invariant differential operators on $\mathcal{B}(G/K \; ; \; \tau_l)$ and $L_l \in \mathbb{D}_l(G/K)$ denote the Laplace-Beltrami operator acting on $\mathcal{B}(G/K \; ; \; \tau_l)$. We have the Harish-Chandra isomorphism

$$\gamma_l : \mathbb{D}_l(G/K) \tilde{\rightarrow} S(\mathfrak{a}_{\mathbb{C}})^W,$$

where $S(\mathfrak{a}_{\mathbb{C}})^W$ denotes the set of W-invariant elements in the symmetric algebra $S(\mathfrak{a}_{\mathbb{C}})$. For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $l \in \mathbb{Z}$ let $\mathcal{A}(G/K, \mathcal{M}_{\lambda,l})$ denote the space of all real analytic functions in $\mathcal{B}(G/K, \tau_l)$ satisfying the system of differential equations,

$$\mathcal{M}_{\lambda,l} : D_u = \gamma_l(D)(\lambda) u, \quad D \in \mathbb{D}_l(G/K).$$

For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $l \in \mathbb{Z}$ we define

$$e_{\lambda,l}^{-1} = \prod_{1 \leq j < k \leq n} \Gamma(\frac{1}{2}(1 + \lambda_j + \lambda_k)) \Gamma(\frac{1}{2}(1 + \lambda_j - \lambda_k)) \\ \times \prod_{1 \leq i \leq n} \Gamma(\frac{1}{2}(\lambda_i + 1 + l)) \Gamma(\frac{1}{2}(\lambda_i + 1 - l)).$$

Theorem 2. 1 (Shimeno[7)]) *If* $\lambda \in \mathfrak{a}_{\mathbf{C}}^{*}$ *satisfies the condition*

$$-2\frac{\langle\lambda, \alpha\rangle}{\langle\alpha, \alpha\rangle} \notin \{1, 2, 3, \cdots\} \text{ for all } \alpha \in \Sigma^{+} \tag{2. 1}$$

$$e_{\lambda, l} \neq 0, \tag{2. 2}$$

then the Poisson transform $\mathcal{P}_{\lambda, l}$ *is a* G*-isomorphism of* $\mathcal{B}(G/P\ ;\ L_{\lambda, l})$ *onto* $\mathcal{A}(G/K, \mathcal{M}_{\lambda, l})$.

Under the condition of the above theorem, the inverse of the Poisson transform is given by the boundary value map up to a non-zero constant multiple.

3 The Poisson transforms and the Hua equations

3. 1 Poisson transform for degenerate series representations

Let $s \in \mathbf{C}$ and $\varepsilon \in \{0, 1\}$. Let σ_ε denote the one-dimensional representation of M_Ξ that corresponds to the character $g \rightarrow (\mathrm{sgn}(\det g))^{\varepsilon}$ $(g \in GL(n, \mathbb{R}))$ by the isomorphism $M_\Xi \simeq GL(n, \mathbb{R})$. Let ρ' be the linear form on $\mathfrak{a}_\Xi$ defined by $\rho'(X_0) = n$. We define

$$\mathcal{B}(G/P_\Xi\ ;\ \varepsilon, s) = \{f \in \mathcal{B}(G)\ ;\ f(gman) = \sigma_\varepsilon(m)^{-1} e^{(s-\frac{1}{2}(n+1))\rho'(\log a)} f(g),$$
$$g \in G,\ m \in M_\Xi,\ a \in A_\Xi,\ n \in N_\Xi^{+}\}.$$

For $s \in \mathbf{C}$ we define $\lambda_s^\Xi \in \mathfrak{a}_{\mathbf{C}}^{*}$ by

$$\lambda_s^\Xi = \left(s + \frac{1}{2}(n-1),\ s + \frac{1}{2}(n-3), \cdots,\ s - \frac{1}{2}(n-1)\right). \tag{3. 1}$$

Then we have

$$\mathcal{B}(G/P_\Xi\ ;\ \varepsilon, s) \subset \mathcal{B}(G/P, L_{\lambda_s^\Xi, l}),$$
$$\mathcal{P}_s(\mathcal{B}(G/P_\Xi\ ;\ \varepsilon, s)) \text{ and } \subset \mathcal{A}(G/K, \mathcal{M}_{\lambda_s^\Xi, l}). \tag{3. 2}$$

For $f \in \mathcal{B}(G/P_\Xi\ ;\ \varepsilon, s)$ and $l \in \mathbb{Z}$ with $l \equiv \varepsilon$ mod 2 we define

$$\mathcal{P}_{\Xi, s, l} f(x) = \int_K f(xk) \tau_l(k) dk.$$

Let $P_{s,l}^\Xi$ be a unique function on G such that $g \mapsto P_{s,l}^\Xi(g^{-1})$ is an element of $\mathcal{B}(G/P_\Xi\ ;\ \varepsilon, -\bar{s})$ and $P_{s,l}^\Xi|K = \tau_{-l}$. A straightforward calculation shows that

$$(\mathcal{P}_{\Xi, s, l} f)(x) = \int_K P_{s,l}^\Xi(k^{-1}x) f(k) dk. \tag{3. 3}$$

Let $\{E_i\}$ be a basis of $\mathfrak{p}^{+}$ and $\{E_j^{*}\}$ be the dual basis of $\mathfrak{p}^{-}$ with respect to B. Let $U(\mathfrak{g}_{\mathbf{C}})$ denote the universal enveloping algebra of $\mathfrak{g}_{\mathbf{C}}$. We cosider the element of $U(\mathfrak{g}_{\mathbf{C}}) \otimes \mathfrak{k}_{\mathbf{C}}$ defined by

$$\mathcal{H}_{\mathfrak{k}_s}^{\Xi} = \sum_{i,j} E_i E_j^{*} \otimes p([E_j, E_i^{*}]), \tag{3. 4}$$

where p denotes the orthogonal projection of $\mathfrak{k}_{\mathbf{C}}$ onto $(\mathfrak{k}_s)_{\mathbf{C}} = [\mathfrak{k}_{\mathbf{C}}, \mathfrak{k}_{\mathbf{C}}]$. Notice that $\mathcal{H}_{\mathfrak{k}_s}^{\Xi}$ defines a homogeneous differential operator from $C^\infty(G/K\ ;\ r_l)$ to

$C^\infty(G/K, \tau_l \otimes \mathrm{Ad}_K|(\mathfrak{k}_s)_c)$, which does not depend on the choice of basis. We call $\mathcal{H}_{\mathfrak{k}_s}^{\Xi}$ the Hua operator.

We state the main result of this article :

Theorem 3. 1 *Assume $s \in \mathbb{C}$ and $l \in \mathbb{Z}$ satisfy the condition,*

$$-s+\frac{n-1}{2} \notin \{1-|l|, 2-|l|, 3-|l|, \cdots\} \cup \{3/2, 5/2, 7/2, \cdots\}. \tag{3. 5}$$

Then the Poisson transform is a G-isomorphism of $\mathcal{B}(G/P_\Xi\ ;\ \varepsilon, s)$ onto the space of analytic functions u on X that satisfy

$$\mathcal{H}_{\mathfrak{k}_s}^{\Xi}\, u = 0, \tag{3. 6}$$

$$L_l u = \frac{n}{4(n+1)}(s+\frac{n+1}{2})(s-\frac{n+1}{2})u. \tag{3. 7}$$

Notice that condition (3. 5) is equivalent to conditions (2. 1) and (2. 2) for λ_s^{Ξ}. The proof of the theorem is divided into three steps ;

1 . Any element in the image of $\mathcal{P}_{\Xi,s,l}$ satisfies (3. 6) (Proposition 3. 2),

2 . Solutions of (3. 6) and (3. 7) satisfy $\mathcal{M}_{\lambda_s^{\Xi},l}$ (Proposition 3. 3),

3 . Under condition (2, 1) and (2. 2) boundary values of solutions of (3. 6) are contained in $\mathcal{B}(G/P_\Xi\ ;\ \varepsilon, s)$ (Section3. 3).

Proposition 3. 2 *For any f in $\mathcal{B}(G/P_\Xi\ ;\ \varepsilon, s)$, $u = \mathcal{P}_{s,l}^{\Xi} f$ satisfies* (3. 6).

Proof. It is sufficient to show that $u = P_{s,l}^{\Xi}$ satisfies (3. 6). We put $F = \mathcal{H}_{\mathfrak{k}_s}^{\Xi} P_{s,l}^{\Xi}$. It is a function from G to $(\mathfrak{k}_s)_c$ such that

$$F(namk) = \sigma_\varepsilon(m)e^{(\bar{s}+\frac{1}{2}(n+1))\rho'(\log a)}\tau_l(k)^{-1}\mathrm{Ad}(k)F(e),$$

for all $k \in K$, $m \in M_\Xi$, $a \in A_\Xi$ and $n \in N_\Xi$, and in particular,

$$F(mk) = \sigma_\varepsilon(m)\tau_l(k)^{-1}\mathrm{Ad}(k)F(e),$$

for all $k \in K$ and $m \in M_\Xi \cap K$. We will show that F must be identically zero. The existence of such F is equivalent to the existence of vectors in $\mathfrak{su}(n)$ that follow the representation σ_ε under the adjoint action of $O(n)$. Since $\mathfrak{su}(n)$ is the irreducible representation of $U(n)$ with highest weight $\varepsilon_1-\varepsilon_n$, it has no non-zero $SO(n)$-fixed vector for $n \geq 3$ by the Cartan-Helgason theorem (cf. Helgason[3], Chapter 5, Corollary 4. 2). For $n = 2$,

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is a unique $SO(2)$ fixed vector in $\mathfrak{su}(2)$ up to constant. But this vector does not follow $\sigma_\varepsilon \equiv 1$ under the adjoint action of

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in O(2).$$

3. 2 Radial parts of the Hua equations

Proposition 3. 3 *Any solution of* (3. 6) *and* (3. 7) *is contained in* $\mathcal{A}(G/K, \mathcal{M}_{\lambda_s^\varepsilon, l})$. We shall prove that $\phi_{\lambda_s^\varepsilon, l}$ is a unique solution of (3. 6) and (3. 7) such that $u(kx) = \tau_l(k)^{-1}u(x)$ for all $k \in K$ and $x \in G$ (Corollary 3. 5). Then we can prove Proposition 3. 3 in the same way as the proof of Shimeno[9], Theorem 3. 3. In the proof of Theorem 3. 3[9] we use a characterization of joint eigenfunctions of $\mathbb{D}(G/K)$ by means of an integral formula (Helgason[3], Ch IV, Proposition 2. 4), which can be generalized to the case of a homogeneous line bundle easily.

We define elements $T_i (i = 1, \cdots, n)$, $X_{\pm 2\varepsilon_i} \in \mathfrak{g}_{\pm 2\varepsilon_i} (i = 1, \cdots, n)$ and $X_{\pm\varepsilon_j \pm \varepsilon_k} \in \mathfrak{g}_{\pm\varepsilon_j\pm\varepsilon_k}$ $(1 \leq j \neq k \leq n)$ by

$$T_i = \begin{pmatrix} 0 & E_{ii} \\ -E_{ii} & 0 \end{pmatrix}, \quad (1 \leq i \leq n)$$

$$X_{2\varepsilon_i} = \begin{pmatrix} E_{ii} & \sqrt{-1}E_{ii} \\ \sqrt{-1}E_{ii} & -E_{ii} \end{pmatrix} \quad (1 \leq i \leq n),$$

$$X_{\varepsilon_j+\varepsilon_k} = \begin{pmatrix} E_{jk}+E_{kj} & \sqrt{-1}(E_{jk}+E_{kj}) \\ \sqrt{-1}(E_{jk}+E_{kj}) & -E_{jk}-E_{kj} \end{pmatrix} \quad (1 \leq j < k \leq n),$$

$$X_{\varepsilon_j-\varepsilon_k} = \begin{pmatrix} E_{jk}-E_{kj} & -\sqrt{-1}(E_{jk}+E_{kj}) \\ \sqrt{-1}(E_{jk}+E_{kj}) & E_{jk}-E_{kj} \end{pmatrix} \quad (1 \leq j < k \leq n),$$

and $X_{-\beta} = \bar{X}_\beta$ $(\beta = 2\varepsilon_1, 2\varepsilon_2, \varepsilon_1 \pm \varepsilon_2)$.

A function u on G is called (τ_{-m}, τ_{-l})-spherical when it satisfies

$$u(k_1 g k_2) = \tau_m(k_1)^{-1} u(g) \tau_l(k_2)^{-1} \text{ for all } g \in G,\ k_1, k_2 \in K.$$

Proposition 3. 4 *If* $u \in C^\infty(G)$ *is* (τ_{-m}, τ_{-1})*-spherical and satisfies* $\mathcal{H}_{t_s}^{\varepsilon} u = 0$, *then the function*

$$\phi(t) = h(t\,;\, m,\, l) u(\exp(\textstyle\sum_{j=1}^n t_j X_j)) \tag{3. 8}$$

satisfies

$$\begin{aligned}(\partial_{tj}^2 + (2(1-l-m)\coth 2t_j + 2m \coth t_j)\partial_{tj} \\ -\partial_{tk}{}^2 - (2(1-l-m)\coth 2t_k + 2m \coth t_k)\partial_{tk})\phi = 0\end{aligned} \tag{3. 9}$$

for all $1 \leq j < k \leq n$, *where* $h(t\,;\, m,\, l)$ *is given by*

$$h(t\,;\, m,\, l) = \left(\prod_{j=1}^n \cosh t_j\right)^{\frac{1}{2}(m+l)} \left(\prod_{j=1}^n \sinh t_j\right)^{\frac{1}{2}(l-m)}$$

Proof. The coefficient of $T_j - T_k$ $(1 \leq j < k \leq n)$ in $\mathcal{H}_{t_s}^{\varepsilon}$ is given by

$$\sum_i [T_j - T_k, E_i] E_i^*,$$

which is a non-zero constant multiple of

$$X_{\gamma_j}X_{-\gamma_j}-X_{\gamma_k}X_{-\gamma_k}. \tag{3. 10}$$

By Iida[4], Lemma 5. 1, (τ_{-m}, τ_{-l})-radial part of (3. 10) is given by

$$\partial_{t_j}^2-\partial_{t_k}^2+2\coth 2t_j\partial_{t_j}-2\coth 2t_k\partial_{t_k}$$
$$-(l \coth 2x-m \sinh^{-1} 2x)^2+(l \coth 2y-m \sinh^{-1} 2y)^2.$$

Thus (3. 9) follows from a straightforward computation. □

It follows from Proposition 3. 4 and Shimeno[8], Proposition 2. 6 that the function u is a (τ_{-m}, τ_{-l})-spherical solution of (3. 6) and (3. 7) if and only if the function ϕ given by (3. 8) is a solution of

$$\begin{aligned}&(\partial_{t_i}^2+(2m \coth t_i+2(1-l-m)\coth 2t_i)\partial_{t_i}\\ &\quad+\sum_{j\neq i}(\coth(t_i+t_j)(\partial_{t_i}+\partial_{t_j})+(\coth(t_i-t_j)(\partial_{t_i}-\partial_{t_j}))\phi\\ &=(s^2-(\frac{1}{2}(n+1)-l)^2)\phi\end{aligned} \tag{3. 11}$$

for $1\leq i\leq n$.

Corollary 3. 5 *Let u be a (τ_{-m}, τ_{-l})-spherical solution of* (3. 6) *and* (3. 7). *Then the function ϕ given by* (3. 8) *is a constant multiple of the hypergeometric function* $F(\exp(\sum_{i=1}^n t_iX_i)\ ;\ \lambda_s^{\Xi}, k)$ *of Heckman and Opdam, where k is given by* $k_{\pm e_j\pm e_k}=1/2\ (1\leq j\neq k\leq n)$ *and* $k_{e_i}=m$, $k_{2e_i}=1/2(1-l-m)\ (1\leq i\leq n)$. *In particular, if* $l=m$, *then $\phi(t)$ is a constant multiple of* $h(t\ ;\ l, l)\varphi_{\lambda_s^{\varepsilon}, l}(\exp(\sum_{i=1}^n t_iX_i))$.

Proof. By Yan[10], Theorem 2. 1, there is a unique solution of (3. 11) for $1\leq i\leq n$ up to constant subject to condition that it is W-invariant and analytic at $t=0$. Moreover, by Beerends-Opdam[1], Theorem 4. 2, it is the hypergeometric function of Heckman and Opdam. The latter statement follows from Shimeno[8], Remark 3. 8. □

3. 3 Induced equations

Put

$$Y_j=\frac{\sqrt{-1}}{2}(X_{\varepsilon_j-\varepsilon_{j+1}}-X_{\varepsilon_{j+1}-\varepsilon_j})\in\mathfrak{k}\quad(1\leq j\leq n-1).$$

The coefficient of Y_j in $\mathcal{H}_{t_s}^{\Xi}$ is given by

$$\sum_i[Y_j, E_i]E_i^*=\sqrt{-1}\{X_{\varepsilon_j+\varepsilon_{j+1}}(X_{2\varepsilon_j}^*+X_{2\varepsilon_{j+1}}^*)+2(X_{2\varepsilon_j}+X_{2\varepsilon_{j+1}})X_{\varepsilon_j+\varepsilon_{j+1}}^*\},$$

which is a non-zero constant multiple of

$$X_{\varepsilon_j+\varepsilon_{j+1}}(X_{-2\varepsilon_j}+X_{-2\varepsilon_{j+1}})+(X_{2\varepsilon_j}+X_{2\varepsilon_{j+1}})X_{-\varepsilon_j-\varepsilon_{j+1}}. \tag{3.12}$$

It can be seen by direct computations that operator (3. 12) induces on $\mathcal{B}(G/P\ ;\ L_{\lambda_s^{\varepsilon}, l})$ differential operator $(2n+2-2s-j)E_{e_{j+1}-e_j}$, where $E_{e_{j+1}-e_j}\in\mathfrak{g}^{e_{j+1}-e_j}$ is a non-zero root vector. Indeed (3. 12) is equivalent to an element of $U(\mathfrak{a}_{\mathbf{C}}+\mathfrak{n}_{\bar{\mathbf{C}}})$ modulo $U(\mathfrak{g}_{\mathbf{C}})(\mathfrak{k}_s)_{\mathbf{C}}$ and computations does not depend on l. Since $\mathfrak{g}^{e_{j+1}-e_j}(1\leq j\leq n-1)$ generate $\mathfrak{n}(\Xi)_{\bar{\mathbf{C}}}$, the boundary value of solutions of (3. 6) and (3. 7) are contained in $\mathcal{B}(G/P_{\Xi}\ ;\ \varepsilon, s)$ for $\varepsilon\equiv l$ modulo 2 and Theorem 3. 1 follows. See Shimeno[9] for details.

Remark 3. 6 We can generalize Theorem 3. 1 for all Hermitian symmetric spaces of tube type. We will discuss this point in a forthcoming paper.

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