Physico-Mathematical Consideration of the Multi-Relaxation in Solids

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Abstract

An axiomatic theory is developed on the multi-relaxation in crystalline solids. Mathematically, it is a theory on a general linear ordinary differential equation of two infinitely differentiable functions: here, the stress and strain. The differential equation is, on the other hand, a defining equation of the general linear solid which represents the multi-relaxation of strain after the applied stress. Properties found in the mathematical theory are interpreted from a physical point of view, by putting emphasis on natural correspondence between physico-mathematical properties.

1. Introduction

One of the important anelastic relaxations in solids is the point-defect relaxation[1]. The point defects are classified into several types; the simplest are one, called a vacancy, produced in a crystal by removing an atom, a substitutional by substituting an atom of a different species, and an interstitial by bringing an extra atom either of the same or a different species into an interstitial position between normal lattice sites. In contrast to these elementaly point defects, there are composite defects made up with a cluster of foreign, extra or missing atoms extending over a few of lattice sites. The existence of the defect breaks the translational symmetry and may lower the original crystalline symmetry belonging to the point group of the perfect crystal. The point-group symmetry around the point defect is called the defect symmetry to distinguish it from the site symmetry of the perfect crystal.

The anelastic relaxation is defined as a relaxation of the strain after the applied stress. The defect symmetry lower than the site symmetry induces an anelastic relaxation called the point-defect relaxation. Each species of the point defect may cause an anelastic relaxation of different relaxation time. There appears a multi-relaxation due to several kinds of point defects present in the crystal.

The defective crystal is considered homogeneous; that is, the point defects are uniformly distributed all over the crystal. The applied stress, furthermore, is taken to be homogeneous, so that the strain relaxation around the defect is a function of time,

indifferent of the defect position. The stress and strain are, therefore, given as a fuction of time only, independent of their position.

The purpose of the present paper is to develop a mathematical theory of the multi-relaxation in crystalline solids, in comparison with the physical meaning of the corresponding mathematical properties.

2. The General Linear Solid

Let ε and σ be functions of time in class C^{∞}: the class of all infinitely differentiable real fuctions, called the strain and stress, respectively. A solid is a continuous body whose properties are specified by the stress-strain equation.

Definition 1 A solid is said to be elastic, if

$$\varepsilon = b_0\sigma, \tag{2.1}$$

with a positive real constant b_0 called the compliance, usually written as J_U.

The stress-strain relation of eqn. (2.1) is the so-called "Hooke's law"[2,3]. The elastic solid is represented by a mechanical model of the Hookean spring.

Definition 2 A solid satisfying the equation with a positive coefficient b_0

$$\dot{\varepsilon} = b_0\sigma \tag{2.2}$$

is said to be Newtonian, whose mechanical model is the Newtonian dashpot. The quantity of $1/b_0$ is a physical quantity called the viscosity, denoted by η.

Definition 3 The Voigt solid is defined as a solid where the stress and strain follow the equation:

$$\dot{\varepsilon}+a_1\varepsilon = b_0\sigma. \tag{2.3}$$

Here, a_1 and b_0 belong to the set of all positive real numbers $\mathbf{R}^+$.

The constant $1/a_1$ has the dimension of time, written as τ_σ; i.e., $\tau_\sigma = 1/a_1$, which is called the relaxation time.

$$\delta J_R = \tau_\sigma b_0 = \frac{b_0}{a_1} \tag{2.4}$$

is the relaxation of the compliance. The stress-strain equation of the Voigt solid is, then, given by

$$\dot{\varepsilon}+\frac{1}{\tau_\sigma}\varepsilon = \frac{\delta J_R}{\tau_\sigma}\sigma. \tag{2.5}$$

Definition 4 The standard linear solid (abbreviated to SLS) is specified by the differential stress-strain equation:

$$\dot{\varepsilon}+a_1\varepsilon = b_0\dot{\sigma}+b_1\sigma \qquad (a_1,\ b_0,\ b_1 \in \mathbf{R}^+) \tag{2.6}$$

or

$$\dot{\varepsilon}+\frac{1}{\tau_\sigma}\varepsilon = J_U\dot{\sigma}+\frac{J_U+\delta J_R}{\tau_\sigma}\sigma \quad , \quad \dot{\varepsilon}+\frac{1}{\tau_\sigma}\varepsilon = J_U\left(\dot{\sigma}+\frac{1+\Delta}{\tau_\sigma}\sigma\right) \tag{2.7}$$

with $\tau_\sigma = 1/a_1$, $J_U = b_0$, $\delta J_R = b_1/a_1 - b_0$ and the relaxation strength $\Delta = \delta J_R/J_U$.

The differential stress-strain equation of the SLS is extended to a differential general linear equation.

Definition 5 The general linear solid is defined as a solid following the differential general linear equation:

$$\sum_{i=0}^{n} a_i\varepsilon^{(n-i)} = \sum_{i=0}^{n} b_i\sigma^{(n-i)} \qquad (a_i, b_i \in \mathbf{R}^+, a_0 = 1), \tag{2.8}$$

where $\varepsilon^{(n-i)}$, $\sigma^{(n-i)}$ are $(n-i)$ -fold time derivatives of the strain and stress.

3. Diffusion Operator

A differential operator D_i

$$D_i \overset{\text{def}}{=} \frac{d}{dt}+\frac{1}{\tau_\sigma^{(i)}}I = d_t+\frac{1}{\tau_\sigma^{(i)}}I \tag{3.1}$$

is introduced with $d_t \overset{\text{def}}{=} \frac{d}{dt}$ and the identity operator I. The oprator D_i is called the diffusion operator which appears in the equation of diffusion of the density[3]. The relaxation functions are C^∞ functions: φ_i and ψ_i, defined as

$$\varphi_i+\psi_i = 1, \tag{3.2}$$

$$\psi_i(t) = \exp(-t/\tau_\sigma^{(i)}). \tag{3.3}$$

Several propositions on the diffusion operator and the relaxation functions are summarized without proof (see refs. 3 or 4).

Proposition 1

(1) D_i is a linear operator of C^∞ into C^∞.

(2) $D_i\psi_i = 0$.

(2′) $D_i\varphi_i = D_i 1 = \frac{1}{\tau_\sigma^{(i)}}$.

(3) $D_j\psi_i = \left(\frac{1}{\tau_\sigma^{(j)}}-\frac{1}{\tau_\sigma^{(i)}}\right)\psi_i$.

(3′) $D_j\varphi_i = \frac{1}{\tau_\sigma^{(j)}}\varphi_i+\frac{1}{\tau_\sigma^{(i)}}\psi_i$.

For $f, g \in C^\infty$,

(4) $D_j(fg) = gD_jf+fD_jg-\frac{fg}{\tau_\sigma^{(j)}}$

(4′) $D_j(\psi_i f) = \psi_i D_j f-\frac{\psi_i f}{\tau_\sigma^{(j)}}$

(4″) $D_j(\varphi_i f) = \varphi_i D_j f+\frac{\psi_i f}{\tau_\sigma^{(j)}}$.

Definition 6

(1) $\mathscr{D} \overset{\text{def}}{=} \left\{ D_\tau = d_t + \frac{1}{\tau} I \;\middle|\; \tau \in \mathbf{R}^+ \cup \{0, \infty\} \right\}, \quad D_0 \overset{\text{def}}{=} I, \quad D_\infty \overset{\text{def}}{=} d_t.$

(2) For $\forall D_{\tau_1}, D_{\tau_2}, \cdots, D_{\tau_n} \in \mathscr{D}$, the composition of operators is defined as

$$(D_{\tau_1} \circ D_{\tau_2}) f \overset{\text{def}}{=} D_{\tau_1}(D_{\tau_2} f)$$

$$\left(\prod_{i=1}^{n} D_{\tau_i} \right) \overset{\text{def}}{=} \left(\prod_{i=1}^{n-1} D_{\tau_i} \right) \circ D_{\tau_n}.$$

(3) $d_t^n \overset{\text{def}}{=} \frac{d^n}{dt^n}, \quad d_t^0 \overset{\text{def}}{=} I.$

Proposition 2 The composition satisfies the commutative and associative laws, i. e.,

(1) $D_{\tau_1} \circ D_{\tau_2} = D_{\tau_2} \circ D_{\tau_1},$

(2) $(D_{\tau_1} \circ D_{\tau_2}) \circ D_{\tau_3} = D_{\tau_1} \circ (D_{\tau_2} \circ D_{\tau_3}).$

Hereafter, $D_{\tau_1} \circ D_{\tau_2}$ is simply written as $D_{\tau_1} D_{\tau_2}$ and the identity operator I is omitted unless required to emphasize this operator.

Proposition 3

(1) $D_{\tau_1} D_{\tau_2} = d_t^2 + \left(\frac{1}{\tau_1} + \frac{1}{\tau_2} \right) d_t + \frac{1}{\tau_1 \tau_2}.$

(2) $$\prod_{i=1}^{n} D_{\tau_i} = \left(\prod_{i=1}^{n-1} D_{\tau_i} \right) D_{\tau_n} = \left(\prod_{i=1}^{n-1} D_{\tau_i} \right) d_t + \left(\prod_{i=1}^{n-1} D_{\tau_i} \right) \frac{1}{\tau_n}$$
$$= d_t^n + \left(\sum_{i=1}^{n} \frac{1}{\tau_i} \right) d_t^{n-1} + \cdots + \left(\prod_{i=1}^{n} \frac{1}{\tau_i} \right) = \sum_{i=0}^{n} \left(\sum_{\Omega_i \subset \Omega_n} \prod_{j \in \Omega_i} \frac{1}{\tau_j} \right) d_t^{n-i},$$

where Ω_i is a subset of i elements of $\Omega_n = \{1, 2, \cdots, n\}$, called a subset of order i of Ω_n, the summation $\sum_{\Omega_i \subset \Omega_k}$ is made over all the subset of order i of Ω_n, and $\prod_{j \in \Omega_i}$ means the product over all the elements of Ω_i, and, furthermore, $\Omega_0 \overset{\text{def}}{=} \emptyset, \prod_{j \in \emptyset} \frac{1}{\tau_i} \overset{\text{def}}{=} 1, \sum_{\Omega_i \subset \emptyset} 1 \overset{\text{def}}{=} 1.$

The kernel and image of the diffusion operator which is a linear operator, are further studied. For brevity, the kernel and image of the diffusion operator D_i are denoted by Ker D_i and Im D_i, respectively.

Proposition 4

(1) Ker $D_i = \langle \psi_i \rangle = \{ a\psi_i | a \in \mathbf{R} \}.$

(2) $\langle \text{Ker } D_i \cup \text{Ker } D_j \rangle = \text{Ker } D_i \oplus \text{Ker } D_j = \{ a_i \psi_i + a_j \psi_j | a_i, a_j \in \mathbf{R} \}.$

(3) Ker $D_i D_j$ = Ker $D_i \oplus$ Ker D_j.

(4) Ker D_i^m = Ker D_i $(m \geq 2)$.

Here, $\mathbf{R}$ is the set of all real numbers. The angle brackets $\langle \bullet \rangle$ exhibits a linear subspace of the linear space C^∞ over $\mathbf{R}$, generated from elements or a subset of C^∞, and $L_1 \oplus L_2$ a direct sum of two linear subspaces L_1 and L_2 disjoint exeept for 0.

Definnition 7 Product or general diffusion operator of D_i's$(i = 1, 2, \cdots, n)$ is defined as

$$D \overset{\text{def}}{=} \prod_{i=1}^{n} D_i.$$

Proposition 5

$$\text{Ker } D = \sum_{i=1}^{n} \oplus \text{Ker } D_i = \left\{ \sum_{i=1}^{n} a_i \psi_i | a_i \in \mathbf{R} \right\}.$$

Definition 8 Inverse oprator of D_i is formally defined as

$$D_i^{-1} \overset{\text{def}}{=} \tau_\sigma^{(i)} \sum_{j=0}^{\infty} (-\tau_\sigma^{(i)} d_t)^j, \tag{3.4}$$

with $$(-\tau_\sigma^{(i)} d_t)^0 \overset{\text{def}}{=} I.$$

The inverse operator is meaningless, if the sum of the infinite series in eqn. (3.4) diverges on operation to a function of class C^∞. For instance,

$$D_i^{-1}\psi_i = \tau_\sigma^{(i)} \sum_{j=0}^{\infty} (-\tau_\sigma^{(i)} d_t)^j \psi_i = \tau_\sigma^{(i)} \sum_{j=0}^{\infty} \psi_i = \tau_\sigma^{(i)} \psi_i \sum_{j=0}^{\infty} 1$$

diverges.

Proposition 6

$$\psi_i \notin \text{Im } D_i.$$

Proof.

Suppose that $\psi_i \in \text{Im } D_i$. There exists a function $f \in C^\infty$ such that $\psi_i = D_i f$. Then,

$$D_i^{-1}\psi_i = D_i^{-1}(D_i f) = f,$$

which contradicts the divergence of $D_i^{-1}\psi_i$. □

Corollary 1

$$\varphi_i \notin \text{Im } D_i.$$

Proof.

If $\varphi_i \in \text{Im } D_i$, then $\psi_i = 1 - \varphi_i \in \text{Im } D_i$, which contradicts Prop. 6. □

From Prop. 1 (3), (3′) it follows that

Proposition 7 If $\tau_\sigma^{(i)} \neq \tau_\sigma^{(j)}$, then

$$\psi_i, \varphi_i \in \text{Im } D_j.$$

Proposition 8 If $D_i^{-1}f$ converges for $f \in C^\infty$, then $D_i^{-1}f \in C^\infty$ and $f \in \text{Im } D_i$.

Proof.

For any positive integer m,

$$d_t^m(D_i^{-1}f) = \sum_{j=0}^{m-1} \left(-\frac{1}{\tau_\sigma^{(i)}}\right)^j d_t^{m-1-j} f + \left(-\frac{1}{\tau_\sigma^{(i)}}\right)^m D_i^{-1} f,$$

whence if $D_i^{-1}f$ converges, then $d_t^m(D_i^{-1}f)$ also converges. Thus, $D_i^{-1}f \in C^\infty$. Since $f = D_i(D_i^{-1}f)$, $f \in \text{Im } D_i$. □

Proposition 9 Let m be any positive integer.

(1) $$D_i^{-1} t^m = \tau_\sigma^{(i)} \sum_{j=0}^{m} (-\tau_\sigma^{(i)})^j \frac{m!}{(m-j)!} t^{m-j}.$$

$$(2)\quad D_i^{-1}\left(\sum_{m=0}^{n} a_m t^m\right) = \tau_\sigma^{(i)} \sum_{m=0}^{n} a_m\left(\sum_{j=0}^{m} (-\tau_\sigma^{(i)})^j \frac{m!}{(m-j)!} t^{m-j}\right)$$
$$= \tau_\sigma^{(i)} \sum_{\nu=0}^{n} \left(\sum_{\mu=0}^{n-\nu} (-\tau_\sigma^{(i)})^\mu a_{\mu+\nu}(\mu+\nu)!\right)\frac{t^\nu}{\nu!}$$

Since $D_i^{-1}t^m \in C^\infty$ and $t^m = D_i(D_i^{-1}t^m)$, the following proposition is asserted.

Proposition 10 For any positive integer m,

$$\langle 1, t, t^2, \cdots, t^m\rangle \subset \mathrm{Im}\ D_\tau \quad (\forall D_\tau \in \mathfrak{D}),\ \text{or}\ \langle 1, t, t^2, \cdots, t^m\rangle \subset \bigcap_{D_\tau \in \mathfrak{D}} \mathrm{Im}\ D_\tau$$

Proposition 11

(1) $\mathrm{Im}\ D_iD_j \subset \mathrm{Im}\ D_i$.

(2) $\mathrm{Im}\ D_iD_j \subset \mathrm{Im}\ D_i \cap \mathrm{Im}\ D_j$.

(3) $\mathrm{Im}\ D_iD_j = \mathrm{Im}\ D_j \cap \mathrm{Im}\ D_j$.

(4) $\mathrm{Im}\ D = \bigcap_{i=1}^{n} \mathrm{Im}\ D_i$.

4. The Differential General Linear Equation (DGLE)

Let $p(x)$ be a polynomial whose coefficients are a_i's in eqn. (2.8):

$$p(x) = \sum_{i=0}^{n} a_i x^{n-i}, \tag{4.1}$$

and k be the number of different roots of the equation:

$$p(x) = 0. \tag{4.2}$$

The number k is called the order of the general linear solid. Now, let $-\alpha_i$ be the roots of eqn. (4.2). Then, the product diffusion operator D is given by

$$D = \prod_{i=1}^{n} (d_t + \alpha_i) = \sum_{i=0}^{n} a_i d_t^{n-i}. \tag{4.3}$$

The left-hand side of eqn. (2.8) is, then, written as $D\varepsilon$. If $\alpha_i > 0 (i = 1, 2, \cdots, n)$, the relaxation time $\tau_\sigma^{(i)}$ is taken as $1/\alpha_i$ and the operator D is written as a product of the n diffusion operators D_i's:

$$D = \prod_{i=1}^{n} \left(d_t + \frac{1}{\tau_\sigma^{(i)}}\right) = \prod_{i=1}^{n} D_i. \tag{4.4}$$

The k different roots of eqn. (4.2) are now denoted by $-\hat{\alpha}_i (i = 1, 2, \cdots, k)$. The product diffusion operator $\hat{D}$ corresponding to $\hat{\alpha}_i$'s is given by

$$\hat{D} \overset{\text{def}}{=} \prod_{i=1}^{k} (d_t + \hat{\alpha}_i) = \sum_{i=0}^{k} \hat{a}_i d_t^{k-i}. \tag{4.5}$$

From Prop. 5, it follows that

Proposition 12

(1) $\mathrm{Ker}\ D = \mathrm{Ker}\ \hat{D}$.

(2) $\dim\ (\mathrm{Ker}\ D) = k$.

It is anticipated that the strain of the GLS which is the solution of its defining DGLE is written as

$$\varepsilon = \sum_{i=0}^{n} \varepsilon_i. \tag{4.6}$$

Here ε_0 is the elastic strain of the GLS, denoted by ε_e, and other ε_i's are the anelastic strains, usually denoted by $\varepsilon_{a,i}$, satisfying

$$D_i\varepsilon_i = \frac{\delta J_{R,i}}{\tau_\sigma^{(i)}}\sigma. \tag{4.7}$$

By applying the product diffusion operator to eqn. (4.6) and combined with eqn. (4.7), the following equation is attained.

$$D\varepsilon = J_U\left\{D + \sum_{i=1}^{n} \frac{\Delta_i}{\tau_\sigma^{(i)}}\left(\prod_{j=1, j\neq i}^{n} D_j\right)\right\}\sigma, \tag{4.8}$$

with $\varepsilon_e = J_U\sigma$ and $\Delta_i = \delta J_{R,i}/J_U$. This equation is rearranged in another form

$$\left[\sum_{i=0}^{n}\left\{\sum_{|\Omega|=i}\left(\prod_{j\in\Omega}\frac{1}{\tau_\sigma^{(j)}}\right)\right\}d_t^{n-i}\right]\varepsilon = J_U\left[\sum_{i=0}^{n}\left\{\sum_{|\Omega|=i}\left(\prod_{j\in\Omega}\frac{1}{\tau_\sigma^{(j)}}\right)\left(1+\sum_{j\in\Omega}\Delta_j\right)\right\}d_t^{n-i}\right]\sigma, \tag{4.9}$$

which is of the same form as eqn. (2.8). Then,

$$a_i = \sum_{|\Omega|=i}\left(\prod_{j\in\Omega}\frac{1}{\tau_\sigma^{(j)}}\right) \tag{4.10}$$

$$b_i = J_U\left\{\sum_{|\Omega|=i}\left(\prod_{j\in\Omega}\frac{1}{\tau_\sigma^{(j)}}\right)\left(1+\sum_{j\in\Omega}\Delta_j\right)\right\}. \tag{4.11}$$

Now, consider a GLS of $n = 2$. The defining stress-strain equation is

$$D_iD_j\varepsilon = J_U\left\{D_iD_j + \frac{\Delta_i}{\tau_\sigma^{(i)}}D_j + \frac{\Delta_j}{\tau_\sigma^{(j)}}D_i\right\}\sigma$$

In the case of $D_i = D_j$, this equation becomes

$$D_i^2\varepsilon = J_UD_i\left\{D_i + \frac{\Delta_i+\Delta_j}{\tau_\sigma^{(i)}}\right\}\sigma,$$

whence

$$D_i\varepsilon - J_U\left\{D_i + \frac{\Delta_i+\Delta_j}{\tau_\sigma^{(i)}}\right\}\sigma \in \mathrm{Ker}\ D_i.$$

Proposition 13 In the case of $D_i = D_j$ or $\tau_\sigma^{(i)} = \tau_\sigma^{(j)}$, the defining stress-strain equation of the GLS for $n = 2$ reduces to

$$D_i\varepsilon = J_U\left\{D_i + \frac{\Delta_i+\Delta_j}{\tau_\sigma^{(i)}}\right\}\sigma.$$

Proof.

From Prop. 4 (1),

$$D_i\varepsilon - J_U\left\{D_i + \frac{\Delta_i + \Delta_j}{\tau_\sigma^{(i)}}\right\}\sigma = a\psi_i \quad (a \in \mathbf{R}). \tag{4.12}$$

Equation (4.7) asserts that

$$J_U\frac{\Delta_i + \Delta_j}{\tau_\sigma^{(i)}}\sigma = \frac{\delta J_{R,i}}{\tau_\sigma^{(i)}}\sigma + \frac{\delta J_{R,j}}{\tau_\sigma^{(i)}}\sigma \in \mathrm{Im}\ D_i.$$

The left-hand side of eqn. (4.12), then, belongs to Im D_i, while $\psi_i \notin$ Im D_i (Prop. 6). Thus, the coefficient a should be equal to 0. □

A relaxation time yields a diffusion operator and vice versa. Proposition 13 further implies that a relaxation time determines the Voigt solid except for the relaxation strength or the relaxation of the compliance. By generalizing Prop. 13, it follows that

Theorem 1

$$D\varepsilon = J_U\left\{D + \sum_{i=0}^{n}\frac{\Delta_i}{\tau_\sigma^{(i)}}\left(\prod_{j=1, j\neq i}^{n} D_j\right)\right\}\sigma \tag{4.13}$$

reduces to

$$\hat{D}\varepsilon = J_U\left\{\hat{D} + \sum_{i=1}^{k}\frac{\tilde{\Delta}_i}{\tau_\sigma^{(i)}}\left(\prod_{j=1, j\neq i}^{k} \hat{D}_j\right)\right\}\sigma \tag{4.14}$$

where $\hat{D}_j$ is a diffusion operator and $\tilde{\Delta}_i$ a relaxation strength or a sum of relaxation strengths, associated with the relaxation time of $\hat{\tau}_\sigma^{(j)} = 1/\hat{a}_j$.

Definition 9 The product diffusion operator D or the GLDE is said to be irreducible, if all the roots of equation $p(x) = 0$ are different one another, and reducible, otherwise.

The Voigt solid is called the Voigt unit in the GLS. From Theorem 1 it is, accordingly, impossible to separate the resultant relaxation of the compliance into the relaxations of the compliance for the individual Voigt units. Only the relaxations of the compliance for different relaxation times are explicitly represented. In the theory of the point-defect relaxation in crystalline solids, such case often occurs when there involve point defects of a species and equivalent symmetries in the point group of the defect symmetries. In such case the relaxations of the compliance are regarded to be equal for each component Voigt unit, so that the relaxation of the compliance for each component Voigt unit is dealt out in equal portion.

5. Solution of DGLE

Consider to solve the irreducible differential general linear equation defining the GLS. From Theorem 1, the DGLE is given by

$$\sum_{i=0}^{k} a_i\varepsilon^{(k-i)} = \sum_{i=0}^{k} b_i\sigma^{(k-i)} \quad (a_i, b_i \in \mathbf{R}^+, a_0 = 1), \tag{5.1}$$

where a_i and b_i are employed instead of $\hat{a}_i$ and $\hat{b}_i$. The different roots of eqn. (4.2) are also written as $-a_i (i = 1, 2, \cdots, k)$ without hat. Let a matrix A_a be

$$\mathbf{A}_\alpha = \begin{pmatrix} 1 & 1 & \cdots & \cdots & 1 \\ \sum_{i_1 \neq 1} \alpha_{i_1} & \sum_{i_1 \neq 2} \alpha_{i_1} & \cdots & \cdots & \sum_{i_1 \neq k} \alpha_{i_1} \\ \sum_{i_1, i_2 \neq 1, i_1 < i_2} \alpha_{i_1}\alpha_{i_2} & \sum_{i_1, i_2 \neq 2, i_1 < i_2} \alpha_{i_1}\alpha_{i_2} & \cdots & \cdots & \sum_{i_1, i_2 \neq k, i_1 < i_2} \alpha_{i_1}\alpha_{i_2} \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ \alpha_2\alpha_3\cdots\alpha_k & \alpha_1\alpha_3\cdots\alpha_k & & & \alpha_1\alpha_2\cdots\alpha_{k-1} \end{pmatrix},$$

the determinant of which is the simplest alternating function of $\alpha_i (i = 1, 2, \cdots, k)$[5], i.e.,

$$|\mathbf{A}_\alpha| = \prod_{i<j} (\alpha_i - \alpha_j)$$

Since α_i's$(i = 1, 2, \cdots, k)$ are different one another , the matrix A_α is regular. Now by setting

$$\begin{aligned} J_{\mathrm{U}} &= b_0 \\ \Delta_i &= \tau_\sigma^{(i)} (\mathbf{A}_\alpha^{-1} \vec{\beta})_i \\ \tau_\sigma^{(i)} &= 1/\alpha_i \\ (\vec{\beta})_i &= \frac{b_i}{b_0} - \sum_{|\Omega|=i} \left(\prod_{j \in \Omega} \alpha_j \right), \end{aligned}$$

eqn. (5.1) becomes eqn. (4.8).

Lemma 1

$$D_i(\varphi_i \sigma) = \frac{1}{\tau_\sigma^{(i)}} \sigma \iff \sigma = \text{constant}$$

Proof.

From Prop. 1 (4″),

$$D_i(\varphi_i \sigma) = \varphi_i D_i \sigma + \frac{\psi_i \sigma}{\tau_\sigma^{(i)}}.$$

Combined with the assumption,

$$\varphi_i D_i \sigma = \frac{\varphi_i}{\tau_\sigma^{(i)}} \sigma.$$

Thus, $d_i \sigma = 0$; that is, σ is constant. The converse proposition is readily proved by following the first part of the proof inversely. □

Theorem 2 If σ is constant, then

$$\varepsilon = J_{\mathrm{U}} \left(1 + \sum_{i=1}^{k} \Delta_i \varphi_i \right) \sigma \tag{5.2}$$

is a special solution of the irreducible DGLE (5.1). Combined with the general solution, the solution of eqn. (5.1) is given by

$$\varepsilon = J_{\mathrm{U}} \left(1 + \sum_{i=1}^{k} \Delta_i \varphi_i \right) \sigma + \sum_{i=1}^{k} a_i \psi_i \quad (a_i \in \mathbf{R}).$$

Proof.

By operating D on ε of eqn. (5.2),

$$D\varepsilon = J_U\left\{D + \sum_{i=1}^{n} \Delta_i\left(\prod_{j=1, j\neq i}^{n} D_j\right) D_i\varphi_i\right\}\sigma.$$

From Lemma 1, it follows that this equation becomes eqn. (4.8). The strain given by eqn. (5.2) is, thus, a solution of the GLDE (5.1). Then,

$$\varepsilon - J_U\left(1 + \sum_{i=1}^{k} \Delta_i\varphi_i\right)\sigma \in \mathrm{Ker}\ D.$$

From Prop. 5,

$$\varepsilon - J_U\left(1 + \sum_{i=1}^{k} \Delta_i\varphi_i\right)\sigma = \sum_{i=1}^{k} a_i\psi_i \quad (a_i \in \mathbf{R}). \qquad \square$$

6. Mechanical Model of the GLS

The Voigt solid is modelled on the basis of the Hookean spring and the Newtonian dashpot, by connecting the latter two models in parallel. This model of the Voigt solid is represented by the same stress-strain equation as eqn. (2.5), so that the model is identified with the Voigt solid. Attaching a Hookean spring to the Voigt model in series yields a model specified by the same stress-strain equation as the standard linear solid. The model of the SLS is extended to a model of the general linear solid, when a Hookean spring and n Voigt models are connected all in series[6]. The model of the GLS is evidently characterized by the differential stress-strain equation identical to the defining differential equation of the GLS[4].

7. Correspondence between Mathematical and Physical Properties of the GLS

The Voigt unit is a component of the GLS model. A relaxation time $\tau_\sigma^{(i)}$ is associated with the Voigt unit, which is defined by the differential stress-strain equation as eqn. (2.5). A Voigt unit of the GLS yields a diffusion operator specified by the relaxation time of the Voigt unit; there is a one-to-one correspondence between a Voigt unit and a diffusion operator through the relaxation time. Serial connection of the two Voigt units in the GLS corresponds to the composition of the two corresponding diffusion operators to the Voigt units.

7.1 Commutativity of Diffusion Operators

Two diffusion operators commutes each other for the composition of the two operators (Prop. 2 (1)). Let the two diffusion operators be D_1 and D_2. The composition $D_1 \circ D_2$ is, in the GLS model, regarded as a serial connection of the corresponding two Voigt units V_1 and V_2 in the order of the suffix number, for instance. In the general linear solid, there produces no substantial difference in mechanical property even if the order of connection of the two Voigt units is inverted. Such mechanical property in the GLS reflects the commutativity of the diffusion operators.

7.2 Property of Ker D_i^2 = Ker D_i

Two Voigt units V_1 and V_2 in the GLS are said to be equivalent, if their relaxation times are the same. The units V_1 and V_2 are specified by the stress-strain equations:

$$D_i\varepsilon_{\mathrm{a},i} = \frac{\delta J_{\mathrm{R},i}}{\tau_\sigma^{(i)}}\sigma \quad (i = 1, 2) \tag{7.1}$$

If V_1 is equivalent to V_2,

$$\tau_\sigma^{(1)} = \tau_\sigma^{(2)} \stackrel{\mathrm{def}}{=} \tau_\sigma^{(1+2)}.$$

Thus, $D_1 = D_2 \stackrel{\mathrm{def}}{=} D_{1+2},$

and
$$D_{1+2}\varepsilon_{\mathrm{a},1+2} = \frac{\delta J_{\mathrm{R},1+2}}{\tau_\sigma^{(1+2)}}\sigma \tag{7.2}$$

with $\varepsilon_{\mathrm{a},1+2} \stackrel{\mathrm{def}}{=} \varepsilon_{\mathrm{a},1} + \varepsilon_{\mathrm{a},2}$, $\delta J_{\mathrm{R},1+2} \stackrel{\mathrm{def}}{=} \delta J_{\mathrm{R},1} + \delta J_{\mathrm{R},2}.$

Equation (7.2) implies that two Voigt units of the same relaxation time is identical to a Voigt unit of the same relaxation time with the relaxation of the compliance as large as $\delta J_{\mathrm{R},1+2} = \delta J_{\mathrm{R},1} + \delta J_{\mathrm{R},2}$. Equivalent two Voigt units relax simultaneously in the GLS, so that the two units is replaced with a Voigt unit of the same relaxation time though the relaxation of the compliance is the sum of those for the component two units. Then, the resultant Voigt unit of the two equivalent Voigt units is nothing but a Voigt unit of the same relaxation time. In eqns. (7.1) and (7.2), the left-hand side is always the same, and so the general solutions of eqns. (7.1), (7.2) are the same as well. As for the diffusion operator, composition of the same diffusion operators corresponding to the connection of two equivalent Voigt units in the GLS, therefore, yields only a single diffusion operator corresponding to the Voigt unit resulting from the two equivalent Voigt units. It follows that Ker D_i^2 = Ker D_i, and furthermore, Ker D_i^m = Ker D_i ($m \geq 2$).

References

1) A. S. Nowick and B. S. Berry: Anelastic Relaxation in Crystalline Solids, Academic Press, New York, (1972).

2) Y. Iwasaki: Application of the theory of electric circuit to the analysis of anelastic relaxation, *Bul. Okayama Univ. Sci.*, **31**, 27-41, (1996).

3) Y. Iwasaki: Mathematical theory of thermodynamics in multi-relaxation in solids with discrete relaxation spectra, *ibid.*, **32**, 9-18, (1997).

4) Y. Iwasaki: General theory of multi-relaxation in solids with discrete relaxation spectra, (submitted).

5) Y. Iwasaki: Representation of the simplest alternating function in a form of determinant other than Vandermonde's and its application, Proc. 26th Domestic Symposium on Numerical Analysis, pp. 75-78, (1997).

6) Y. Iwasaki and K. Fujimoto: Resonant system of a torsion pendulum with anelastic solids of discrete relaxation spectra, *J. Phys. D: Appl.Phys.*, **13**, 823-34, (1980).