The Computation of the Character Table of the Johnson Scheme by Using a Generating Function Method

Michihiko HASHIZUME and Yoshiyuki MORI

Department of Applied Mathematics, Faculty of Science
Okayama University of Science
Ridaicho 1-1, Okayama 700-0005, Japan

(Received October 6, 1997)

1 Introduction

In this paper, we compute the character table of the Johnson scheme by using a generating function method. The main idea of our approach is to see that the generating function of the characters of the Johnson scheme can be expressed by the Gauss hypergeometric function. This enables us to obtain the explicit formula of the characters of the Johnson scheme. The character table of the Johnson scheme is already determined in [1] by an algebraic method, more precisely, by using a deep result of the representation theory of symmetric groups. On the contrary, our approach is analytic and elementary, so that it is possible to have many applications.

2 Review of association schemes and distance-regular graphs

Let V be a finite set and d be a natural number. An association scheme with d classes is a pair $(V,(R_j)_{0\le j\le d})$ such that

(i) $(R_j)_{0\le j\le d}$ is a partition of $V\times V$,

(ii) $R_0=\{(v,v);v\in V\}$,

(iii) $(u,v)\in R_j$ if and only if $(v,u)\in R_j (0\le j\le d)$,

(iv) there are numbers b_{ij}^k such that for any pair $(u,v)\in R_k$ the number of $w\in V$ with $(u,w)\in R_i$ and $(w,v)\in R_j$ equals b_{ij}^k.

Define the 0-1 matrices $(A_j)_{0\le j\le d}$ of size $|V|\times|V|$ indexed by the elements of V as follows:

$$(2.1)\qquad A_j=(A_j(u,v))_{(u,v)\in V\times V},\ A_j(u,v)=\begin{cases}1 & \text{if } (u,v)\in R_j,\\ 0 & \text{otherwise.}\end{cases}$$

Then the axiom (i)-(iv) of an association scheme are described by

$$(2.2)\qquad \text{(i)}\sum_{0\le j\le d}A_j=J,\ \text{(ii)}A_0=I,\ \text{(iii)}{}^tA_j=A_j,\ \text{(iv)}A_iA_j=\sum_{0\le k\le d}b_{ij}^kA_k,$$

where I is the identity matrix and J is the matrix such that all the entries are 1. Since $b_{ij}^k=b_{ji}^k$, it follows that $A_iA_j=A_jA_i$. Thus $(A_j)_{0\le j\le d}$ are linearly independent and generate a $(d+1)$-dimensional commutative semisimple algebra $\mathcal{A}$. This algebra is called the Bose-Mesner algebra of the association scheme. Let $(E_j)_{0\leq j\leq d}$ be the set of all minimal idempotents of $\mathcal{A}$. Then there exist $p_{ij}\in \mathbb{C}(0\leq i,j\leq d)$ such that

$$A_j=\sum_{0\leq i\leq d} p_{ij}E_i \qquad (0\leq j\leq d). \tag{2.3}$$

The matrix $(p_{ij})_{0\leq i,j\leq d}$ is called the character table of the association scheme.

Let G be a connected graph with vertex set V. We denote the graph distance of vertices $u, v\in V$ by $\pi(u,v)$ and denote the diameter by d. For $u\in V$ and $j(0\leq j\leq d)$, put $S_j(u)=\{v\in V;\pi(u,v)=j\}$. We say that G is a distance-regular graph if there exist $b_j, c_j\in \mathbb{Z}_+(0\leq j\leq d)$ such that $|S_1(v)\cap S_{j+1}(u)|=b_j$ and $|S_1(v)\cap S_{j-1}(u)|=c_j$ for any $u, v\in V$ at distance $\pi(u,v)=j$. Note that $c_0=b_d=0$ and G is a regular graph with valency b_0. Put $a_j=b_0-b_j-c_j(0\leq j\leq d)$. Then $a_j=|S_1(v)\cap S_j(u)|$ where $\pi(u,v)=j$. Define $R_j=\{(u,v)\in V\times V;\pi(u,v)=j\}$ for $0\leq j\leq d$. Then the distance-regular graph G yields an association scheme $(V,(R_j)_{0\leq j\leq d})$[2]. In this case the matrix A_1 coincides with the adjacency matrix of G. Furthermore since $a_j=b^j_{1j}$, $b_j=b^j_{1j+1}$ and $c_j=b^j_{1j-1}$, it follow that

$$A_1A_j=b_{j-1}A_{j-1}+a_jA_j+c_{j+1}A_{j+1} \qquad (0\leq j\leq d) \tag{2.4}$$

where $b_{-1}=c_{d+1}=0$. Therefore if we introduce a sequence of polynimials $(p_j(\lambda))_{0\leq j\leq d}$ recursively by $p_0(\lambda)=1$, $p_1(\lambda)=\lambda$ and

$$c_{j+1}p_{j+1}(\lambda)+a_jp_j(\lambda)+b_{j-1}p_{j-1}(\lambda)=\lambda p_j(\lambda) \qquad (1\leq j\leq d-1), \tag{2.5}$$

then we can write

$$A_j=p_j(A_1) \qquad (0\leq j\leq d). \tag{2.6}$$

Moreover we can choose, as the set of all minimal idempotants, the set of the orthogonal projections $(E_j)_{0\leq j\leq d}$ to the eigenspaces of A_1. Note that the adjacency matrix A_1 of a distance-regular graph has $d+1$ distinct eigenvalues $\lambda_0, \lambda_1,\cdots,\lambda_d$. From (2.3) and (2.6), we conclude that

$$A_j=\sum_{0\leq i\leq d} p_j(\lambda_i)E_i \qquad (0\leq j\leq d). \tag{2.7}$$

Consequently the character table of the association scheme coming from a distance-regular graph is given by

$$(p_j(\lambda_i))_{0\leq i,j\leq d}, \tag{2.8}$$

where $\lambda_0, \lambda_1,\cdots,\lambda_d$ are distinct eigenvalues of the adjacency matrix and $(p_j(\lambda))_{0\leq j\leq d}$ are polynomials defined by (2.5).

3 The Johnson association schemes and their character tables

Let n be an integer greater than 1, and let d be an integer such that $1\leq d\leq n-1$. We denote by V the set of all subsets v of $\{1,2,\cdots,n\}$ with cardinality $|v|=d$;

(3.1) $$V = \{v \subset \{1, 2, \cdots, n\}; |v| = d\}.$$

The Johnson graph $J(n, d)$ has vertex set V, where two vertices $u, v \in V$ are adjacent whenever $|u \cap v| = d-1$. Note that $J(n, d)$ and $J(n, n-d)$ are isomorphic under the map which assigns v to the complement v^c of v. So we may assume $1 \leq d \leq [n/2]$. The Johnson graph $J(n, d)$ is not only a connected regular graph with valency $d(n-d)$ but also a distance-regular graph with diameter d[2] such that

(3.2) $$a_j = nj - 2j^2,\ b_j = (d-j)(n-d-j),\ c_j = j^2 \qquad (0 \leq j \leq d).$$

Hence the sequence of polynomials $(p_j(\lambda))_{0 \leq j \leq d}$ in (2.5) is defined by

(3.3) $$\begin{cases} p_0(\lambda) = 1,\ p_1(\lambda) = \lambda, \\ (j+1)^2 p_{j+1}(\lambda) + (nj - 2j^2) p_j(\lambda) + (d-j+1)(n-d-j+1) p_{j-1}(\lambda) = \lambda p_j(\lambda) \\ \hfill (1 \leq j \leq d-1), \\ (nd - 2d^2) p_d(\lambda) + (n-2d+1) p_{d-1}(\lambda) = \lambda p_d(\lambda). \end{cases}$$

In the sequel, let λ be an eigenvalue of the adjacency matrix A_1 of the Johnson graph $J(n, d)$. We introduce the generating function $G_\lambda(z)$ of the sequence $(p_j(\lambda))_{0 \leq j \leq d}$ by

(3.4) $$G_\lambda(z) = \sum_{0 \leq j \leq d} p_j(\lambda) z^j.$$

This is a polynomial in z of degree d. From the recurrence relations (3.3), we obtain the following results.

Theorem 1. *The generating function $G_\lambda(z)$ satisfies the ordinary differential equation of Fuchsian type;*

(3.5) $$z(1-z)^2 \frac{d^2 G_\lambda}{dz^2} + (1-z)((n-1)z+1) \frac{dG_\lambda}{dz} + (d(n-d) - \lambda) G_\lambda = 0.$$

Proof. From (3.4), we have

(3.6) $$G_\lambda'(z) = \sum_{1 \leq j \leq d} j p_j(\lambda) z^{j-1}$$

and

(3.7) $$G_\lambda''(z) = \sum_{2 \leq j \leq d} j(j-1) p_j(\lambda) z^{j-2}.$$

Consequently we have

(3.8) $$z G_\lambda''(z) + G_\lambda'(z) - \lambda = \sum_{1 \leq j \leq d-1} (j+1)^2 p_{j+1}(\lambda) z^j.$$

Applying (3.3) to the right side of (3.8), we have

(3.9) $$\sum_{1 \leq j \leq d-1} (j+1)^2 p_{j+1}(\lambda) z^j = \sum_{1 \leq j \leq d-1} (\lambda + 2j^2 - nj) p_j(\lambda) z^j - \sum_{1 \leq j \leq d-1} (d-j+1)(n-d-j+1) p_{j-1}(\lambda) z^j.$$

The first term of the right side of (3.9) can be written as

$$\lambda \sum_{1\le j\le d-1} p_j(\lambda)z^j + 2\sum_{2\le j\le d-1} j(j-1)p_j(\lambda)z^j + (2-n)\sum_{1\le j\le d-1} jp_j(\lambda)z^j.$$

Using (3.4), (3.6) and (3.7), we have

$$\lambda(G_\lambda(z)-1-p_d(\lambda)z^d)+2z^2(G_\lambda''(z)-d(d-1)p_d(\lambda)z^{d-2})+(2-n)z(G_\lambda'(z)-dp_d(\lambda)z^{d-1}).$$

Hence the first term is equal to

$$2z^2G_\lambda''(z)-(n-2)zG_\lambda'(z)+\lambda G_\lambda(z)-\lambda+(nd-2d^2-\lambda)p_d(\lambda)z^d. \tag{3.10}$$

The second term of the right side of (3.9) can be written as

$$-\sum_{0\le j\le d-2}(d(n-d)-nj+j^2)p_j(\lambda)z^{j+1},$$

which is equal to

$$-d(n-d)z\sum_{0\le j\le d-2} p_j(\lambda)z^j+(n-1)z^2\sum_{1\le j\le d-2} jp_j(\lambda)z^{j-1}-z^3\sum_{2\le j\le d-2} j(j-1)p_j(\lambda)z^{j-2}.$$

Using (3.4), (3.6) and (3.7), we have

$$\begin{aligned}&-d(n-d)z(G_\lambda(z)-p_{d-1}(\lambda)z^{d-1}-p_d(\lambda)z^d)+(n-1)z^2(G_\lambda'(z)-(d-1)p_{d-1}(\lambda)z^{d-2}\\&\quad -dp_d(\lambda)z^{d-1})-z^3(G_\lambda''(z)-(d-1)(d-2)p_{d-1}z^{d-3}-d(d-1)p_d(\lambda)z^{d-2}).\end{aligned}$$

Hence the second term is equal to

$$-z^3G_\lambda''(z)+(n-1)z^2G_\lambda'(z)-d(n-d)G_\lambda(z)+(n-2d+1)p_{d-1}(\lambda)z^d. \tag{3.11}$$

From (3.3), (3.10) and (3.11), it follows that the right side of (3.9) is equal to

$$(2z^2-z^3)G_\lambda''(z)+((n-1)z^2-(n-2)z)G_\lambda'(z)+(\lambda-d(n-d))G_\lambda(z)-\lambda.$$

Consequently from (3.8), we conclude that

$$z(1-z)^2G_\lambda''(z)+(1-z)((n-1)z+1)G_\lambda'(z)+(\lambda-d(n-d))G_\lambda(z)=0.$$

So the theorem is proved.//

Corollary. *If we put*

$$\mu=2^{-1}(n+1-((n+1)^2+4(\lambda-d(n-d)))^{1/2}) \tag{3.12}$$

and define $F_\lambda(z)$ *by*

$$G_\lambda(z)=(1-z)^\mu F_\lambda(z), \tag{3.13}$$

then $F_\lambda(z)$ *satisfies the Gauss hypergeometric differential equation*

$$z(1-z)\frac{d^2F_\lambda}{dz^2}+(1-(2\mu-n+1)z)\frac{dF_\lambda}{dz}-(\mu^2-n\mu+d(n-d))F_\lambda=0. \tag{3.14}$$

Since $G_\lambda(z)$ is a polynomial in z of degree d, it is a solution of (3.5) which are regular at $z=0$. Furthermore since $G_\lambda(0)=1$, it follows from the corollary that

$$(3.15) \qquad G_\lambda(z) = (1-z)^{\mu}{}_2F_1(-(d-\mu), -(n-d-\mu), 1; z),$$

where in general ${}_2F_1(\alpha, \beta, \gamma; z)$ means the Gauss hypergeometric series. As is well known,

$${}_2F_1(-(d-\mu), -(n-d-\mu), 1; z) = \sum_{j\geq 0} \frac{(-(d-\mu))_j(-(n-d-\mu))_j}{(j!)^2} z^j,$$

where in general

$$(\alpha)_j = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+j-1).$$

Since $G_\lambda(z)$ is a polynominal in z of degree d, it follows that μ must be a nonnegative integer and $(-(d-\mu))_j = 0$ for some $j (0 \leq j \leq d)$. This implies that μ must be an integer such that $0 \leq \mu \leq d$. Therefore from (3.12) the eigenvalue λ of the adjacency matrix is of the form $\lambda = \lambda_\mu$ where

$$(3.16) \qquad \lambda_\mu = \mu^2 - (n+1)\mu + d(n-d) \qquad (0 \leq \mu \leq d).$$

So we have obtained all the eigenvalues of the adjacency matrix of $J(n, d)$ by an analytic method. Furthermore from (3.15), we have

$$(3.17) \qquad G_{\lambda_\mu}(z) = \sum_{0\leq j\leq d} \sum_{0\leq i\leq j} (-1)^i \binom{\mu}{i} \binom{d-\mu}{j-i} \binom{n-d-\mu}{j-i} z^j \qquad (0 \leq \mu \leq d),$$

and consequently we have

$$p_j(\lambda_\mu) = \sum_{0\leq i\leq j} (-1)^i \binom{\mu}{i} \binom{d-\mu}{j-i} \binom{n-d-\mu}{j-i} \quad \text{for } 0 \leq j, \mu \leq d.$$

Summaring the above results, we have the following theorem.

Theorem 2. *The character table of the Johnson association scheme $J(n, d)$ is given by $(p_j(\lambda_\mu))_{0\leq j,\mu\leq d}$, where*

$$(3.18) \qquad p_j(\lambda_\mu) = \sum_{0\leq i\leq j} (-1)^i \binom{\mu}{i} \binom{d-\mu}{j-i} \binom{n-d-\mu}{j-i} \qquad (0 \leq j, \mu \leq d).$$

References

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