On the Normality and the Seminormality of Subrings of Simple Ring Extension

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Let R be a Noetherian integral domain with quotient field K and let $\bar{R}$ denote the integral closure of R in K. Assume that $\bar{R}$ is a finite R-module. Let A be a subring between R and $\bar{R}$. In [T], Traverso defined the seminormalization ${}^{+}_{A}R$ of R in A as follows :

${}^{+}_{A}R = \{a \in A | a_p \in R_p + J(A_p)$ for all $p \in$ Spec $(R)\}$, where $J(A_p)$ is the Jacobs on radical of A_p. Then he proved that the ring ${}^{+}_{A}R$ is characterized as the greatest subring B between R and A which satisfies the following two properties ;

(1) The contraction map Spec $(B) \to$ Spec (R) is an injection.

(2) For all $P \in$ Spec (B), k $(P \cap R)$ = k (P), where k (P) denotes the residue field of P.

Furthemore, in [BC], Brewer and Costa showed that R is seminormal in A (i. e., R = ${}^{+}_{A}R$) if and only if R is (2, 3)-closed in A, (i. e., whenever a $\in A$ satisfies that a^2, $a^3 \in R$, then $a \in R$).

Let $R[X]$ denote a polynomial ring, Let α be an element of an algebraic field extension L of K, and let $\pi : R[\mathrm{X}] \to R[\alpha]$ be the R-algebra homomorphism sending X to α. Let $\varphi_\alpha(X)$ be the monic minimal polynomial of α over K with $\deg(\varphi_\alpha(X)) = d$ and write

$$\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d.$$

Let $I_{[\alpha]} := \bigcap_{i=1}^{d} I_{\eta_i}$ where $I_{\eta_i} := (R :_R \eta_i)$. For $f(X) \in R(X)$, let $C(f(X))$ denote the R-submodule of K generated by the coefficients of f(X). Let $J_{[\alpha]} := I_{[\alpha]} C(\varphi_\alpha(X))$, which is an ideal of R and contains $I_{[\alpha]}$. The element α is called an **anti-integral** element of degreed over R if $\mathrm{Ker}(\pi) = I_{[\alpha]}\varphi_\alpha(X)R[X]$. When α is an anti-integral element over R, R$[\alpha]$is called an anti-integral extension of R. Let $\mathrm{Dp}_1(R) := \{p \in \mathrm{Spec}(R) | \mathrm{depth}(R_p) = 1\}$. The element α is called a **super-primitive** element of degree d over R ih $J_{[\alpha]} \not\subset p$ for any $p \in \mathrm{Dp}_1(R)$,

Let α be an anti-integral element and let $A = R[\alpha] \supset B_\alpha := R[\alpha] \cap R[\alpha^{-1}]$. We define the element ζ_i of A by $\zeta_i = \alpha^i + \eta_1\alpha^{i-1} + \cdots + \eta_{i-1}\alpha$ for any $i = 1, \cdots, d$-1. We put

$\zeta_d = 0$. Then, in [KY], Kanemitsu and Yoshida proved that $B_a = R \oplus I_{[a]}\zeta_1 \oplus \cdots \oplus I_{[a]}\zeta_{d-1}$ as R-modules. In this paper, we shall show that if B_a is seminomal in A, then $I_{[a]}$ is (2, 3)-closed as ideals. Futhemore we add some results derived from the condition that $I_{[a]}$ is integrally closed.

Throughout this paper, all rings are assumed to be commutative Noetherian with unity.

DEFINITION 1. Let I be an ideal of R. We say that I is **(2, 3)-closed** if whenever $x \in R$ satisfies $x^2 \in I^2$, $x^3 \in I^3$, then $x \in I$.

PROPOSITION 2. If B_a is seminomal in A, then $I_{[a]}$ is (2, 3)-closed.

PROOF. Suppose that $x \in R$ satisfies x^2, $x^3 \in I_{[a]}$.

(i) $d = 1$. In this case, $B_a = R$. Since $\zeta_1 = a$, we have

$$(x\zeta_1)^2 = x^2\zeta_1^2 \in I_{[a]}{}^2\zeta_1^2 \subset R,\ (x\zeta_1)^3 = x^3\zeta_1^3 \in I_{[a]}{}^3\zeta_1^3 \subset R.$$

By assumption, R is seminomal in A. We have $x\zeta_1 \in R$, and hence $x \in I_{[a]}$.

(ii) $d = 2$. Note that $(x\zeta_1)^2 = -x^2\eta_1\zeta_1 - x^2\eta_2$, $(x\zeta_1)^3 = x^3(\eta_1^2 - \eta_2) + x^3\eta_1\eta_2$. Since $x^2\eta_1 \in I_{[a]}{}^2\eta_1 \subset I_{[a]}$ and $x^2\eta_2 \in R$, we see that $(x\zeta_1)^2 \in B_a$. Furhemore, since $x^3\eta_1^2 \in I_{[a]}{}^3\eta_1^2 \subset I_{[a]}$, $x^3\eta_1 \in I_{[a]}{}^3\eta_1 \subset I_{[a]}$ and $x^3\eta_1\eta_2 \in I_{[a]}{}^3\eta_1\eta_2 \subset I_{[a]}$, it follows that $(x\zeta_1)^3 \in B_a$. Therefore we have $x\zeta_1 \in B_a$. Thus we get $x \in I_{[a]}$.

(iii) $d = 3$. In this case, we have $(x\zeta_1)^2 = x^2\zeta_2 - x^2\zeta_1\eta_1$. Since $x^2\zeta_2 \in I_{[a]}$ and $x^3\eta_1^2 \in I_{[a]}{}^3\eta_1^2 \subset I_{[a]}$, $x^2\eta_1 \in I_{[a]}{}^2\eta_1 \subset I_{[a]}$, we see that $(x\zeta_1)^2 \in B_a$. On the other hand, it holds that $(x\zeta_1)^3 = -x^3\zeta_2 + x^3(\eta_1^2 - \eta_2)\zeta_1 - x^3\eta_3$. Since $x^3\eta_1^2 \in I_{[a]}{}^3\eta_1^2 \subset I_{[a]}$, $x^3\eta_2 \in I_{[a]}{}^3\eta_2 \subset I_{[a]}$, we see that $(x\zeta_1)^3 \in B_a$. Thus it follows that $x\zeta_1 \in B_a$ by assumption. Therefore we get $x \in I_{[a]}$.

(iv) $d \geqq 4$. Note that $(x\zeta_1)^2 = x^2\zeta_2 - x^2\zeta_1\eta_1$, $(x\zeta_1)^3 = x^3\zeta_3 - (x^3\eta_1)\zeta_2 + x^3(\eta_1^2 - \eta_2)\zeta_1$. Since $x^2\zeta_2 \in I_{[a]}$, $x^2\eta_1 \in I_{[a]}{}^2\eta_1 \subset I_{[a]}$, then we conclude $(x\zeta_1)^2 \in B_a$. On the other hand, since $x^3\zeta_3 \in I_{[a]}$, $x^3\eta_1 \in I_{[a]}{}^3\eta_1 \subset I_{[a]}$, $x^3\eta_1^2 \in I_{[a]}{}^3\eta_1^2 \subset I_{[a]}$, and $x^3\eta_2 \in I_{[a]}{}^3\eta_2 \subset I_{[a]}$, then we see that $(x\zeta_1)^3 \in B_a$. Hence it holds $x\zeta_1 \in B_a$ by seminomality. This means that $x \in I_{[a]}$. With this, we are done.

Q. E. D.

The following proposition shows that (2, 3) closedness of ideals is a local property.

PROPOSITION 3. For an ideal I of R, the following statements are equivalent :

(1) I is (2, 3)-closed,

(2) I_p is (2, 3)-closed for all $p \in \mathrm{Ass}\,(R/I)$.

PROOF. (1)⇒(2) : Let $p \in \mathrm{Ass}\,(R/I)$ and let $x \in R_p$. Assume now that $x^2 \in I_p{}^2$, $x^3 \in I_p{}^3$. Then there exists some $a \in R \backslash p$ such that $ax^2 \in I^2$, $ax^3 \in I^3$. Hence we have that $(ax)^2 \in I^2$, $(ax)^3 \in I^3$. Since I is (2, 3)-closed, it holds that $ax \in I$, and thus $x \in I_p$.

(2)⇒(1) : Let $\mathrm{Ass}(R/I) = \{p_1, \cdots, p_n\}$. Then, for any $x \in R$, we see that $x \in I$ is equivalent to $x \in I_{p_i}\,(1 \leq i \leq n)$. Suppose that $x^2 \in I^2$ and $x^3 \in I^3$. Then it holds that $x^2 \in I_{p_i}{}^2$, $x^3 \in I_{p_i}{}^3\,(1 \leq i \leq n)$. Hence $x \in I_{p_i}\,(1 \leq i \leq n)$ by assumption, and thus $x \in \mathrm{I}$.

Q. E. D.

REMARK 4. (1) Since each $I_{\eta_i} := (R : {}_R\eta_i) = \{a \in R | a\eta_i \in R\}$ is a divisorial ideal of R, then we conclude that $I_{[\alpha]} = \bigcap_{i=1}^{d} I_{\eta_i}$ is a divisorial of R.
(2) It is well-known that the prime divisor p of a divisorial ideal are of depth one, that is $p \in \mathrm{Dp}_1(R)$ (cf. [Y]). If α is super-primitive over R, then we can prove that $(I_{[\alpha]})_p$ is a principal ideal of R_p for all $p \in \mathrm{Dp}_1(R)$ (cf. [OSY]).
(3) If R is seminormal, then we can see easily that aR is (2, 3)-closed for all $a \in R$. In fact, if $x^2 \in (aR)^2$, $x^3 \in (aR)^3$ for any $x \in R$, then it holds $(x/a)^2$, $(x/a)^3 \in R$. Since R is seminormal, we have $x/a \in R$, and hence $x \in aR$.

PROPOSITION 5. Let R be a Noetherian domain with quotient field K and let α be a super-primitive element of degree d over R. If R is seminormal, then $I_{[\alpha]}$ is (2, 3)-closed.

PROOF. We have only to show that $(I_{[\alpha]})_p$ is (2, 3)-closed for any $p \in \mathrm{Ass}(R/I_{[\alpha]})$. Note that $(I_{[\alpha]})_p$ is a principal ideal of R_p by *REMARK* 4 (2). Since, by assumption, R is seminormal, then R_p is also seminormal (cf. [GH]). Hence $((I_{[\alpha]})_p$ is (2, 3)-closed by REMARK 4 (3). Therefore we have that $I_{[\alpha]}$ is (2, 3)-closed.

Q. E. D.

REMARK 6. For a super-primitive element α, the following result is shown in [OY] :

$$A \cap K = R \Leftrightarrow \bigcap_{i=1}^{d-1} I_{\eta_i} \subseteq I_{\eta_d}.$$

PROPOSITION 7. Suppose that α is super-primitive of degree d over R and that $\bigcap_{i=1}^{d-1} I_{\eta_i} \subseteq I_{\eta_d}$. If B_α is seminormal, then R is seminormal.

PROOF. Let $\lambda \in$ K and suppose that $\lambda^2, \lambda^3 \in R$. Then we have $\lambda^2, \lambda^3 \in B_\alpha$. Since B_α is seminormal, it holds $\lambda \in B_\alpha \subseteq A$, and so $\lambda \in A \cap K = R$. Hence R is seminormal.

Q. E. D.

Let (V, m) be a discrete valuation ring with maximal ideal m and let J be a non-zero ideal of V. Then m and J are principal ideals tV and $t^{\ell} V$ for some positive integer ℓ. Let ν denote the valuation associated with V. Then we write $\nu(J) = \ell$.

PROPOSITION 8. Assume that R is a discrete valuation ring with maximal ideal m and that d $\geqq 4$. Let ν denote the valuation associated with ν. If B_α is seminormal and $\eta_2 \in R$, then we have $\nu(I_{[\alpha]}) = 1$ or $\nu(I_{[\alpha]}) \geq \nu(I_{\eta_i}) \geq \nu(I_{[\alpha]}) - 1$.

PROOF. (1) $\nu(I_{\eta_1}) > 0$. Suppose that $\nu(I_{\eta_1}) \leq \nu(I_{[\alpha]}) - 2$. Put $e = \nu(I_{[\alpha]}) - 1$. Then we have $e \geq 2$. Take an element a of R with v $(a) =$ e. Since ν $(a) < \nu(I_{[\alpha]})$, we see that $a \notin I_{[\alpha]}$. By the equality $(a\zeta_1)^2 = a^2\zeta_2 + (a^2\eta_1)\zeta_1 - (a^2\eta_2)$, we have

$$\nu(a^2) = 2\nu(a) = 2e \geq e+1 = \nu(I_{[\alpha]}),\ \nu(a^2\eta_1) = 2e - \nu(I_{\eta_1}) = e + (e - \nu(I_{\eta_1})) \geq e+1 = \nu(I_{[\alpha]}).$$

Hence $a^2, a^2\eta_1 \in I_{[\alpha]}$. Since $a^2\eta_2 \in R$, it follows that $(a\zeta_1)^2 \in B_\alpha$. On the other hand, by the equality

$$(a\zeta_1)^3 = a^3\zeta_3 + 2(a^3\eta_1)\zeta_2 + (\eta_1{}^2 - \eta_2)a^3\zeta_1 - (\eta_1\eta_2 + \eta_3)a^3,$$

we can conclude that

$$\nu(a^3) = 3\nu(a) = 3e \geq e+1 = \nu(I_{[a]}).$$
$$\nu(a^3\eta_1) = 3e - \nu(I_{\eta_1}) = 2e + (e - \nu(I_{\eta_1})) \geq 2e+1 \geq e+1 \geq \nu(I_{[a]}).$$
$$\nu(a^3\eta_1{}^2) = 3e - 2\nu(I_{\eta_1}) = 2e + 2(e - \nu(I_{\eta_1})) \geq e+2 \geq e+1 \geq \nu(I_{[a]}).$$
$$\nu(a^3\eta_i) = 3e - \nu(I_{\eta_i}) = e + (2e - \nu(I_{\eta_i})) \geq e + (2e - \nu(I_{[a]}))$$
$$= e + (e-1) = 2e=1 \geq e+1 \geq \nu(I_{[a]}) \qquad (i = 2,\ 3)$$

Therefore $(a\zeta_1)^3 \in B_a$. Since B_a is seminormal, this impliet $a\zeta_1 \in B_a$. Hence we have $a \in I_{[a]}$, a contradiction.

(2) $\nu(I_{\eta_1}) = 0$. In this case, we have $\eta_1 \in R$. It then follows that "B_a is seminormal in A" is equivalent to "B_a is integrally closed in A" (cf. [SY]). Then we have $\nu(I_{[a]}) = 1$ or $\nu(I_{[a]}) = \nu(I_{\eta_1})$ (cf. [TOY]). This completes the proof.

Q. E. D.

Let $A \subseteq B$ be integral domains. We say that the dimension formula holds between A and B if ht (P) = ht (p) + Tr. $\deg_A B$-Tr. $\deg_{k(p)} k(P)$ for every $P \in \mathrm{Spec}(B)$, where $p = P \cap A$. In this case, it is known that if $I_{[a]}$ is integrally closed, then R_p is a DVR for all $p \in \mathrm{Ass}_R(R/I_{[a]})$ (cf. [TOY]). Combining with the above results, we have the following :

THEOREM 9. Let R be a Noetherian domain with quotient field K. Assume that $\overline{R}$ is a finitely generated R-module and that the dimension formula holds between R and $\overline{R}$. Assume that α is a anti-integral element of degree d over R and that $I_{[a]}$ is integrally closed. Let $A = R[\alpha] \supset B_a = R[\alpha] \cap R[\alpha^{-1}]$. For any $p \in \mathrm{Ass}_R(R/I_{[a]})$, let ν_p be the associated discrete valuation with (R_p, pR_p). Then it holds that $\nu(I_{[a]}) = 1$ or $\nu(I_{\eta_1}) \geq \nu(I_{[a]}) - 1$ if B_a is seminormal. Hence we have the following possibilities :

(1) $\nu(I_{[a]}) = 1$,

(2) $\nu(I_{[a]}) = \nu(I_{\eta_1})$,

(3) $\nu(I_{\eta_1}) = \nu(I_{[a]}) - 1$.

References

[BC] J. Brewer and D. Costa : Seminormality and projective modules over polynomial rings, J. Algebra, **58** (1979), 208-216.

[GH] R. Gilmer and R. Heitmann : On Pic (R[X]) for R seminomal, J. Pure Appl. Alg., **16** (1980), 251-257.

[KY] M. Kanemitsu and K. Yoshida : Some properties of extensions $R[\alpha] \cap R[\alpha^{-1}]$ over Noetherian domains R, Communications in Algebra, **23** (12)(1995), 4501-4507.

[OSY] S. Oda, J. Sato and K. Yoshida : High degree anti-integral extensions of Noetherian domains, Osaka. J. Math., **30** (1993), 119-135.

[OY] S. Oda and K. Yoshida : Remarks on an exclusive extension generated by a super-primitive. Osaka. J. Math., **32** (1995), 495-499.

[TOY] S. TAEDA, S. ODA and K. YOSHIDA : On the integral closure of the ring $R[\alpha] \cap R[1/\alpha]$, to appear in Journal of Algelra.