Super-Primitive Ideals and Sharma Polynomials in Polynomial Rings

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Abstract

In [OSY] , we investigated the following : Let R be a Noetherian integral domain, let $R[X]$ be a polynomial ring and let K be the quotient field of R. Let α be an element of an algebraic field extension L of K and let $\pi : R[X] \to R[\alpha]$ denote the R-algebra homomorphism sending X to α. Let $\varphi_\alpha(X)$ be the monic minimal polynomial of α over K with deg $\varphi_\alpha(X) = d$ and write $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$. Let $I_{[\alpha]} := \cap_{i=1}^{d}(R :_R \eta_i)(= R[X] :_R \varphi(X))$. For $f(X) \in R[X]$, let $C(f(X))$ denote the ideal generated by the coefficients of $f(X)$. Put $J_{[\alpha]} := I_{[\alpha]}C(\varphi_\alpha(X))$, which is an ideal of R and contains $I_{[\alpha]}$. We say that the element α is an anti-integral element of degree d over R if $Ker\ \pi = I_{[\alpha]}\varphi_\alpha(X)R[X]$. When α is an anti-integral element over R, $R[\alpha]$ is called an anti-integral extension of R. In the case $K(\alpha) = K$, an anti-integral element α is the same as an anti-integral element (i. e., $R = R[\alpha] \cap R[1/\alpha]$) defined in[OY]. The element α is called a super-primitive element of degree d over R if $J_{[\alpha]} \not\subset p$ for all primes p of depth one.

As was seen in the above, [OSY] concerned a simple extension with certain properties by use of the ideal $Ker\ \pi$. This paper deals with an ideal H of $R[X]$ such that $P \cap R = (0)$ for all $P \in Ass_{R[X]}(R[X]/H)$(i. e., exclusive) or a monic polynomial $\varphi(X)$ in $K[X]$. We define the super-primitiveness, anti-integralness and flatness of the ideal H or a polynomial $\varphi(X)$. If necessary, we can consider a simple ring-extension $R[X]/H$ or $R[X]/(\varphi(X)K[X] \cap R[X])$. When H is a prime ideal or $\varphi(X) \in K[X]$ is an irreducible polynomial, we come back to the case treated in [OSY]. When H is not a prime ideal or $\varphi(X) \in K(X)$ is not an irreducible polynomial, we can extend the super-primitiveness, anti-integralness and flatness to a simple extension which is not necessarily an integral domain.

We use the following notation throughout this paper unless otherwise specified :

Let R be a Noetherian integral domain and $R[X]$ a polynomial ring and let K is the quotient field of R. Let H be an ideal of $R[X]$ and let $\varphi(X)$ be a monic polynomial in $K[X]$.

Our unexplained technical terms are standard and are seen in [M1] and [M2].

1. Anti-Integral Ideals and Super-Primitive Ideals

Let $R[X]$ be a polynomial ring over R and let H denote an ideal of $R[X]$. We say that H is an *exclusive* ideal if $P \cap R = (0)$ for every prime ideal $P \in Ass_{R[X]}(R[X]/H)$. When H is exclusive, $H_K := H \otimes_R K \subsetneqq K[X]$. Since H_K is a principal ideal of $K[X]$, we can write $H_K = \varphi_H(X)K[X]$ for some monic polynomial $\varphi_H(X) \in K[X]$. Let $d = \deg\varphi_H(X)$ and put $\varphi_H(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$ with $\eta_i \in K$. Let $I_H := \bigcap_{i=1}^d I_{\eta_i} (= R[X]:_R \varphi_H(X))$. Then $I_H\varphi_H(X)R[X] \subseteq R[X]$. For $f(X) \in K[X]$, $C(f(X))$ denotes the content ideal of $f(X)$, the fractional ideal of R generated by the coefficients of $f(X)$.

Definition 1.1. Let H be an exclusive ideal of $R[X]$.

(1) The ideal H is an *anti-integral ideal* of $R[X]$ or is of *anti-integral type* if $H = I_H\varphi_H(X)R[X]$.

(2) The ideal H is a *super-primitive ideal* of $R[X]$ or is of *super-primitive type* if grade $(I_H C(\varphi_H(X))) > 1$.

Remark 1.2. Assume that the ideal H of $R[X]$ is exclusive. Then $H_K \cap R[X] = H$. In particular, the inclusion $I_H\varphi_H(X)R[X] \subseteq H$ holds. Indeed, since $H_K \supseteq H$ asserts the inclusion $H \subseteq H_K \cap R[X]$. Conversely, take any $f(X) \in \varphi_H K[X] \cap R[X]$. Then $f(X) = \varphi_H(X)\zeta(X)$ for some $\zeta(X) \in K[X]$. Since $f(X) \in H_K$, there exists a non-zero $a \in R$ such that $af(X) \in H$. Since H is exclusive over R, we have that $a \notin P$ for each $P \in Ass_{R[X]}(R[X]/H)$. Thus $f(X) \in H$, which implies that $H \supseteq H_K \cap R[X]$.

Proposition 1.3. *Assume that H is an exclusive ideal of $R[X]$. If H is generated by some polynomials of the least degree, then H is anti-integral type.*

Proof. Let d denote the least degree of a polynomial in H. Then $\varphi_H(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d (\eta_i \in K)$. Take $f(X) = a_0 X^d + a_1 X^{d-1} + \cdots + a_d \in H$ with $a_i \in R$. Since $f(X) \in H$, we have $a_i/a_0 = \eta_i (1 \le i \le d)$. So $a_0 \in I_H$ and hence $f(X) = a_0\varphi_H(X)$. Thus $H \subseteq I_H\varphi_H(X)R[X]$. Since $I_H\varphi_H(X)R[X] \subseteq H$ by Remark 1. 2, we have $H = I_H\varphi_H(X)R[X]$, which means that H is an anti-integral ideal. □

Let $f(X) = a_0 X^n + a_1 X^{n-1} + \cdots + a_n$ be a polynomial in $R[X]$. We say that $f(X)$ is a *Sharma polynomial* in $R[X]$ if there does not exist $t \in R$ with $t \notin a_0 R$ such that $ta_i \in a_0 R$ for $1 \le i \le n$.

We require the following result seen in [OSY] :

Lemma 1.4. ([OSY,(1.2)]). *Let $f(X)$ be a polynomjal in $R[X]$. Then $f(X)$ is a Sharma polynomial if and only if $C(f(X)) \nsubseteq P$ for any $P \in \mathrm{Dp}_1(R) := \{Q \in \mathrm{Spec}(R) | \mathrm{depth}_{R_Q} = 1\}$.*

Lemma 1.5. (cf. [OSY, (1. 3)]). *Assume that an ideal H is exclusive and that deg*

$\varphi_H(X) = d$. *The following statements are equivalent*:

(i) *H is a principle ideal of $R[X]$*;

(ii) *I_H is a principle ideal of R*;

(iii) *there exists a Sharma polynomial in H of degree d.*

If one of the above conditions holds, then H is generated by a Sharma polynomial.

Proof. (iii)$\Rightarrow$(i) : Let $f(X)$ be a Sharma polynomial in H of degree d. Since $\deg_{\varphi H}(X) = d$, this Sharma polynomial has the least degree. So by [S], H is principal.

(i)$\Rightarrow$(ii) : Let $H = f(X)R[X]$. Then $f(X)R[X] \supseteq I_H\varphi_H(X)R[X]$. Note that $H \otimes_R K = f(X)K[X] = \varphi_H(X)K[X]$ and hence $\deg f(X) = \deg \varphi_H(X) = d$. Take $a \in I_H$. Then $a\varphi_H(X) = bf(X)$ for some $b \in R$. Let $f(X) = a_0X^d + \cdots + a_d$ with $a_i \in R$. Then $a = ba_0$, so that $I_H \supseteq a_0R$ for some $b \in R$. Since $ba_0\eta_i = a\eta_i = ba_i (1 \le i \le d)$, we have $a_0\eta_i = a_i \in R$. Hence $a_0 = I_H$, which implies that $I_H = a_0R$.

(ii)$\Rightarrow$(iii) : Let $I_H = bR$. Then $I_H\varphi_H(X)R[X] = b\varphi_H(X)R[X] \subseteq H$ and $b\eta_i \in R (1 \le i \le d)$. Suppose that there exists $t \notin bR$ with $tb\eta_i \in bR (1 \le i \le d)$. Then $t\eta_i \in R$ and hence $t \in I_H = bR$, a contradiction. Thus $b\varphi_H(X) \in R[X]$ is a Sharma polynomial of degree d. □

Proposition 1.6. *Assume that H is an exclusive ideal. If H is a super-primitive ideal of $R[X]$, then H is an anti-integral ideal of $R[X]$.*

Proof. Since H is exclusive, $H \supseteq I_{H\varphi H}(X)R[X]$ by Remark 1. 2. Take $f(X) \in H$. Then since $f(X) \in H_K$, there exists a non-zero element $a \in R$ such that $af(X) \in I_H\varphi_H(X)R[X]$. Take an arbitrary $p \in \mathrm{Dp}_1(R)$. Then $I_HC(\varphi_H(X)) \nsubseteq p$ because grade $(I_H\varphi_H(X)R[X]) > 1$. So there exists a Sharma polynomial in H_p of the least degree. By Lemma 1. 4, we have $H_p = g(X)R_p[X]$ for some $g(X) = b\varphi_H(X)$ with $b \in R$. Since $f(X) \in H_p$, there exists $c \in R \backslash p$ such that $cf(X) \in I_H\varphi_H(X)R[X]$. Thus $f(X) \in I_H\varphi_H(X)R[X]$. □

Remark 1.7. If H is a super-primitive ideal of $R[X]$, then grade $(C(H)) > 1$. But the converse statement is not necessarily valid as is seen in the example below.

Example 1.8. Consider an integral domain R satisfying the condition : $R \subsetneqq \overline{R}$, where $\overline{R}$ denotes the integral closure. Take $\alpha \in \overline{R}$ but $\alpha \notin R$. Then we have an exact sequence :

$$0 \to H \to R[X] \to R[\alpha] \to 0 \text{ (exact)}$$

Note that H is a prime ideal of $R[X]$. In this case, H contains a monic polynomial (i, e., a polynomial giving an integral dependence of α). Hence $C(H) = R$. Suppose that H is a super-primitive ideal, then H is an anti-integral ideal by Proposition 1. 6. Since $H = I^R_{\varphi H(X)}\varphi_H(X)R[X]$, we have $I^R_{\varphi H(X)}C(\varphi_H(X)) = C(H) = R$. Thus α is flat over R (cf. [OSY, (2. 6)]). Since α is an integral flat element in K, $R[\alpha] = R$, that is, $\alpha \in R$, which is a contradiction. Therefore H is not a super-primitive ideal.

Theorem 1.9. *Assume that H is an exclusive ideal of $R[X]$.*

(1) *If H is of anti-integral type and* grade $(C(H)) > 1$, *then H is of super-primitive type.*

(2) *If H is of anti-integral type and H contains a Sharma polynomial, then H is of*

super-primitive type.

Proof. Note that $H = I_H\varphi_H(X)R[X]$ by the assumption. Since $C(H) = I_H C(\varphi_H(X))$, (1) is valid. The statement (2) follows from Lemma 1. 4. □

Let H be an exclusive ideal of $R[X]$. Define $J_H := I_H C(\varphi_H(X))$, an ideal of R.

Proposition 1. 10. *Let H be an exclusive ideal of $R[X]$.*

(1) *If $J_H = R$, then $R[X]/H$ is flat over R.*

(2) *If $J_H = R$, then H is of super-primitive type and I_H is an invertible ideal of R.*

Proof. (1) Take $p \in \mathrm{Spec}(R)$. Since $R = J_H = I_H C(\varphi_H(X))$, there exists $b \in I_H$ such that $(I_H)_p$ is a principal ideal bR_p. So $(I_H)_p = b\varphi_H(X)R_p[X]$. It follows that $R_p[X]/(I_H)_p = R_p[X]/b\varphi_H(X)R_p[X]$. Thus $R_p[X]/b\varphi_H(X)R_p[X]$ is flat over R_p by [M1, (20. F)] because $R_p = (J_H)_p = C(b\varphi_H(X))_p$. Hence $R[X]/H$ is flat over R.

(2) Since $R = J_H = I_H C(\varphi_H(X))$, I_H is an invertible ideal of R and H is a super-primitive ideal by definition. □

Corollary 1. 10. 1. *Let H be an exclusive ideal of $R[X]$. If $J_H = R$, then $H = I_H\varphi_H(X)R[X]$ and H is an invertible ideal of $R[X]$. Furthermore H^ℓ ($\ell > 0$) is also an invertible ideal of $R[X]$.*

Proof. By Proposition 1. 10 (2), H is of super-primitive type and hence H is of anti-integral type by Proposition 1. 6. So $H = I_H\varphi_H(X)R[X]$, which is an invertible ideal of $R[X]$. □

Remark 1. 11. Assume that H satisfies the condition in Corollary 12. 1. Let $J := \{a \in R \mid a\varphi_H(X) \in R[X]\}$, an ideal of R. Then $H^\ell = J\varphi_H(X)^\ell R[X]$ and $J = (I_H)^\ell$.

2. Sharma Polynomials

In this section, we investigate how a Sharma polynomial works.

Proposition 2. 1. *Let $f(X)$ be a Sharma polynomial. Then $f(X)R[X]$ does not have any embedded prime divisor.*

Proof. Let P be a prime divisor of $f(X)R[X]$. Then depth $R(X)_p = 1$. Suppose that $p := P \cap R \neq (0)$. Then depth $R_p = 1$ and $P = pR[X]$. Thus $f(X) \in P = pR[X]$, which implies that $C(f(X)) \subseteq p$. Hence grade $(C(f(X))) = 1$. This contradicts the assumption that $f(X)$ is a Sharma polynomial. So we have $P \cap R = (0)$. Since $P = P_K \cap R[X]$ and $\mathrm{ht}(P) = 1$, $f(X)R[X]$ has no embedded prime divisor. □

Remark 2. 2. Let $f(X)$ be a Sharma polynomial in $R(X)$ and let P be a prime divisor of $f(X)R[X]$. It does not necessarily follow that P is of super-primitive type. But if P is of anti-integral type, then P is of super-primitive type. Indeed, consider the exact sequence :

$$0 \to P \to R[X] \to R[\alpha] \to 0,$$

where α denotes Xmod. P. In this case, since P contains a Sharma polynomial $f(X)$, α is a super-primitive element over R by [OSY, (1. 12)] . Next consider $R = \overline{R}$. If $P \cap R = (0)$, then P is of super-primitive type (cf. [OSY,(1. 13)]).

Proposition 2. 3. *Let $f(X) \in R[X]$ be a Sharma polynomial. Assume that $f(X)$ is irreducible in $K[X]$. Then $f(X)R[X]$ is a prime ideal in $R[X]$.*

Proof. Let P be a prime divisor of $f(X)\ R[X]$. Then $f(X) \in \varphi_P(X)K[X]$. Since $f(X)$ is irreducible, there exists $a \in R$ such that $f(X) = a\varphi_P(X)(a \in P)$. Since grade $(C(f(X))) > 1$, P is of super-primitive type. So P is of anti-integral type by Proposition 1. 6 and hence $P = I_P\varphi_P(X)R[X]$. Thus $f(X)R[X] = P$. □

Proposition 2. 4. *Let $f(X)$ be a Sharma polynomial in $R[X]$. Let P be a prime ideal in $R[X]$ such that P is a unique prime divisor of the ideal $f(X)R[X]$. Then P is of super-primitive type.*

Proof. By the assumption, $f(X)K[X] = \varphi_P(X)^\ell\ K[X]$. Since $f(X)$ is a Sharma polynomial, we have $f(X)K[X] \cap R[X] = f(X)R[X]$ (cf. [S, Remark 5]). Hence $f(X)R[X] \subseteq (I_P\varphi_P(X))^\ell R[X]$. So since grade $(C(f(X))) > 1$, we have grade $(I_P\varphi_P(X)) > 1$. Hence P is of super-primitive type. □

Lemma 2. 5. *Let $\varphi(X) = X^d + \eta_1 X^{d-1} + \cdots \eta_d (\eta_i \in K)$ be a monic irreducible polynomial in $K[X]$. Put $I^R_{\varphi(X)} := \bigcap_{i=1}^d I_{\eta_i} (= R[X] :_R \varphi(X))$. If* grade $(I^R_{\varphi(X)}C(\varphi(X)))$ > 1, *then* $\varphi(X)K[X] \cap R[X] = I^R_{\varphi(X)}\varphi(X)R[X] \in$ Spec $(R[X])$.

Proof. Since $\varphi(X)K[X] \in$ Spec $(K[X])$, we have $P := \varphi(X)K[X] \cap R[X] \in$ Spec $(R[X])$. By construction, $P \supseteq I^R_{\varphi(X)}R[X]$. By the assumption,

$$\text{grade } (I^R_{\varphi(X)}C(\varphi(X))) > 1.$$

Thus P is of super-primitive type. So we obtain $P = I^R_{\varphi(X)}\varphi(X)R[X] \in$ Spec $(R[X])$. □

Theorem 2. 6. *Let $f(X)$ be a Sharma polynomial in $R[X]$ and let $f(X) = a\varphi_1(X)^{e_1} \cdots \varphi_t(X)^{e_t}$ be a products of irreducible polynomials $\varphi_i(X) \in K[X]$ and $a \in R$. Then $f(X)R[X] = (\varphi_1(X)^{e_1}\ K[X] \cap R[X]) \cap \cdots \cap (\varphi_t(X)^{e_t}\ K[X] \cap R[X])$ is a primary decomposition.*

Proof. Since $f(X)$ is a Sharma polynomial, we have $f(X)R[X] = f(X)K[X] \cap R[X] = \varphi_1(X)^{e_1} \cdots \varphi_t(X)^{e_t}K[X] \cap R[X] = (\varphi_1(X)^{e_1}K[X] \cap \cdots \cap (\varphi_t(X)^{e_t}\ K[X]) \cap R[X] = (\varphi_1(X)^{e_1}\ K[X] \cap R[X]) \cap \cdots \cap (\varphi_t(X)^{e_t}\ K[X] \cap R[X])$. Since $f(X) \in \varphi_i(X)K[X] \cap R[X]$, the prime ideal $\varphi_i(X)K[X] \cap R[X]$ is of super-primitive type. So $\varphi_i(X)K[X] \cap R[X] = I^R_{\varphi_i(X)}\ R[X]$ by Lemma 2. 5. Thus we see that $\varphi_i(X)^{e_i}K[X] \cap R[X]$ is an $I^R_{\varphi_i(X)}\ R[X]$-primary ideal. □

Proposition 2. 7. *Let $I \subseteq J$ be ideals of $R[X]$. Assume that $I \otimes_R K = J \otimes_R K$ and that for each $p \in \mathrm{Dp}_1(R)$, JR_p contains a Sharma polynomial over R_p of the least degree. ThenI $= J$.*

Proof. Take $f(X) \in I$. Then $f(X) \in I \subseteq I \otimes_R K = J \otimes_R K$. Hence there exists $a \in R$ such that $af(X) \in J$. Let p be a prime divisor of aR. Then $p \in \mathrm{Dp}_1(R)$. By the assumption, $J_p = g(X)R_p[X]$, $C(g(X)) = R_p$ for some $g(X) \in J$. Put $f(X) = g(X)\zeta(X)$ with $\zeta(X) \in R_p[X]$. Then there exists $b \in R \backslash p$ such that $bf(X) \in J$. Since a, b is a regular sequence, we have $f(X) \in J$. □

3. Super-Primitive Polynomials and Sharma Polynomials

Definition 3. 1. Let $\varphi(X)$ be a monic polynomial in $K[X]$ and let $I^R_{\varphi(X)} := R[X] :_R \varphi(X)$. The polynomial $\varphi(X)$ is called a *super-primitive polynomial* if grade

$(I_{\varphi}^{R}C(\varphi(X))) > 1$.

Let $\bar{R}$ denote the integral closure of R in K and let $C(\bar{R}/R)$ denote the conductor ideal between R and $\bar{R}$. For an element $\eta \in K$, we put $I_{\eta} := \{a \in R | a\eta \in R\}$.

Remark 3. 2. Assume that $R \subsetneqq \bar{R}$.

(1) There exists a polynomial in $K[X]$ which is not super-primitive. In fact, take $\eta \in \bar{R} \backslash R$ and let $\varphi(X) := X - \eta$. Then $\varphi(X)$ is not super-primitive (cf. Example 3. 9. below).

(2) When $\eta \in K$ satisfies grade $(I_{\eta} + C(\bar{R}/R)) > 1$, the polynomial $X - \eta$ is super-primitive. More generally, let $\varphi(X) = X^{d} + \eta_1 X^{d-1} + \cdots + \eta_d (\eta_i \in K)$ satisfy grade $(I_{\eta_i} + C(\bar{R}/R)) > 1$ for all i. Then grade $(\bigcap_{i=1}^{d} I_{\eta_i} + C(\bar{R}/R)) =$ grade $(I_{\varphi(X)}^{R} + C(\bar{R}/R)) > 1$. Then $\varphi(X)$ is super-primitive (cf. Proposition 3. 10).

Proposition 3. 3. *Let B be an ideal of R. Assume that every prime divisor of B is contained in* $\mathrm{Dp}_1(R)$. *Let a, b be a regular sequence and $f \in R$. If af, $bf \in B$, then $f \in B$.*

Proof. Consider a primary decomposition of B :

$$B = q_1 \cap \cdots \cap q_n,$$

where $\sqrt{q_i} = p_i$. Then $p_i \in \mathrm{Dp}_1(R)$ for all $1 \leq i \leq n$. Take p_i. Then a, b is an R-regular sequence. So $a \notin p_i$ or $b \notin p_i$. If $a \notin p_i$, then $f \in q_i R_{p_i} \cap R = q_i$. A similar argument is applicable to the case $b \notin p_i$. Hence we conclude that $f \in B$. □

Proposition 3. 4. *Let $\varphi(X) \in K[x]$ be an irreducible super-primitive polynomial. Let $I := I_{\varphi(X)}^{R}$ and $J := I_{\varphi(X)^{\ell}}^{R}$ $(\ell > 0)$. Then $I\varphi(X)R[X]$ is a prime ideal of $R[X]$ and $J\varphi(X)^{\ell}R[X]$ is $I\varphi(X)R[X]$-primary ideal.*

Proof. Since $\varphi(X)^{\ell}K[X]$ is $\varphi(X)K[X]$-primary, the ideal $\varphi(X)^{\ell}K[X] \cap R[X]$ is $\varphi(X)K[X] \cap R[X]$-primary. Since $\varphi(X)$ is super-primitive and hence anti-integral. So $\varphi(X)K[X] \cap R[X] = I\varphi(X)R[X]$. Thus we have only to prove that $\varphi(X)^{\ell}K[X] \cap R[X] = J\varphi(X)^{\ell}R[X]$. The implication $(\supseteq)$ is obvious. We shall show that the implication $(\subseteq)$ holds. Since J is a denominator ideal in R, J is a divisorial ideal. So $JR[X] (\subseteq R[X])$ is also a divisorial ideal. Hence $J\varphi(X)^{\ell} R[X] (\subseteq R[X])$ is a divisorial ideal. Thus any prime divisor of $J\varphi(X)^{\ell}R[X]$ is of depth one. Take $f(X) \in \varphi(X)^{\ell} K[X] \cap R[X]$. Then $f(X)/\varphi(X)^{\ell} \in K[X]$. We must show that $f(X)/\varphi(X)^{\ell} \in JR[X]$. Let p be any prime divisor of J. Then $p \in \mathrm{Dp}_1(R)$. In this case, we have to show $f(X)/\varphi(X)^{\ell} \in (JR[X])_p$. Since $(I\varphi(X))^{\ell}R[X] \subseteq J_{\varphi}(X)^{\ell}R[X]$, we have grade $(JC(\varphi(X)^{\ell})) > 1$. Hence $J_p = aR_p$ for some $a \in J$. Put $g(X) = a\varphi(X)^{\ell}$. Then $g(X) \in J$ and $C(g(X))R_p = R_p$, which shows $f(X) \in g(X)R_p[X]$. Thus $f(X)/\varphi(X)^{\ell} \in (g(X/\varphi(X)^{\ell})R_p[X] = aR_p[X] = (JR[X])_p$. From this, we get $\varphi(X)^{\ell} K[X] \cap R[X] = J\varphi^{\ell} R[X]$. By Lemma 2. 5, $I\varphi(X)R[X] \in \mathrm{Spec}\,(R[X])$. Therefore we have $J\varphi(X)^{\ell} R[X]$ is an $I\varphi(X)$-primary ideal. □

Theorem 3. 5. *Let $\varphi_1(X), \cdots, \varphi_t(X) \in K[X]$ be irreducible super-primitive polynomials. Let $I_i := I_{\varphi_i(X)}^{R} (1 \leq i \leq t)$ and let $J_i := I_{\varphi_i(X)^{e_i}}^{R} (1 \leq i \leq t,\ e_i > 0)$. Then*

$$\varphi_1(X)^{e1}\cdots\varphi_t(X)^{et}\ K[X]\cap R[X]=\bigcap_{i=1}^{t}(J_i\varphi_i(X)^{e_i}\ R[X])$$

and this is a primary decomposition of the left side ideal, where $J_i\varphi_i(X)^{e_i}\ R[X]$ is an $I_i\varphi_i(X)R(X)$-primary ideal for each $1\le i\le t$.

Proof. It follows that $\varphi_1(X)^{e_1}\cdots\varphi_t(X)^{e_t}\ K[X]\cap R[X]=(\varphi_1(X)^{e_1}\ K[X]\cap\cdots\cap\varphi_t(X)^{e_t}\ K[X])\cap R[X]=(\varphi_1(X)^{e_1}\ K[X]\cap R[X])\cap\cdots\cap(\varphi_t(X)^{e_t}\ K[X]\cap R[X]\cap R[X])=J_1\varphi_1(X)^{e_1}\ R[X]\cap\cdots\cap J_t\varphi_t(X)^{e_t}\ R[X]$ by Proposition 1. 6. By Proposition 3. 4, $J_i\varphi_i(X)^{e_i}R[X]$ is an $I_i\varphi_i(X)R[X]$-primary ideal for each $1\le i\le t$. □

Proposition 3. 6. *Let $\varphi_1(X),\cdots,\ \varphi_t(X)\in K[X]$ be irreducible super-primitive polynomials. Then $\varphi_1(X)^{e_1}\cdots\varphi_t(X)^{e_t}\ K[X]\cap R[X]$ is a prinicipal ideal at each $P\in$* $\mathrm{Dp}_1(R)$.

Proof. Let $J:=R[X]:{}_R\varphi_1(X)^{e_1}\cdots\varphi_t(X)^{e_t}$ and let $I_i:=R[X]:{}_R\varphi_i(X)$. Since $\varphi_i(X)$ is a super-primitive polynomial, we have grade $(I_iC(\varphi_i(X)))>1$ for each $1\le i\le t$. Note that $(I_1)^{e_i}\cdots(I_t)^{e_t}\subseteq J$, which yields

$$\text{grade }(JC(\varphi_1(X)^{e_1}\cdots\varphi_t(X)^{e_t}))>1.$$

Take $p\in\mathrm{Dp}_1(R)$. There exists $f_i(X)\in I_i\varphi_i(X)$ such that $C(f_i(X))\nsubseteq p$ for $1\le i\le t$. Put $f(X):=f_1(X)^{e_1}\cdots f_t(X)^{e_t}$. Then $C(f_i(X))\nsubseteq p$. Since $f(X)\in\varphi_1(X)^{e_1}\cdots\varphi_t(X)^{e_t}K[X]\cap R[X]$ and $f(X)$ has the least degree, it follows that $(\varphi_1(X)^{e_1}\cdots\varphi_t(X)^{e_t}\ K[X]\cap R[X])_p=f(X)R[X]_p$ by [S, Remark 5]. □

Remark 3. 7. Let $\varphi(X)\in K[X]$ be a monic polynomial and let $I:=I^R_{\varphi(X)}$. If $I\varphi(X)R[X]$ contains a Sharma polynomial, then $\varphi(X)$ is a super-primitive polynomial. Indeed, take a Sharma polynomial $f(X)\in I\varphi(X)R[X]$. Suppose there exists $p\in\mathrm{Dp}_1(R)$ such that $IC(\varphi(X))\subseteq p$. Then $I\varphi(X)R[X]\subseteq pR[X]$ and hence $f(X)\in pR[X]$. Thus $C(f(X))\subseteq p$, which contradicts the assumption that $f(X)$ is a Sharma polynomial.

Proposition 3. 8. *Let $\varphi(X)\in K[X]$ be a monic polynomial. Then $\varphi(X)$ is a super-primitive polynomial if and only if $I^R_{\varphi(X)}\ \varphi(X)R[X]$ contains a Sharma polynomial.*

Proof. The implication ($\Leftarrow$) follows from Remark 3. 7.

($\Rightarrow$) Take a non-zero element $a\in I^R_{\varphi(X)}$ and put $f_1(X)=a\varphi(X)$. Let $p_1,\cdots,\ p_n\in\mathrm{Dp}_1(R)$ such that $p_i\supseteq C(f_1(X))$. Since $\varphi(X)$ is a super-primitive polynomial, we have $I^R_{\varphi(X)}\ C(\varphi(X))\nsubseteq p_1,\cdots,\ p_n$. There exists $g_i(X)\in I^R_{\varphi(X)}\ \varphi(X)R[X]$ such that $C(g_i(X))\nsubseteq p_i(1\le i\le n)$. Put $f(X)=f_1(X)g_i(X)\cdots g_n(X)$. In this case, $C(f(X))\subseteq C(f_1(X))$, $C(g_i(X))(1\le i\le n)$. Thus for each $p\in\mathrm{Dp}_1(R)$, $C(f(X))\nsubseteq p$. Hence $f(X)$ is a Sharma polynomial and $f(X)\in I^R_{\varphi(X)}\ \varphi(X)R[X]$. □

The following example shows that an irreducible factor $\varphi(X)$ in $K[X]$ of a Sharma polynomial $f(X)$ is not always a super-primitive polynomial.

Example 3. 9. Assume that $R\subsetneq\overline{R}$. Take $\alpha\in\overline{R}\backslash R$. Since α is integral over R, there exists a monic polynomial $f(X)$ in $R[X]$ such that $f(\alpha)=0$. Note that $C(f(X))=R$ because $f(X)$ is monic. Thus $f(X)$ is a Sharma polynomial. It is obvious that $X-\alpha$ is a factor of $f(X)$ in $K[X]$. Suppose $X-\alpha$ is anti-integral. Then $f(X)\in(X$

$-\alpha)K[X] \cap R[X] = I_\alpha(X-\alpha)R[X]$. So $R = C(f(X)) \subseteq I_\alpha C(X-\alpha)$, which means that $R[\alpha] \cong R[X]/(X-\alpha)$ is flat over R (cf. [OSY, (2. 6)]). Since α is integral over R, we have $R[\alpha] = R$, that is, $\alpha \in R$, which is a contradiction. Therefore the polynomial $X-\alpha$ is not an anti-integral polynomial, and hence $X-\alpha$ is not a super-primitive polynomial.

Proposition 3. 10. *Let $\varphi(X)$ be a monic polynomial in $K[X]$.*

(1) *If* grade $(I^R_{\varphi(X)}+C(\overline{R}/R)) > 1$, *then* grade $(I^R_{\varphi(X)}C(\varphi(X))) > 1$.

(2) grade $(I^R_{\varphi(X)}C(\varphi(X))) > 1$ *if and only if $I^R_{\varphi(X)}\varphi(X)R[X]$ contains a Sharma polynomial.*

(3) *If $R = \overline{R}$, then* grade $(I^R_{\varphi(X)}C(\varphi(X))) > 1$.

Proof. (1) Take $p \in \mathrm{Dp}_1(R)$. If $I^R_{\varphi(X)} \nsubseteq p$, then $I^R_{\varphi(X)} \subseteq I^R_{\varphi(X)}C(\varphi(X))$. So $I^R_{\varphi(X)} \nsubseteq p$. On the other hand, if $I^R_{\varphi(X)} \subseteq p$, then $C(\overline{R}/R) \nsubseteq p$ by the assumption and hence R_p is a DVR. Let $v_p(\ )$ denote the valuation of R_p. If $v_p(I^R_{\varphi(X)}) = e$, then$(I^R_{\varphi(X)})_p = p^eR_p$ and there exists i such that $V_p(\eta_i) = -e > 0$, where $\varphi(X) = X^d+\eta_1X^{d-1}+\cdots+\eta_d$ with $\eta_j \in K$. Thus $v_p(I^R_{\varphi(X)}\eta_i) = 0$. Hence $(I^R_{\varphi(X)}\eta_i)_p \nsubseteq pR_p$. Thus $(I^R_{\varphi(X)}C(\varphi(X)))_p \nsubseteq pR_p$. This show that $I^R_{\varphi(X)}C(\varphi(X)) \nsubseteq p$. Therefore we obtain that grade $(I^R_{\varphi(X)}C(\varphi(X))) > 1$.

(2) $(\Rightarrow)$: Take $a \in I^R_{\varphi(X)}$ and put $f(X) = a\varphi(X)$. If $f(X)$ is a Sharma polynomial, then there is nothing to prove. Suppose that $f(X)$ is not a Sharma polynomial. Then there exists a prime ideal p in $\mathrm{Dp}_1(R)$ such that $C(f(X)) \subseteq p$. Take a non-zero element $b \in C(f(X))$. Then p is a prime divisor of the ideal bR because $p \in \mathrm{Dp}_1(R)$. Since the number of prime divisors of bR is finite, the number of prime ideals in $\mathrm{Dp}_1(R)$ containing $C(f(X))$ is also finite. Let $p_1, \cdots, p_n$ be the prime ideals containing $C(f(X))$. Since $I^R_{\varphi(X)}C(\varphi(X)) \nsubseteq p_i$ for all $1 \leq i \leq n$ by the assumption, there exists $g_i(X) \in I^R_{\varphi(X)}\varphi(X)$ such that $g_i(X) \notin p_iR[X]$ for each $1 \leq i \leq n$. Put $g(X) := f(X)+X^dg_1(X)+\cdots+X^{nd}g_n(X)$. Then $C(g(X)) = \sum_{i=1}^n(C(g_i(x))+C(f(X)) \nsubseteq p$ for all $p \in \mathrm{Dp}_1(R)$. Thus $g(X)$ is a Sharma polynomial.

(2) $(\Leftarrow)$: Let $f(X)$ be a Sharma polynomial in $I^R_{\varphi(X)}\varphi(X)R[X]$. Then we can express $f(X) = \sum f_i(X)g_i(X)$ with $f(X) \in I^R_{\varphi(X)}\varphi(X)$ and $g_i(X) \in R[X]$. Suppose that there exists $p \in \mathrm{Dp}_1(R)$ such that $I^R_{\varphi(X)}C(\varphi(X)) \subseteq p$. Then $f_i(X) \in pR[X]$ and hence $f(X) \in pR[X]$, which contradicts the assumption that $f(X)$ is a Sharma polynomial.

(3) follows from $C(\overline{R}/R) = R$ and (1). □

In the above Proposition 3 . 10, the converse implication of (1) does not always valid as is seen in following example.

Example 3. 11. Let k be a field and let t be an indeterminate. Let $R := k[t^2, t^3]$. Then $\overline{R} = k[t]$. Let $\varphi(X) := X^2+X+1/t^2 \in K[X] := k(t)[X]$. Then $I^R_{\varphi(X)} = t^2R$ and $I^R_{\varphi(X)}C(\varphi(X)) = t^2(1, 1/t^2)R = R$. So we have grade$(I^R_{\varphi(X)}C(\varphi(X)))$ = grade$(R) >$ 1, but grade$(I^R_{\varphi(X)}+C(\overline{R}/R))$ = grade(t^2R+tR) = grade$(tR) = 1$.

Remark 3. 12. Consider a Noetherian domain R satisfying $R \subsetneq \overline{R}$ and let $\varphi(X) = X^d+\eta_1X^{d-1}+\cdots+\eta_d \in K[X]$. Assume that every $\eta_i \in \overline{R}$ and that $\eta_j \notin R$ for some j. Then $I^R_{\varphi(X)}\varphi(X)R[X]$ does not cotain any Sharma polynomial.

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