Finitely Generated Ring-Extensions of Anti-Integral Type

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Let R be a Noetherian domais with quotient field K. We say that an element $\beta \in K$ is a *flat element* over R if $R[\beta]$ is flat over R (see [4]). In [4], it was shown that if $\beta_1, \ldots, \beta_n \in K$ are all flat elements then $R[\beta_1, \ldots, \beta_n]$is flat over R. Example 17 in [4] shows that this assertion does not always hold if $\beta_i \notin K$ for some i, that is, the assertion above is not necessarily valid in the non-birational case.

In this paper, we treat the case that some of $\beta_1, \ldots\ldots \beta_n$ are not contained in K and show that the similar assertion holds even in the non-birational case if $\beta_1, \ldots, \beta_n$ satisfy certain conditions.

Let A be an integral domain containing R. Assume that A and L are in some large fixed field. Let $K(A)$denote the quotient field of A. Let L be an algebraic field extension of K and $\alpha \in L$. Let $\pi^{(A)}: A[X] \longrightarrow A[\alpha]$ be the canonical A-algebra homomorphism sending X to α. Let $\varphi_\alpha^{(A)}(X)$ denotes the minimal polynomial of α over $K(A)$ and let deg $\varphi_\alpha^{(A)}(X)$ denotes the minimal polynomial of α over $K(A)$and let deg $\varphi_\alpha^{(A)}(X) = d_A$. Write

$$\varphi_\alpha^{(A)}(X) = X^{dA} + \eta_1 X^{dA-1} + \ldots + \eta_{dA},$$

($\eta_i \in K(A)$). Let $I_{\eta_i}^{(A)} := A :_A \eta_i$ and let $I_{[\alpha]}^{(A)} := \cap_{i=1}^{d_A} I_{\eta_i}^{(A)}$. If $A = R$, we put $d := d_A$, $I_{ni} := I_{ni}^{(A)}$ and $I_{[\alpha]} := I_{[\alpha]}^{(A)}$. It is easy to see that $I_{[\alpha]}$(resp. $I_{[\alpha]}^{(A)}$) is an ideal of R (resp. A).

An element $\alpha \in L$ is called an *anti-integral element* over R of degree d if the equality Ker $\pi = I_{[\alpha]}\varphi_\alpha(X)R[X]$. holds. For a polynomial $f(X) \in R[X]$, $C(f(X))$denotes the ideal of R generated by the coefficients of $f(X)$. Let J be a of $R[X]$, we denote by $C(J)$ the ideal of R generated by the coefficients of the polynomials in J. If $\alpha \in L$ is anti-integral, it follows that $C(\text{Ker } \pi) = C(I_{[\alpha]}\varphi_\alpha(X)R[X]) = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$. We put $J_{[\alpha]} := I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$, which is an ideal of R. Similarly let $J_{[\alpha]}^{(A)} := I[^{A}_{d}](1, \eta_1, \ldots, \eta_{d_A})$. When α is an element in K, $\varphi_\alpha(X) = X - \alpha$. So we have $J_{[\alpha]} := I_{[\alpha]}(1, \eta_1, \ldots, \eta_d) = I_{[\alpha]} + \alpha I_{[\alpha]}$.

It is knows that if α is anti-integral and integral over R of degree d, then $R[\alpha]$ is a free R-module of rank d(cf. [3]).

Throughout this paper, we use the above notation unless otherwise specified.

§1. Several Properties of the Ideal $J_{[\alpha]}$

We start with the following definition.

Definition 1.1. Assume that $\alpha \in L$ is anti-integral and integral over R of degree d. For an element $\beta \in K(\alpha) = K[\alpha]$, putting: $\beta = \varsigma_0 + \varsigma_1\alpha + \ldots + \varsigma_{d-1}\alpha^{d-1}$ with $\varsigma_i \in K$, let

$$T_\beta := \bigcap_{i=0}^{d-1} I_{\varsigma_i} = R[\alpha]:_R \beta.$$

Lemma 1.2. *Assume that $\alpha \in L$ is anti-integral over A of degree d_A. Put*

$$F_\ell^{(A)} := \{f(X) \in \operatorname{Ker}\pi^{(A)} | \deg f(X) = \ell\}$$

for a non-negative integer ℓ, where $\pi^{(A)}: A[X] \longrightarrow A[\alpha]$ is the canonical A-algebra homomorphism sending X to α. If $\ell \geq d_A := [K(A)(\alpha):K(A)]$ then $C(F_\ell) = J_{[\alpha]}^{(A)}$.

Proof. Since α is an anti-integral element of degree d_A over A, we have

$$\operatorname{Ker}\pi^{(A)} = I_{[\alpha]}^{(A)} \varphi_\alpha^{(A)}(X) A[X],$$

and hence

$$J_{[\alpha]}^{(A)} = C(\operatorname{Ker}\pi^{(A)}) = C(I_{[\alpha]}^{(A)} \varphi_\alpha^{(A)}(X) A[X]).$$

Take $f(X) \in I_{[\alpha]}^{(A)} \varphi_\alpha^{(A)}(X) A[X]$ with $\deg f(X) = d_A$. For an integer $\ell \geq d_A$, we see that $X^{l-d_A} f(X) \in F_l^{(A)}$. Thus $C(f(X)) = C(X^{l-d_A} f(X)) = C(f(X)) \subseteq C(F_l^{(A)})$. So we obtain that $J_{[\alpha]}^{(A)} \subseteq C(F_l^{(A)})$. Conversely, take $f(X) \in F_l^{(A)}$. Then $f(X) \in \operatorname{Ker}\pi^{(A)} = I_{[\alpha]}^{(A)} \varphi_\alpha^{(A)}(X) A[X]$. Hence we can write:

$$f(X) = \sum f_i(X) g_i(X),$$

with $\deg f_i(X) = d_A$, $f_i(\alpha) = 0$. From this, we have $C(f(X)) \subseteq \sum C(f_i(X)) A \subseteq J_{[\alpha]}^{(A)}$. Thus $C(F_l^{(A)}) \subseteq J_{[\alpha]}^{(A)}$. So we conclude that $J_{[\alpha]}^{(A)} = C(F_l^{(A)})$. Q.E.D.

Lemma 1.3. *Let $\beta \in L$ be an anti-integral element over R and let A be an Noetherian domain containing R. If β is an anti-integral element over A of degree e ($\leq d$), then $J_{[\beta]} A \subseteq J_{[\beta]}^{(A)}$.*

Proof. Take $f(X) \in I_{[\beta]} \varphi_\beta(X) R[X]$ with $\deg f(X) = d$. Consider the natural exact sequence:

$$0 \longrightarrow \operatorname{Ker}\pi^{(A)} \longrightarrow A[X] \xrightarrow{\pi(A)} A[\beta] \longrightarrow 0.$$

Then $f(X) \in \ker\pi^{(A)}$. So by Lemma 1. 2, we have $C(f(X)) \subseteq C(F_d^{(A)}) = J_{[\beta]}^{(A)}$, which yields that $J_{[\beta]}A \subseteq J_{[\beta]}^{(A)}$. Q. E. D.

Making these preparations, we have the following proposition.

Proposition 1.4. *Assume that $\alpha \in L$ is an anti-integral element over R of degree d. Let $\beta \in K[\alpha]$. Assume that β is anti-integral over both R and $R[\alpha]$. If $R[\beta]$ is flat R then $R[\alpha, \beta]$ is flat over $R[\alpha]$.*

Proof. Put $A = R[\alpha]$. By [3, Proposition 2.6], we have $J_{[\beta]} = R$. Hence $J_{[\beta]}^{(A)} = A$ by Lemma 1.3. Using[3, Proposition 2.6] again, we have that $A[\beta]$ is flat over A. Hence $R[\alpha, \beta]$ is flat over $R[\alpha]$. Q.E.D.

Remark 1.5. In [2], it is shown that $\alpha \in L$ is anti-integral over R if and only if so is α^{-1}.

§2. Flat Elements

In this section, we show when $\beta \in L$ is flat over $R[\alpha]$ in terms of the ideal T_β. For this purpose, we study some properties of the ideal T_β.

Proposition 2.1. *Assume that $\alpha \in L$ is anti-integral and integral over R of degree d. Then the following statements holds:*
(i) For $\beta_1, \dots, \beta_n \in K[\alpha]$, put $A = R[\alpha, \beta_1, \dots, \beta_n]$. If $(T_{\beta_1} \cap \dots \cap T_{\beta_n})A = A$ then A is flat over R.
(ii) For $\beta \in K[\alpha]$, the equality $I_{[\beta]}^{(R[\alpha])} \cap R = T_\beta$ holds and hence $T_\beta R[\alpha] \subseteq I_{[\beta]}^{(R[\alpha])}$.

Proof. (i) By the definitions of T_β and $I_{[\beta]}^{(R[\alpha])}$, we have $T_\beta \subseteq I_{[\beta]}^{(R[\alpha])}$. So the assumption yields $T_{\beta i}A = A (1 \leq i \leq n)$. Thus we have $I_{[\beta i]}^{(R[\alpha])}A = A (1 \leq i \leq n)$. From this, it follows that $(I_{[\beta_1]}^{(R[\alpha])} \cap \dots \cap I_{[\beta n]}^{(R[\alpha])})A = A$, and hence A is flat over R (cf. [4]). Since $R[\alpha]$ is flat over R, A is flat over R.
(ii) Note that $\beta \in K(\alpha)$. Since $I_{[\beta]}^{(R[\alpha])} = R[\alpha] :_{R[\alpha]} \beta$ and $T_\beta = R[\alpha] :_R \beta$, we have $I_{[\beta]}^{(R[\alpha])} \cap R = T_\beta$. From this, it follows that $T_\beta R[\alpha] \subseteq I_{[\beta]}^{(R[\alpha])}$. Q. E. D.

Remark 2.2. (i) For $\beta \in K[\alpha]$, the rings $R[\alpha]$ and $R[\alpha][\beta]$ are birational. So it holds that $I_{[\beta^{-1}]}^{(R[\alpha])} = \beta I_{[\beta]}^{(R[\alpha])}$.
(ii) By use of (i), we have the following:

$$\begin{aligned} J_{[\beta]}^{(R[\alpha])} &= I_{[\beta]}^{(R[\alpha])} + \beta I_{[\beta]}^{(R[\alpha])} \\ &= I_{[\beta]}^{(R[\alpha])} + I_{[\beta^{-1}]}^{(R[\alpha])} \\ &\supseteq (T_\beta + T_{\beta^{-1}})R[\alpha] \end{aligned}$$

and

$$J_{[\alpha]}^{(R[\alpha])} \cap R \supseteq T_\beta + T_{\beta^{-1}}.$$

The next theorem is our main result in this section.

Theorem 2.3. *Assume that α is anti-integral and integral over R of degree d. Let $\beta_1, \ldots, \beta_n$ be elements in $K[\alpha]$. If $J_{[\beta_i]} = R$ and each β_i is anti-integral over both R and $R[\alpha](1 \leq i \leq n)$, then $R[\alpha, \beta_1, \ldots, \beta_n]$ is flat over R.*

Proof. Since $\beta_1, \ldots, \beta_n \in K[\alpha]$, $A := R[\alpha, \beta_1, \ldots, \beta_n]$ is birational over $R[\alpha]$. By Proposition 1.4, each $\beta_i (1 \leq i \leq n)$ is a flat element over $R[\alpha]$. Thus A is flat over $R[\alpha]$(cf. [4]). Since $R[\alpha]$ is flat over R by the assumption, A is flat over R. Q.E.D.

Proposition 2.4. *Assume that α is anti-integral and integral over R. Let $\beta_1, \ldots, \beta_n \in K[\alpha]$. If $(T_{\beta i} + T_{\beta i-1})R[\alpha] = R[\alpha]$ for each i, then $R[\alpha, \beta_1, \ldots, \beta_n]$ is flat over R.*

Proof. By using of Remark 2.2, we have $J_{[\beta i]}^{(R[\alpha])} \supseteq (T_{\beta i} + T_{\beta i^{-1}})R[\alpha] = R[\alpha]$ for each i because $R[\alpha]$ is faithfully flat over R. So $J_{[\beta i]}^{(R[\alpha])} = R[\alpha]$. Thus each β_i is a flat element over $R[\alpha]$. Since $\beta_i \in K(\alpha)$, $R[\alpha, \beta_1, \ldots, \beta_n]$ is flat over $R[\alpha]$(cf. [4]). Hence $R[\alpha, \beta_1, \ldots, \beta_n]$ is flat over R. Q.E.D.

In the preceding results, the condition that β is anti-integral over $R[\alpha]$ works effectively. We attempt to characterize this condition in terms of the ideal T_β.

Let $\mathrm{Dp}_1(R) := \{p \in \mathrm{Spec}(R) | \mathrm{depth} R_p = 1\}$. Let α be an element in an algebraic field extension L of K. We say that α is a *super-primitive element* of degree d if $J_{[\alpha]} \nsubseteq p$ for all $\in \mathrm{Dp}_1(R)$. Let A be a Noetherian domain containing R. An element $\beta \in K[\alpha]$ is a super-primitive element over A if and only if grade $(J_{[\beta]}^{(A)}) > 1$. It is known in [3] that the super-primitive elements are anti-integral.

Proposition 2.5. *Assume that α is anti-integral and integral over R of degree d. Let β be an element in $K[\alpha]$.*
(i) β is super-primitive over $R[\alpha]$ if grade $(T_\beta + T_{\beta-1}) > 1$. *In particular, if* grade $(T_\beta + T_{\beta-1}) > 1$, *then β is anti-integral over $R[\alpha]$.*
(ii) β is a flat element over $R[\alpha]$ if $T_\beta + T_{\beta-1} = R$.

Proof. (i) Since $R[\alpha]$ is faithfully flat and integral over R, grade $(T_\beta + T_{\beta-1}) > 1 \Longrightarrow$ grade$(J_{[\beta]}^{(R[\alpha])} \cap R) > 1 \Longrightarrow$grade$(J_{[\beta]}^{(R[\alpha])}) > 1$. Our conclusion follows.
(ii) Since β is a flat element over $R[\alpha]$, we have $J_{[\beta]}^{(R[\alpha])} = R[\alpha]$ and hence $J_{[\beta]}^{(R[\alpha])} \cap R = R$. So we have $T_\beta + T_{\beta-1} = R$ by the same way as in (i) Hence we have $J_{[\beta]}^{(R[\alpha])} = R[\alpha]$. Q.E.D

Proposition 2.6. *Assume that α is anti-integral and integral over R of degree d. Let $\beta \in K[\alpha]$ which is anti-integral over both R and $R[\alpha]$. Then $\sqrt{I_{[\beta]}} = \sqrt{T_\beta} = \sqrt{I_{[\beta]}^{(R[\alpha])} \cap R}$.*

Proof. First we shall show the inclusion $I_{[\beta]} \subseteq I_{[\beta]}^{(R[\alpha])} \cap R$. Let $\varphi_\beta(X) = X^d + \varsigma_1 X^{d-1} + \ldots + \varsigma_d$ with $\varsigma_i \in K$. Then for a non-zero element a $\in I_{[\beta]}$, there exists an algebraic dependence: $a\beta^d + a_1\beta^{d-1} + \ldots + a_d = 0$, $a_i = a\varsigma_i \in R$. From this, we have $a_d(\beta^{-1})^d + \ldots + a_1\beta^{-1} + a = 0$. Put $f(X) := a_d X^d + \ldots + a_1 X + a$. Then $f(\beta^{-1}) = 0$. By the assumption, β is anti-integral over both R and $R[\alpha]$. Hence by Remark 1.5, β^{-1} is also anti-integral over both R and $R[\alpha]$. So we have $f(X) = \sum f_i(X) g_i(X)$, where $\deg f_i(X) = 1$, $f_i(\beta^{-1}) = 0$ with $f_i(X), g_i(X) \in R[\alpha][X]$. Consider the constant term, we see that $a \in \beta^{-1} I_{[\beta^{-1}]}^{(R[\alpha])} = \beta^{-1} \times \beta I_{[\beta]}^{(R[\alpha])} = I_{[\beta]}^{(R[\alpha])}$. Thus $I_{[\beta]} \subseteq I_{[\beta]}^{(R[\alpha])} \cap R$. But since $I_{[\beta]}^{(R[\alpha])} \cap R = T_\beta$ by proposition 2.1, we have $I_{[\beta]} \subseteq T_\beta$. Thus $\sqrt{I_{[\beta]}} \subseteq \sqrt{T_\beta}$. Conversely, take $a \in T_\beta$. Then $a\beta \in R[\alpha]$. Since $R[\alpha]$ is a free R-module of rank d, $a\beta$ has a monic relation of degree d: $(a\beta)^d + b_1(a\beta)^{d-1} + \ldots + b_d = 0$, $b_i \in R$. From this, we have: $\beta^d + (b_1/a)\beta^{d-1} + \ldots + b_d/a^d = 0$. Thus $a^d \in I_{[\beta]}$. Therefore for a sufficiently large integer N, $T_\beta^N \subseteq I_{[\beta]}$ and hence $\sqrt{T_\beta} \subseteq \sqrt{I_{[\beta]}}$. The second equality follows from Profosition 2.1 (ii). Q.E.D.

§3. Unramified Extensions

In this section, we discuss certain relationships between flat extensions and unramified extensions. Let β be an anti-integral element K. Then the module of differentials $\Omega_{R[\beta]/R}$ of $R[\beta]$ over R is given by

$$\Omega_{R[\beta]/R} = R[\beta]/I_{[\beta]}\varphi'_\beta(\beta)R[\beta].$$

Since $\varphi_\beta(X) = X - \beta (\beta \in K)$, we have $I_{[\beta]}\varphi'_\beta(\beta) = I_{[\beta]}$. Thus

$$\begin{aligned} R[\beta] \text{ is unramified over } R &\Longleftrightarrow \Omega_{R[\beta]R} = (0) \\ &\Longleftrightarrow I_{[\beta]}R[\beta] = R[\beta] \\ &\Longleftrightarrow R[\beta] \text{ is flat over } R \end{aligned}$$

(See [1] for detail). This means that in the birational case, the flatness is equivalent to the unramifiedness. But in the non-birational case, there exists an counter-example to this assertion an follows:

Example. Let $\beta \in L$ satisfy $\beta^d = a \in R$ and $a \notin R^\times$ with $[K(\beta):K] = d$ for $d > 1$. Then $\varphi_\beta(X) = X^d - a$ and $\varphi'_\beta(\beta) = d\beta^{d-1} \notin R[\beta]^\times$. Thus $R[\beta]$ is integral and hence flat over R by [3] but not unramified over R.

Theorem 3.1. *Assume that α is anti-integral and integral over R of degree d. Let $\beta \in K[\alpha]$ and assume that β is anti-integral over both R and $R[\alpha]$.*
(i) $R[\beta]$ is flat over R if and only if $R[\alpha, \beta]$ is unramified over $R[\alpha]$.
(ii) If $R[\beta]$ is unramified over R then $R[\beta]$ is flat over R.

Proof. (i) By Proposition 1.4, $R[\beta]$ is flat over R if and only if $R[\alpha, \beta]$ is flat over $R[\alpha]$. Since $\beta \in K[\alpha]$, that is, $R[\alpha, \beta]$ and $R[\alpha]$ are birational, we have the equivalence: $R[\alpha, \beta]$ is flat over $R[\alpha]$ if and only if $R[\alpha, \beta]$ is unramified over $R[\alpha]$.

(ii) Assume that $R[\beta]$ is unranified over R. Then the set $\{f'(X) | f(X) \in I_{[\beta]}\varphi_\beta(X)R[X]\}$ generates the unit ideal in $R[\beta]$. Since $f(\beta) = 0$ and β is anti-integral over $R[\alpha]$, we have $f(X) \in I_{[\beta]}^{(R[\alpha])}\varphi_\beta^{(R[\alpha])}(X)R[\alpha][X]$, where $\varphi_\beta^{(R[\alpha])}(X)$ denotes the monic minimal polynomial of β over $K(\alpha)$. Hence

$$f(X) = \sum F_i(X)G_i(X),\ F_i(X) \in I_{[\beta]}^{(R[\alpha])}\varphi_\beta^{(R[\alpha])}(X)R[\alpha][X],$$

where $G_i(X) \in R[\alpha][X]$, $\deg F_i(X) = 1$. Noting that $F_i'(X) \in I_{[\beta]}^{(R[\alpha])}$, we have $f'(\beta) = \sum F_i'(\beta)G_i(\beta) \in I_{[\beta]}^{(R[\alpha])}R[\alpha, \beta)$. Thus since $I_{[\beta]}^{(R[\alpha])}R[\alpha, \beta] = R[\alpha, \beta]$, $R[\alpha, \beta]$ is flat over $R[\alpha]$. So by Proposition 1.4, we conclude that $R[\beta]$ is flat over R. Q.E.D.

Theorem 3.1. yields the following corollary.

Corollary 3.2. *Under the same assumption in Theorem 3.1, if $R[\beta]$ is unramified over R then $R[\alpha, \beta]$ is unramified over $R[\alpha]$.*

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