On the diagonally dominant ratio

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It is well known that if the coefficient matrix is a generalized diagonally dominant matrix, then the Jacobi and the Gauss-Seidel methods converge. We know in our experiments that the number of iteration of these iterative methods is inveresely proportional to the degree of diagonal dominance of the coefficient matrix. In this paper, our is to present the definition of the diagonally dominant ratio as the degree of the convergence rate, and we define the diagonal dominator to improve the diagonally dominant ratio.

1 the diagonally dominant ratio

To solve the liner system

$$Ax = b,$$

the SOR method has been used widely in engineering fields, where A is a real $n \times n$ matrix, b is known $n \times 1$ vector, x is unknown $n \times 1$ vector. For a matrix A satisfying strict or irreducible weak diagonal dominance, the convergence of the Gauss-Seidel and the Jacobi methods is well known [1]. As is well known, an $n \times n$ complex mtrix $A = (a_{ij})$ is said to be strictly diagonally dominant if

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}| \quad for \quad 1 \leq i \leq n, \tag{1}$$

and irreducibly diagonally dominant if irreducible and if

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}| \quad for \quad 1 \leq i \leq n, \tag{2}$$

with strict inequality for at least one i[1][2].

It is well known that if matrix A is not weakly diagonal dominance, and the classical iterative method such as the Jacobi and the Gauss-Seidel methods convergencce, then A is generalized diagonally dominant matrix. We now give Lemma and theorem for generalized diagonally dominant matrices.

Lemma 1. [3] Let $A = (a_{ij})$ be an $n \times n$ matrix. Then A satisfies generalized diagonal dominance by rows if and only if there exist positive vectors $u = [u_1, u_2, \cdots, u_n]^T \in R^n$

satisfying

$$|a_{i,i}|u_i > \sum_{\substack{j \in N \\ j \neq i}} |a_{i,j}|u_j.$$

Theorem 1. [4] Let $A = I - M$ be an irreducible L-matrix. Then the Gauss-Seidel and Jacobi methods are convergence for A only if A satisfies generalized diagonal dominance by rows.

We analyze the relation of between a diagonally dominant ratio and convergence rate, from the properties of diagonal dominance and convergence. So, we define the diagonally dominant ratio.

Definition 1. We define the diagonally dominant ratio (DDR) p_i as follws ;

$$p_i = \frac{|a_{ii}|}{\sum_{j \neq i} |a_{ij}|} \qquad (1 \leq i \leq n),$$

and the average DDR p

$$p_i = \frac{1}{n} \sum_{i=1}^{n} \frac{|a_{ii}|}{\sum_{j \neq i} |a_{ij}|}.$$

From Definition 1 and Lemma 1, if $p_i > 1$ then a coefficent matrix A is diagonal dominance. When $p > 1$, A is generalized diagonal dominance.

Now, we show that when the diagonally dominant ratio increase, the eigenvalue of the iteration matrix decrease. In [4] the upper bounds of spectral radiuses of the Gauss-Seidel and the Jacobi are given.

Corollary 1. [4] In the case of the ordinary Gauss-Seidel and Jacobi iteration matrices for $A = I + L + U$, the upper bounds of eigenvalues are respectively,

$$\beta_{GS} = \max_i \frac{u_i}{1 - l_i}, \quad \beta_J = \max_i (l_i + u_i),$$

where l_i, u_i, are sums of modulus elements in row i of the triangular matrices L, U, respectively. If in addition $\max_i (l_i + u_i) < 1$, then the following inequalities are satisfied,

$$\beta_{GS} < \beta_j < 1.$$

From this Corollary, we obtain the following Theorem for the Jacobi method.

Theorem 2. When the diagonally dominant ratio p_i, $1 \leq i \leq n$ of a coefficient matrix increase, the upper bound of spectral radius of the Jacobi iteration matrix decreases.

Proof We assume $p_i^a < p_i^b$ for $1 \leq i \leq n$, where p_i^a, p_i^b are the diagonally dominant ratio of $A = (a_{ij}) = I^a - L^a - U^a$, $B = (b_{ij}) = I^b - L^b - U^b$, respectively. Then, we obtain the relation of between of A and B as follows

$$\frac{1}{\sum_{j\neq 1}|a_{ij}|} < \frac{1}{\sum_{j\neq 1}|b_{ij}|},$$

$$l_i^a + u_i^a > l_i^b + u_i^b \qquad 1 \leq i \leq n,$$

where l_i^a, u_i^a, l_i^b and u_i^b are sums of modulus elements in row i of L^a, U^a, L^b and U^b, respectively. Therefore, if $p_i^a < p_i^b$ $(1 \leq i \leq n)$ then the upper bound of the spectral radius of the Jacobi iterative matrix for B is smaller than one for A.

2 Examples for the diagonally dominant ratio

To show that the number of iteration depends on the diagonally dominant ratio, we test by numerical examples with random value entries. We also test the skew-matrix with the strictly lower triangular part is negative and another part is positive. The computation results for $p=1.05$ and 2.00 of random Z-matrices and random skew-matrices are shown in Table 1.

It is clear that for n = 50, 100, 200, 300 when the diagonally dominant ratio is same, the numbers of iteration for the Gauss-Seidel method for are about same. Table 1 shows that the number of iteration depends on the diagonally dominant ratio.

The results for the Jacobi (J), the Gauss-Seidel (GS) and the SOR method for various value of p are shown in Tables 2 and 3. The convergence criterion is

$$\frac{\|x^{(k+1)} - x^{(k)}\|}{\|x^{(k+1)}\|} \leq 1.0E-7. \tag{3}$$

From Tables 2 and 3, we know that the number of iteration decreases with increasing p for all iterative methods. Therefore, we try on the improvement of the diagonally dominant ratio.

From Tables 2, 3, it is clear that the number of iterations of iterative methods depend on the diagonally dominant ratio. Moreover, the optimum parameter of SOR method depends on the diagonally dominant ratio. When p is nearly to 1 the number of iteration changes intensely.

Next, we propose the diagonal dominator to improve diagonally dominant ratio.

3 the diagonal dominator

We consider a procedure for increasing the diagonally dominant ratio of A. So we define a diagonal dominator $Q(\beta)$, where β is a positive parameter. We call this parameter the diagonal domination parameter. By pre-multiplying with $Q(\beta)$, we

Table 1 (number of iteration for Gauss-Seidel method)

p	matrix	$n=50$	$n=100$	$n=200$	$n=300$
1.05	Z-matrix	198	211	222	209
	Skew-type	270	258	263	265
2.00	Z-matrix	19	20	20	21
	Skew-type	19	20	21	21

Table 2 (N=50)

p	J	GS	SOR_{opt}	optimum ω
1.016734	1081	589	84	1.75000
1.026800	593	322	56	1.70000
1.036867	414	225	44	1.65000
1.046934	321	175	38	1.60000
1.057000	264	144	36	1.60000
1.067067	224	122	30	1.55000
1.087200	174	95	26	1.50000
1.117400	132	572	23	1.45000
1.147600	107	59	21	1.40000
1.197934	83	46	18	1.35000
1.258334	66	37	16	1.30000
1.348934	51	29	14	1.25000
1.489867	39	23	12	1.15000
2.103934	22	14	9	1.10000
2.909267	16	11	8	1.05000
3.311934	15	10	7	1.05000

Table 2 (N=50)

p	J	GS	SOR	optimum ω
1.013846	1145	625	91	1.75000
1.023884	610	332	54	1.70000
1.033922	422	230	43	1.65000
1.043961	326	178	39	1.60000
1.053999	267	145	34	1.60000
1.064037	226	124	30	1.55000
1.084113	175	96	26	1.50000
1.114227	132	73	22	1.45000
1.154379	101	56	19	1.40000
1.194532	83	46	18	1.35000
1.274836	62	35	15	1.30000
1.345103	51	29	14	1.25000
1.485636	39	23	12	1.20000
2.107997	22	14	9	1.10000
2.579787	18	11	8	1.10000
3.322605	15	10	7	1.05000

obtain the following equation

$$Q(\beta)Ax = Q(\beta)b,$$

where $Q(\beta)$ is the nonsingular. Accordingly, choosing of $Q(\beta)$ is important, but often a diffcult part of the problem. From our computational results, we obtained that we make the choice $Q(\beta) = (I + \beta U)$, where U is the upper triangular part of $-A$ and I is unit matrix, then the diagonally dominant ratie is larger than of original matrix A. If $\beta = 1$ then the diagonal dominator is similar to the preconditioning matrix f the

Adaptive iterative method [5].

Remark 1 The performance of the diagonal dominator is to improve the diagonally dominant ratio of A, but one of the preconditioning matrix is to improve the conditino number of A.

We show the effectiveness of the diagonal dominator for the model problem [1, p. 202] in Table 4.

The DDR of the Gauss-Seidel method means the diagonally dominant ratio for original coefficient matrices, and DDR of the adaptive Gauss-Seidel is the ratio for $(I+U)A$. For $m=5,10,15,20$, the diagonal dominator is available. The Gauss-Seidel method with the diagonal dominator decreases approximately half number of iteration of the Gauss-Seidel iteration method.

Next, we use $Q=(I+\alpha S)$ as the diagonal dominator, where

$$S=\begin{bmatrix} 0 & -a_{12} & 0 & \cdots & 0 \\ 0 & 0 & -a_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & -a_{n-1n} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

transforms the first upper codiagonal part of $(I+S)A$ to zero. If $\alpha=1$ then the diagonal dominator is the preconditioning matrix of the motified iteration method [6]. We use the parameter α as follows

$$\alpha S=\begin{bmatrix} 0 & -\alpha_1 a_{12} & 0 & \cdots & 0 \\ 0 & 0 & -\alpha_2 a_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & -\alpha_{n-1} a_{n-1n} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $\alpha=(\alpha 1,\alpha 2,\cdots,\alpha_{n-1})$.

The behaviors of the spectral radius of the Gauss-Seidel iteration matrix of $(I+\alpha S)A$ versus $\alpha=\alpha_1=\cdots=\alpha_{n-1}$ for the strictly diagonally dominant Z-matrix A is shown in Fig. 1.

In Fig. 1, a variation of the spectral radius of the Gauss-Seidel iteration matrix for $(I+\alpha S)A$ is extremely small as compare with one of the SOR method. Moreover, the

Table 4

m	*Gauss-Seidel*		*Adaptive*	*Gauss-Seidel*
	DDR	ite.	DDR	ite.
5	1.41667	37	1.753531	17
10	1.16461	144	1.28506	64
15	1.10204	317	1.27558	140
20	1.07387	555	1.12670	245

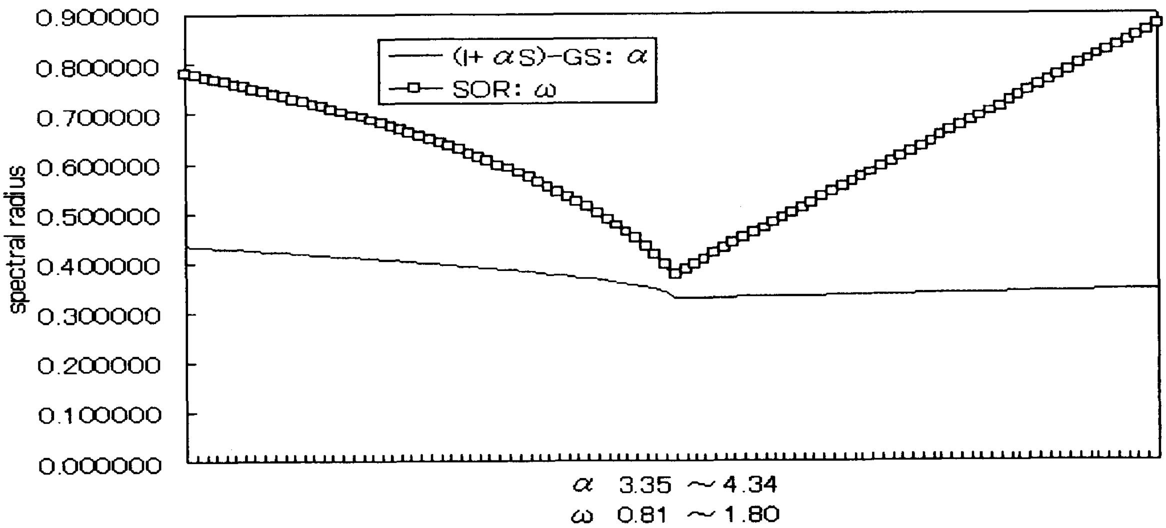

Fig. 1 1 the spectral radiuses of proposed method and the SOR method for random Z-matrix (n = 10).

convergence curve is relatively flat for $\alpha > \alpha_{opt}$. Hence, the rough estimation of the diagonal domination parameter is effective. Moreover, for the diagonal dominator $(I+\alpha S)$, we proposed the estimating technique of the diagonal domination parameter α_i, $i = 1,2,\cdots,n-1$ in [7],

$$\alpha_i = \frac{u_i + 2a_{ii+1}}{2a_{ii+1} - r_i}, \quad 1 \le i \le n-1 \tag{4}$$

4 Examples for the diagonal dominator

We now test the validity of the diagonal dominator and above estimation (4) for a following matrix

$$A = \begin{bmatrix} 1 & c_1 & c_2 & c_3 & c_1 & \cdots \\ c_3 & 1 & c_1 & c_2 & & c_1 \\ c_2 & c_3 & & & & c_3 \\ c_1 & & & 1 & c_1 & c_2 \\ c_3 & & c_2 & c_3 & 1 & c_1 \\ \cdots & c_3 & c_1 & c_2 & c_3 & 1 \end{bmatrix},$$

where $c_1 = \frac{-1}{n}$, $c_2 = \frac{-1}{n+2}$. We set b such that solution is $x^T = (1,2,\cdots,n)$. Let the convergence criterion be $\frac{\|x^{k+1} - x^k\|}{\|x^{k+1}\|} \le 10^{-6}$. We show CPU times and the number of iterations for $n = 20,30,50,100$ in Table 5. For comparison, we also show results for the Gauss-Seidel method $(Q = I)$, the modified Gauss-Seidel method $(Q = (I+S))$[5] and the optimum SOR method (SOR_{opt}).

It is clear that the number of iteration of the Gauss-Seidel method with the diagonal dominator are smaller than of standard the Gauss-Seidel method. An optimum parame-

Table 5 Z-matrix

n	$Q=(I+\alpha S)$ optimum		$Q=(I+\alpha S)$ est.		$Q=I$		$Q=(I+S)$		$Q=I+U$		SOR_{opt}	
	ite. (α_{opt})	*time*	*ite.*	*time*	*ite.*	*time*	*ite.*	*time*	*ite.*	*time*	*ite.* (ω_{opt})	*time*
20	19(10.4)	0.01	31	0.01	65	0.02	59	0.02	85	0.03	20(1.50)	0.01
30	23(17.4)	0.01	48	0.02	93	0.06	87	0.05	50	0.09	25(1.55)	0.01
50	28(32.3)	0.04	80	0.10	146	0.16	141	0.15	79	0.33	30(1.65)	0.03
100	38(72,9)	0.22	156	0.62	269	1.01	265	1.00	148	2.56	42(1.75)	0.18

Table 6 Model problem

m	$Q=(I+\alpha S)$ optimum		$Q=(I+\alpha S)$ est.		$Q=I$		$Q=(I+S)$		$Q=I+U$		SOR_{opt}	
	ite. (α_{opt})	*time*	*ite.*	*time*	*ite.*	*time*	*ite.*	*time*	*ite.*	*time*	*ite.* (ω_{opt})	*time*
10	20(2.65)	0.06	20	0.06	110	0.27	69	0.18	50	0.56	24(1.53)	0.10
15	26(3.0)	0.61	45	1.01	230	4.85	144	3.00	108	7.74	34(1.66)	0.73
20	34(3.2)	2.58	82	6.5	385	29.24	242	18.7	185	54.65	54(1.73)	4.15

ter ω_{opt} of the SOR method is obtained by numerical computation.

Next, we test for the model problem. We use a standard central difference formula on a uniform mesh with length $h = \frac{1}{m}$. Table 6 shows CPU times and the number of iterations. We adopt the theoretical value $\omega_{opt} = \dfrac{2}{1 + \sin\left(\frac{\pi}{\omega}\right)}$ for the SOR method.

5 Conclusion

We found three results of the diagonally dominant ratio from many experiments;

1 When the diagonally domiant ratios equal for given matrices, the number of iterations of Jacobi and Gauss-Seidel method is same number independently to the order of matrix A, respectively.

2 When the diagonally dominant ratios are nearly to different matrices, the value of the optimum parameter of the SOR method is nearly equal to to the order of matrix independently.

3 By calculating an asymptotic convergence rate from the diagonally dominat ratio, we are able to estimate a number of iteration from the diagonally dominant ratio.

For example, from Table 3, for $p = 1.013846$, we have

$$\frac{7}{-\log \dfrac{1}{1.013846}} = 1172.138...$$

It's easy to find that the number of iteration of the Gauss-Seidel method of Z-matrix is nearly equal to one of Skew-matrix, and the number of iterations are strongly depend on the diagonally dominant ratio.

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