Octonion Formulation of Eight-Dimensional Cartan's Pure Spinors

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Abstruct
The complex octonion formulation of Cartan's pure spinors in eight dimensions is described. It is shown that such spinors are equivalent to null octonions. Furthermore, by making use of the null octonions, the geometrical interpretation of pure spinors is given.

1. Introduction

Spinors were introduced into the quantum physics to make it relativistic covariance by Dirac[1]. On the other hand, Cartan developed a geometric theory of spinors, which can be applied to complex Euclidian space of any dimensions[2]. He showed that a null ν-plane in 2ν dimensions or $2\nu+1$ dimensions corresponded to a spinor[2-5]. Such a spinor is called *pure*. Namely, the pure spinors are completely geometrical objects. Furthermore, Cartan observed that, in eight dimensions, there was a beautiful symmetry with respect to the permutation between vectors and spinors. Such a symmetry is called the *principle of triality*[2,3,5-8].

Independently to the theory of spinors, octonions were discovered by Cayley[9], as an extension of Hamilton's quaternions. The algebraic correspondence between octonions and spinors in eight demensions was shown by Gamba[10]. Such a theory was generally discussed by Sudbery[11]. Furthermore, some authors constructed octonionic spinors in ten-dimensional Minkowski space[12-17]. However, in those theories, the algebraic relations between ordinary spinors and octonions are merely described. Namely, pure spinors is not discussed in such theories. In this paper, we will discuss the geometric relations between pure spinors in eight-dimensional complex space and complex octonions.

In section 2, we will describe some fundamental algebraic properties of complex octonions. Such properties are necessary at the least in this paper. The algebraic properties of null octonions which are special cases of complex octonions will be discussed in section 3. In section 4, we will describe the processes which make complex octonions correspond to SO(8) vectors and spinors. The principle of triality will be also described. In section 5, the relations between pure spinors and null octonions will be described. By making use of the null octonions, we will directly give the geometrical interpretation of pure spinors. Namely, such null octonions correspond to totally null four-plane of two types, and such planes are called *α-planes* and *β-planes*, respectively. In section 6, the intersections of α-planes and β-planes will be described.

2. Complex octonions

Before discussions of the algebra of complex octonions, we introduce the complex quaternions. Because we treat in later sections the complex octonions with an extension of the complex quaternions.

A complex quaternion A is an element of a four-dimensional vector space over complex numbers and is defined by

$$A := a_0 + Ia_1 + Ja_2 + Ka_3, \tag{1}$$

where I, J, and K are quaternion imaginary units defined with the relations

$$\begin{aligned} &I^2 = J^2 = K^2 = -1, \\ &IJ = -JI = K, \quad JK = -KJ = I, \quad KI = -IK = J, \end{aligned} \tag{2}$$

and a_0, a_1, a_2, and a_3 are complex numbers. The quaternion conjugate of A, denoted by $\bar{A}$,

$$\bar{A} := a_0 - Ia_1 - Ja_2 - Ka_3. \tag{3}$$

The inner product of any complex quaternions A and B is defined by

$$\begin{aligned} A \cdot B &:= \frac{1}{2}(A\bar{B} + B\bar{A}) = \frac{1}{2}(\bar{A}B + \bar{B}A) \\ &= a_0b_0 + a_1b_1 + a_2b_2 + a_3b_3 \in \mathbf{C}, \end{aligned} \tag{4}$$

where $\mathbf{C}$ is a set of every complex number. Furthermore, the norm of A is given as follows;

$$N(A) := A \cdot A = a_0^2 + a_1^2 + a_2^2 + a_3^3. \tag{5}$$

We denote a set of every complex quaternion by $\mathbf{CH}$. $\mathbf{CH}$ is not a division algebra but a composition algebra.

Let us consider a Cartesian product $\mathbf{CH} \times \mathbf{CH}$. Then we define for such objects the following operations;

$$(A_1, A_2)(B_1, B_2) = (A_1B_1 - \bar{B}_2A_2,\ A_2\bar{B}_1 + B_2A_1) \tag{6}$$

and

$$\overline{(A_1, A_2)} := (\bar{A}_1, -A_2) \tag{7}$$

where A_1, A_2, B_1 and B_2 are any complex quaternions. Note that the operations (6) and (7) are linear. We define a *complex octonion* as an ordered pair of any two complex quaternions with the operations (6) and (7). The operation (6) is called *Cayley-Dickson* *product* and (7) is called *octonion conjugate*. Moreover we denote a set of every octonion by **CO**. From the definitions (6) and (7), we can prove easily

$$\overline{\boldsymbol{ab}}=\bar{\boldsymbol{b}}\bar{\boldsymbol{a}} \tag{8}$$

for any complex octonions $\boldsymbol{a}$ and $\boldsymbol{b}$.

We define octonion units $\boldsymbol{e}_0$, $\boldsymbol{e}_1$, ..., $\boldsymbol{e}_7$ as follows;

$$\begin{aligned}&\boldsymbol{e}_0:=(1,\ 0),\quad \boldsymbol{e}_1:=(I,\ 0),\quad \boldsymbol{e}_2:=(J,\ 0),\quad \boldsymbol{e}_3:=(K,\ 0),\\ &\boldsymbol{e}_4:=(0,\ 1),\quad \boldsymbol{e}_5:=(0,\ I),\quad \boldsymbol{e}_6:=(0,\ J),\quad \boldsymbol{e}_7:=(0,\ K).\end{aligned} \tag{9}$$

According to the multiplication rule (6), we see that the unit $\boldsymbol{e}_0$ is the identity for any complex octonion with respect to the octonion product. Furthermore, according to the mapping rule (7), $\boldsymbol{e}_0$ is invariant with respect to the octonion conjugate operation. Therefore, we can identify $\boldsymbol{e}_0$ with mere "1" in real or complex numbers. Using this fact and (9), we can write any complex octonion $\boldsymbol{a}$ as follows;

$$\boldsymbol{a}=a_0+\boldsymbol{e}_1a_1+\ \dots\ +\boldsymbol{e}_7a_7. \tag{10}$$

Then, its octonion conjugate can be expressed as follows;

$$\bar{\boldsymbol{a}}=a_0-\boldsymbol{e}_1a_1-\ \dots\ -\boldsymbol{e}_7a_7. \tag{11}$$

We can call the octonion units except for $\boldsymbol{e}_0$ the *octonion imaginary units*.

We find that a triple octonion product for any complex octonions does generally not satisfy the associative law. Therefore, for a convenience of calculations of complex octonions, we introduce the *associator* defined as

$$[\boldsymbol{a},\ \boldsymbol{b},\ \boldsymbol{c}]:=(\boldsymbol{ab})\boldsymbol{c}-\boldsymbol{a}(\boldsymbol{bc}), \tag{12}$$

for any complex octonions $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$. Note that the linearity holds for the operation (12). The associator has the following identities;

$$[\boldsymbol{a},\ \boldsymbol{b},\ \boldsymbol{c}]=-[\boldsymbol{b},\ \boldsymbol{a},\ \boldsymbol{c}]=-[\boldsymbol{a},\ \boldsymbol{c},\ \boldsymbol{b}]=-[\bar{\boldsymbol{a}},\ \boldsymbol{b},\ \boldsymbol{c}]=-\overline{[\boldsymbol{a},\ \boldsymbol{b},\ \boldsymbol{c}]}, \tag{13}$$

$$[\boldsymbol{a},\ \boldsymbol{ba},\ \boldsymbol{c}]=[\boldsymbol{a},\bar{\boldsymbol{a}}\boldsymbol{b},\ \boldsymbol{c}]=\boldsymbol{a}[\boldsymbol{a},\ \boldsymbol{b},\ \boldsymbol{c}]=[\boldsymbol{a},\ \boldsymbol{b},\ \boldsymbol{c}]\bar{\boldsymbol{a}}. \tag{14}$$

The proofs of these identities are accomplished by the direct calculations. In an other paper[18], we will give them by making use of a symbolic computation system.

We can construct a complex number from any two complex octonions $\boldsymbol{a}$ and $\boldsymbol{b}$ as follows;

$$\boldsymbol{a}\cdot\boldsymbol{b}:=\frac{1}{2}(\boldsymbol{a}\bar{\boldsymbol{b}}+\boldsymbol{b}\bar{\boldsymbol{a}})=\frac{1}{2}(\bar{\boldsymbol{a}}\boldsymbol{b}+\bar{\boldsymbol{b}}\boldsymbol{a})=a_0b_0+a_1b_1+\ \dots\ +a_7b_7\in \mathrm{C}. \tag{15}$$

We can easily show that such a product has the following symmetries;

$$\boldsymbol{a}\cdot\boldsymbol{b}=\boldsymbol{b}\cdot\boldsymbol{a}=\bar{\boldsymbol{a}}\cdot\bar{\boldsymbol{b}}. \tag{16}$$

The multiplication rule defined by (15) is called the *inner product* of complex octonions $\boldsymbol{a}$ and $\boldsymbol{b}$.

PROPOSITION 1. *Let* $\boldsymbol{a}$, $\boldsymbol{b}$ *and* $\boldsymbol{c}$ *any complex octonions. Then the identities*

$$\boldsymbol{a}\cdot\boldsymbol{bc}=\boldsymbol{b}\cdot\boldsymbol{a}\bar{\boldsymbol{c}}=\boldsymbol{c}\cdot\bar{\boldsymbol{b}}\boldsymbol{a} \tag{17}$$

hold.

Proof. From (13) and (15), we have

$$\begin{aligned}\boldsymbol{a}\cdot\boldsymbol{bc}&=\frac{1}{2}(\boldsymbol{a}\overline{(\boldsymbol{bc})}+(\boldsymbol{bc})\bar{\boldsymbol{a}})\\&=\frac{1}{2}((\boldsymbol{a}\bar{\boldsymbol{c}})\bar{\boldsymbol{b}}+\boldsymbol{b}(\boldsymbol{c}\bar{\boldsymbol{a}}))-[\boldsymbol{a},\ \bar{\boldsymbol{c}},\ \bar{\boldsymbol{b}}]+[\boldsymbol{b},\ \boldsymbol{c},\ \bar{\boldsymbol{a}}])\\&=\boldsymbol{b}\cdot\boldsymbol{a}\bar{\boldsymbol{c}}.\end{aligned}$$

By similar calculations, we can prove the other identities. ■

A self inner product of any complex octonion $\boldsymbol{a}$ is called the *norm* and denoted by $N(\boldsymbol{a})$, i.e.

$$N(\boldsymbol{a}):=\boldsymbol{a}\cdot\boldsymbol{a}=\boldsymbol{a}\bar{\boldsymbol{a}}=\bar{\boldsymbol{a}}\boldsymbol{a}\in\mathbf{C}. \tag{18}$$

We can prove easily that the identity

$$\boldsymbol{a}(\bar{\boldsymbol{a}}\boldsymbol{b})=(\boldsymbol{b}\bar{\boldsymbol{a}})\boldsymbol{a}=N(\boldsymbol{a})\boldsymbol{b} \tag{19}$$

holds for any complex octonions $\boldsymbol{a}$ and $\boldsymbol{b}$.

PROPOSITION 2. *Let* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *be any complex octonions. Then*

$$N(\boldsymbol{ab})=N(\boldsymbol{a})N(\boldsymbol{b}) \tag{20}$$

holds.

Proof. From (13), (14) and (19), we have

$$\begin{aligned}N(\boldsymbol{ab})&=(\boldsymbol{ab})\overline{(\boldsymbol{ab})}=\boldsymbol{a}(\boldsymbol{b}(\bar{\boldsymbol{b}}\bar{\boldsymbol{a}}))+[\boldsymbol{a},\ \boldsymbol{b},\ \overline{\boldsymbol{ab}}]\\&=N(\boldsymbol{b})\boldsymbol{a}\bar{\boldsymbol{a}}-[\boldsymbol{a},\ \boldsymbol{b},\ \boldsymbol{ab}]=N(\boldsymbol{a})N(\boldsymbol{b})-[\boldsymbol{a},\ \boldsymbol{b},\ \boldsymbol{b}]\boldsymbol{a}\\&=N(\boldsymbol{a})N(\boldsymbol{b}).\end{aligned}$$

Thus this proposition is true. ■

PROPOSITION 3. *Any complex octonion* $\boldsymbol{a}$ *can be expressed in the form*

$$\boldsymbol{a}=\boldsymbol{bc} \tag{21}$$

for a suitable choice of complex octonions $\boldsymbol{b}$ *and* $\boldsymbol{c}$.

Proof. Let $N(\boldsymbol{b})\neq 0$. Then, if we put $\mathbf{c}=\bar{\boldsymbol{b}}\boldsymbol{a}/N(\boldsymbol{b})$, we have from (19)

$$\boldsymbol{bc}=\frac{\boldsymbol{b}(\bar{\boldsymbol{b}}\boldsymbol{a})}{N(\boldsymbol{b})}=\frac{N(\boldsymbol{b})\boldsymbol{a}}{N(\boldsymbol{b})}=\boldsymbol{a}.$$

Thus this proposition is true. ■

3. Null octonions

A non zero complex octonion $\boldsymbol{a}$ is called *null* if and only if its norm is zero, i.e. $N(\boldsymbol{a})=0$. Note that for ordinary octonions $\boldsymbol{a}$ (based on real number field) $N(\boldsymbol{a})=0$ is equivalent to $\boldsymbol{a}=0$. It is meaningful that $\boldsymbol{a}$ is a null octonion if and only if it is a complex octonion or a split octonion.

PROPOSITION 4. *A complex octonion $\boldsymbol{a}$ is null if and only if for any decomposition $\boldsymbol{a}=\boldsymbol{bc}$ either $N(\boldsymbol{b})=0$ or $N(\boldsymbol{c})=0$.*

Proof. Suppose $\boldsymbol{a}=\boldsymbol{bc}$. Then, from (20), we have

$$N(\boldsymbol{a})=N(\boldsymbol{b})N(\boldsymbol{c}).$$

When $N(\boldsymbol{a})=0$, the above expression must vanish. Therefore, we must have $N(\boldsymbol{b})=0$ or $N(\boldsymbol{c})=0$. Thus this proposition is true. ■

Before showing an important property of null octonion, we preparatorily describe two lemmas in what follows.

LEMMA1. *Let A, B and C be any complex quaternions. Then*

$$[(A, 0), (B, 0), (C, 0)]=0 \tag{22}$$

holds for the three complex octonions $(A, 0)$, $(B, 0)$ and $(C, 0)$.

Proof. Note that complex quaternions have the associativity. Now, from (6), we have

$$(A, 0)(B, 0)=(AB, 0).$$

Therefore we have

$$[(A, 0), (B, 0), (C, 0)]=([A, B, C], 0)=(0, 0)=0.$$

Thus this proposition is true. ■

LEMMA2. *The set*

$$\{(A, 0)|A\in \mathbf{CH}\}\subset \mathbf{CO} \tag{23}$$

is isomorphic to $\mathbf{CH}$.

Proof. Let A and B be any complex quaternions. Then, from (6) and (7), for two complex octonions $(A, 0)$ and $(B, 0)$ constructed by those complex quaternions, we have

$$(A, 0)(B, 0)=(AB, 0), \quad \overline{(A, 0)}=(\bar{A}, 0).$$

Furthermore, from these expressions and (15) and (18), we have

$$(A, 0)\cdot(B, 0)=A\cdot B, \quad N((A, 0))=N(A).$$

From these facts and LEMMA 1, we see that this proposition is true. ■

From LEMMA 2, we can simply denote a complex octonion $(A, 0)$ by the complex quaternion A in the complex octonion algebra, i.e.

$$(A, 0)=:A. \tag{24}$$

Therefore, for example, we can write $(A, 0)\boldsymbol{b}=A\boldsymbol{b}$ for any complex quaternion A and any complex octonion $\boldsymbol{b}$.

PROPOSITION 5. *Let $\boldsymbol{a}=(A_1, A_2)$ be a null octonion satinfying $N(A_1)=-N(A_2)$*

$\neq 0$. *Then*

$$\boldsymbol{ab} = \boldsymbol{a}U, \quad U \in \mathbf{CH}, \tag{25a}$$

$$\boldsymbol{ba} = V\boldsymbol{a}, \quad V \in \mathbf{CH} \tag{25b}$$

hold for any complex octonion $\boldsymbol{b} = (B_1, B_2)$, *where*

$$U = B_1 - \frac{1}{N(A_1)}\bar{A}_1\bar{B}_2A_2 \tag{26a}$$

$$V = B_1 - \frac{1}{N(A_1)}\bar{A}_2B_2\bar{A}_1. \tag{26b}$$

Proof. From (6), we have

$$(A_1, A_2)(B_1, B_2) = (A_1B_1 - \bar{B}_2A_2,\ A_2\bar{B}_1 + B_2A_1).$$

On the other hand, substituting (26a) into $\boldsymbol{a}U$ and noting $N(A_1) = -N(A_2)$, we obtain

$$\begin{aligned}\boldsymbol{a}U &= (A_1, A_2)(U, 0) = (A_1U,\ A_2\bar{U})\\ &= \left(A_1B_1 - \frac{1}{N(A_1)}N(A_1)\bar{B}_2A_2,\ A_2\bar{B}_1 - \frac{1}{N(A_1)}N(A_2)B_2A_1\right)\\ &= (A_1B_1 - \bar{B}_2A_2,\ A_2\bar{B}_1 + B_2A_1).\end{aligned}$$

Thus we have $\boldsymbol{ab} = \boldsymbol{a}U$. Similarly, we have $\boldsymbol{ba} = V\boldsymbol{a}$. ■

4. SO(8) spinors and complex octonions

The basis treated in this paper will be an eight-dimensional complex linear space $\mathbf{C}^8$. Now we write z_a ($a, b, \ldots = 0, 1, \ldots 7$) for coordinates on $\mathbf{C}^8$. Then the action of SO(8, C) on $\mathbf{C}^8$ is defined as the transformation which preserves the quadratic form z_cz_c. Similarly, the spinor and dual spinor spaces for SO(8) are also each eight-dimensional complex linear spaces. Let u_A and $v_{A'}$ ($A, B, \ldots$ and $A', B', \ldots = 0, 1, 2, \ldots 7$) be coordinates in such spaces, respectively. Each of the spinor spaces is given the invariant quadratic form u_Au_A and $v_{A'}v_{A'}$, respectively.

Furthermore, there is an invariant *trilinear* form $\gamma_{aAA'}z_au_Av_{A'}$. Its coefficient $\gamma_{aAA'}$ can be used to carry out multiplication between vectors and two kinds of spinors. For example, the product of a spinor u_A and a dual spinor $v_{A'}$ is given by $\gamma_{aAA'}u_Av_{A'}$, and the resultant quantity in a vector. Namely, $\gamma_{aAA'}$ maps a diret product between the eight-dimensional spaces of different two types into another eight-dimensional space. Such a $\gamma_{aAA'}$ satisfies the following relations;

$$\gamma_{c(A|A'}\gamma_{c|B)B'} = \gamma_{cA(A'|}\gamma_{cB|B')} = \delta_{AB}\delta_{A'B'} \tag{27a}$$

$$\gamma_{(a|CA'}\gamma_{|b)CB'} = \gamma_{aC(A'|}\gamma_{bC|B')} = \delta_{ab}\delta_{A'B'} \tag{27b}$$

$$\gamma_{(a|AC'}\gamma_{|b)BC'} = \gamma_{a(A|C'}\gamma_{b|B)C'} = \delta_{ab}\delta_{AB}. \tag{27c}$$

Let us consider two octonion units $\boldsymbol{e}_\alpha$ and $\boldsymbol{e}_\beta$ ($\alpha, \beta, \ldots = 0, 1, 2, \ldots 7$), where $\boldsymbol{e}_0 = 1$. Then we define a multiplication of their units as follows;

$$e_\beta e_\gamma := e_\alpha f_{\alpha\beta\gamma}. \tag{28}$$

We call the real numbers $f_{\alpha\beta\gamma}$ the *octonion structure constants*. By taking the inner product of any octonion units and (28), we can easily obtain for the octonion structure constants the following expression;

$$f_{\alpha\beta\gamma} = e_\alpha \cdot e_\beta e_\gamma \tag{29}$$

PROPOSITION 6. *For any complex octonions* $\boldsymbol{a} = e_\alpha a_\alpha$ *and* $\boldsymbol{b} = e_\alpha b_\alpha$,

$$\boldsymbol{ab} = e_\alpha f_{\alpha\beta\gamma} a_\beta b_\gamma, \tag{30a}$$

$$\boldsymbol{a\bar{b}} = e_\beta f_{\alpha\beta\gamma} a_\alpha b_\gamma, \tag{30b}$$

$$\boldsymbol{\bar{a}b} = e_\gamma f_{\alpha\beta\gamma} a_\alpha b_\beta, \tag{30c}$$

hold, where a_α *and* b_α *are any complex numbers.*

Proof. By taking a multiplication between (28) and $a_\beta b_\gamma$, we can obtain easily (30a). Next, from (29) and (17), we have

$$f_{\alpha\beta\gamma} = e_\alpha \cdot e_\beta e_\gamma = e_\beta \cdot e_\alpha \bar{e}_\gamma = e_\gamma \cdot \bar{e}_\beta e_\alpha.$$

Comparing this expression with (28), we have

$$e_\beta \bar{e}_\gamma = e_\alpha f_{\beta\alpha\gamma}, \quad \bar{e}_\beta e_\gamma := e_\alpha f_{\gamma\beta\alpha}.$$

By taking multiplications between these expressions and $a_\beta b_\gamma$, we can obtain (30b) and (30c), respectively. ■

PROPOSITION 7. *The octonion structure constants satisfy the following identities;*

$$\begin{aligned} f_{(\alpha|\beta\varepsilon} f_{|\gamma)\delta\varepsilon} &= f_{\alpha(\beta|\varepsilon} f_{\gamma|\delta)\varepsilon} = f_{(\alpha|\varepsilon\beta} f_{|\gamma)\varepsilon\delta} = f_{\alpha\varepsilon(\beta|} f_{\gamma\varepsilon|\delta)} \\ &= f_{\varepsilon(\alpha|\beta} f_{\varepsilon|\gamma)\delta} = f_{\varepsilon\alpha(\beta|} f_{\varepsilon\gamma|\delta)} = \delta_{\alpha\gamma}\delta_{\beta\delta}. \end{aligned} \tag{31}$$

Proof. Put $\boldsymbol{a} = e_\alpha a_\alpha$, $\boldsymbol{b} = e_\alpha b_\alpha$ and $\boldsymbol{c} = e_\alpha c_\alpha$. Then, from (30a), we obtain

$$(\boldsymbol{ab})\cdot(\boldsymbol{ac}) = f_{\varepsilon\alpha\beta} f_{\varepsilon\gamma\delta} a_\alpha a_\gamma b_\beta c_\delta = f_{\varepsilon(\alpha|\beta} f_{\varepsilon|\gamma)\delta} a_\alpha a_\gamma b_\beta c_\delta.$$

On the other hand, using (17), we have

$$(\boldsymbol{ab})\cdot(\boldsymbol{ac}) = \boldsymbol{c}\cdot(\bar{\boldsymbol{a}}(\boldsymbol{ab})) = N(\boldsymbol{a})\boldsymbol{c}\cdot\boldsymbol{b} = \delta_{\alpha\gamma}\delta_{\beta\delta} a_\alpha a_\gamma b_\beta c_\delta.$$

Since these expressions hold for any a_α, b_α and c_α, we have

$$f_{\varepsilon(\alpha|\beta} f_{\varepsilon|\gamma)\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta}.$$

Similarly, the proofs of the rest identities are evident. ■

Comparing (31) with (27), we can see that the former relations are equivalent to the latter. Thus we can identify the octonion structure constants $f_{\alpha\beta\gamma}$ with the coefficients $\gamma_{aAA'}$ of the trilinear form in eight dimensions. Then the expressions (30) can be regarded as the multiplication rules between vectors and two kinds of spinors. Therefore, we can express a vector z_a, a spinor u_A and a dual spinor $v_{A'}$ by the following octonions;

$$z_a \rightarrow \boldsymbol{z}=z_0+\boldsymbol{e}_1 z_1+\ \dots\ +\boldsymbol{e}_7 z_7, \tag{32a}$$

$$u_A \rightarrow \boldsymbol{u}=u_0+\boldsymbol{e}_1 u_1+\ \dots\ +\boldsymbol{e}_7 u_7, \tag{32b}$$

$$v_{A'} \rightarrow \boldsymbol{v}=v_0+\boldsymbol{e}_1 v_1+\ \dots\ +\boldsymbol{e}_7 v_7. \tag{32c}$$

We denote the spaces of such complex octonions by $\mathbf{CO}_V$, $\mathbf{CO}_S$ and $\mathbf{CO}_C$., respectively.

PROPOSITION 8. *Inner products between* $\mathbf{CO}_V$, $\mathbf{CO}_S$ *and* $\mathbf{CO}_C$ *are invariant under the* SO(8)*-transformations if and only if we have*

$$\boldsymbol{z}\cdot\boldsymbol{z}' \in \mathrm{C}, \quad \text{for } \boldsymbol{z},\ \boldsymbol{z}' \in \mathbf{CO}_V, \tag{33a}$$

$$\boldsymbol{u}\cdot\boldsymbol{u}' \in \mathrm{C}, \quad \text{for } \boldsymbol{u},\ \boldsymbol{u}' \in \mathbf{CO}_S, \tag{33b}$$

$$\boldsymbol{v}\cdot\boldsymbol{v}' \in \mathrm{C}, \quad \text{for } \boldsymbol{v},\ \boldsymbol{v}' \in \mathbf{CO}_C. \tag{33c}$$

Proof. Note that the quadratic forms defined on the spaces on which the SO(8)-transformations act are only $z_a z_a$, $u_A u_A$ and $v_{A'} v_{A'}$. Therefore allowed inner products are only $z_a z'_a$, $u_A u'_A$ and $v_{A'} v'_{A'}$. By this fact and the inner product (15) of complex octonions, we understand that this proposition is true.

PROPOSITION 9. *Octonion products are covariant under the* SO(8)*-transformations if and only if we have*

$$\boldsymbol{uv} \in \mathbf{CO}_V, \text{ for } \boldsymbol{u} \in \mathbf{CO}_S \text{ and } \boldsymbol{v} \in \mathbf{CO}_C, \tag{34a}$$

$$\boldsymbol{z}\bar{\boldsymbol{v}} \in \mathbf{CO}_S, \text{ for } \boldsymbol{z} \in \mathbf{CO}_V \text{ and } \boldsymbol{v} \in \mathbf{CO}_C, \tag{34b}$$

$$\bar{\boldsymbol{u}}\boldsymbol{z} \in \mathbf{CO}_C, \text{ for } \boldsymbol{u} \in \mathbf{CO}_S \text{ and } \boldsymbol{v} \in \mathbf{CO}_V, \tag{34c}$$

Proof. Note that the product of a spinor u_A and a dual spinor $v_{A'}$ is given by $\gamma_{aAA'} u_A v_{A'}$, and the resultant quantity is a vector. From this and the equivalency of $\gamma_{aAA'}$ and $f_{\alpha\beta\gamma}$, we see that the product between each elements of $\mathbf{CO}_S$ and $\mathbf{CO}_C$ has the form of (30a). Therefore (34a) is true. Similarly, (34b) and (34c) are true. Furthermore, since $\gamma_{aAA'}$ has the three indices of different type, allowed octonion products are only three types in (30). Thus this proposition is true. ■

PROPOSITION 9 indicates that there is a remarkable symmetry, the permutation of degree 3, betwees three complex octonion spaces $\mathbf{CO}_V$, $\mathbf{CO}_S$ and $\mathbf{CO}_C$. This symmetry which is peculiar to the eight dimensions is called the *principle of triality*. From this principle, we see that, *for every proposition, we can obtain further propositions as corollaries by a permutation of the three types of complex octonions.*

5. Pure spinors and null octonions

As was shown by Cartasn[2], *pure spinors* can be generally correlated with maximal null plane of certain complex Euclidean space. Dimensions of the maximal null planes are ν if such a Euclidean space in 2ν- or $2\nu+1$-dimension. Therefore pure spinors in eight dimensions correspond to *totally null four-planes* through the origin.

Let $\boldsymbol{z}$ be a null octonion corresponding to a totally null four-plane through the origin in $\mathbf{CO}_V$, i.e.

$$\boldsymbol{z} := t_0\boldsymbol{k} + t_1\boldsymbol{l} + t_2\boldsymbol{m} + t_3\boldsymbol{n} \in \mathbf{CO}_V, \tag{35}$$

where t_0, t_1, t_2 and t_3 are any complex parameters and $\boldsymbol{k}$, , $\boldsymbol{m}$ and $\boldsymbol{n}$ are constant null octonions satisfying

$$N(\boldsymbol{k}) = N(\boldsymbol{l}) = N(\boldsymbol{m}) = N(\boldsymbol{n}) = 0, \tag{36a}$$

$$\boldsymbol{k}\cdot\boldsymbol{l} = \boldsymbol{k}\cdot\boldsymbol{m} = \boldsymbol{k}\cdot\boldsymbol{n} = \boldsymbol{l}\cdot\boldsymbol{m} = \boldsymbol{l}\cdot\boldsymbol{n} = \boldsymbol{m}\cdot\boldsymbol{n} = 0. \tag{36b}$$

By virtue of PROPOSITION 3, we can decompose $\boldsymbol{z}$ into the following octonion product;

$$z = \boldsymbol{u}\boldsymbol{v}, \tag{37}$$

where, according to PROPOSITION 9, $\boldsymbol{u}$ and $\boldsymbol{v}$ are elements of $\mathbf{CO}_S$ and $\mathbf{CO}_C$, respectively. Then, from PROPOSITION 4, we must have either $N(\boldsymbol{u}) = 0$ or $N(\boldsymbol{v}) = 0$.

Put $\boldsymbol{u} = (U_1, U_2)$ and $\boldsymbol{v} = (V_1, V_2)$, where U_1, U_2, V_1 and V_2 are complex quaternions. Now we suppose that $\boldsymbol{u}$ is a constant complex octonion satisfying the null condition $N(\boldsymbol{u}) = 0$, where $N(U_1) = -N(U_2) \neq 0$. Then, from PROPOSITION 5, $\boldsymbol{z}$ can be chosen as follows;

$$\boldsymbol{z} = \boldsymbol{u}\Gamma, \tag{38}$$

where, from (26a), we have

$$\Gamma = V_1 - \frac{1}{N(U_1)}\bar{U}_1\bar{V}_2U_2 \in \mathbf{CH}. \tag{39a}$$

Putting

$$\Gamma = \gamma_0 + I\gamma_1 + J\gamma_2 + K\gamma_3 \tag{39b}$$

where γ_0, γ_1, γ_2 and γ_3 are complex parameters, we obtain from (38)

$$\boldsymbol{z} = \gamma_0\boldsymbol{u} + \gamma_1\boldsymbol{u}I + \gamma_2\boldsymbol{u}J + \gamma_3\boldsymbol{u}K. \tag{40}$$

Note that, from $N(\boldsymbol{u}) = 0$, (17) and (19),

$$N(\boldsymbol{u}) = N(\boldsymbol{u}I) = N(\boldsymbol{u}J) = N(\boldsymbol{u}K) = 0, \tag{41a}$$

$$\boldsymbol{u}\cdot\boldsymbol{u}I = \boldsymbol{u}\cdot\boldsymbol{u}J = \boldsymbol{u}\cdot\boldsymbol{u}K = \boldsymbol{u}I\cdot\boldsymbol{u}J = \boldsymbol{u}I\cdot\boldsymbol{u}K = \boldsymbol{u}J\cdot\boldsymbol{u}K = 0. \tag{41b}$$

Comparing (40), (41a) and (41b) with (35), (36a) and (36b), respectively, we obtain

$$\boldsymbol{k} = \boldsymbol{u}, \quad \boldsymbol{l} = \boldsymbol{u}I, \quad \boldsymbol{m} = \boldsymbol{u}J, \quad \boldsymbol{n} = \boldsymbol{u}K, \tag{42a}$$

$$t_0 = \gamma_0, \quad t_1 = \gamma_1, \quad t_2 = \gamma_2, \quad t_3 = \gamma_3. \tag{42b}$$

Thus we can understand that a totally null four-plane through the origin can be expressed by (38) if and only if $\boldsymbol{u}$ is a null octonion.

On the other hand, if $\boldsymbol{v}$ is a constant complex octonion satisfying the condition $N(\boldsymbol{v})$

$=0$ where $N(V_1)=-N(V_2)\neq 0$, then we have

$$z=\Delta v, \tag{43}$$

where Δ is a quaternion consisting of four complex parameters and, from (26b), it is given by

$$\Delta=U_1-\frac{1}{N(V_1)}\overline{V}_2U_2\overline{V}_1. \tag{44}$$

This is obvious by the principle of triality.

From the above discussions, we obtain the following proposition.

PROPOSITION 10. *Any null octonion which belongs to* $\mathbf{CO}_S$ *or* $\mathbf{CO}_C$ *can be represented, up to proportionality, by a totally null four-plane through the origin in eight-dimensional Euclidean vector space on which* SO(8) *acts. Conversely, a null octonion which belongs to* $\mathbf{CO}_S$ *or* $\mathbf{CO}_C$ *determines a totaly null four-plane through the origin in such a vector space.*

According to Cartan's definition, we call spinors corresponding to the null octonions which belong to $\mathbf{CO}_S$ and $\mathbf{CO}_C$ *pure spinors* and *dual pure spinors*, respectively. Moreover, totally null four-planes corresponding to pure spinors and dual pure spinors are called *α-planes* and *β-planes*, respectively. (See Figure 1.)

6. Intersections of α-planes and β-planes

The geometric relations discussed in previous section can be further pursued by making use of the properties of null octonions. Namely, what we discuss in this section are intersections of α-planes and β-planes.

PROPOSITION 11. *Let* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *be null octonions being elements of* $\mathbf{CO}_S$ *and* $\mathbf{CO}_C$*, respectively. Suppose that there exists an intersection of an α-plane and a β-plane given by such null octonions. Then the intersection is a null line passing through the origin if and only if* $\boldsymbol{uv}\neq 0$. (See Figure 2.)

Proof. Since an α-plane (38) and β-plane (43) have an intersection, we can put

$$z=u\Gamma=\Delta v.$$

From this second expression, (12), (13) and (15), we have

$$\begin{aligned}
z\bar{v}&=(u\Gamma)\bar{v}\\
&=u(\Gamma\bar{v})+[u,\ \Gamma,\ \bar{v}]\\
&=u(2\Gamma\cdot v-v\Gamma)-[u,\ v,\bar{\Gamma}]\\
&=2u(\Gamma\cdot v)-u(v\bar{\Gamma})-(uv)\bar{\Gamma}+u(v\bar{\Gamma})\\
&=2u(\Gamma\cdot v)-(uv)\bar{\Gamma}.
\end{aligned}$$

On the other hand, from the third expression, (19) and $N(v)=0$, we have

$$z\bar{v}=(\Delta v)\bar{v}=\Delta N(v)=0.$$

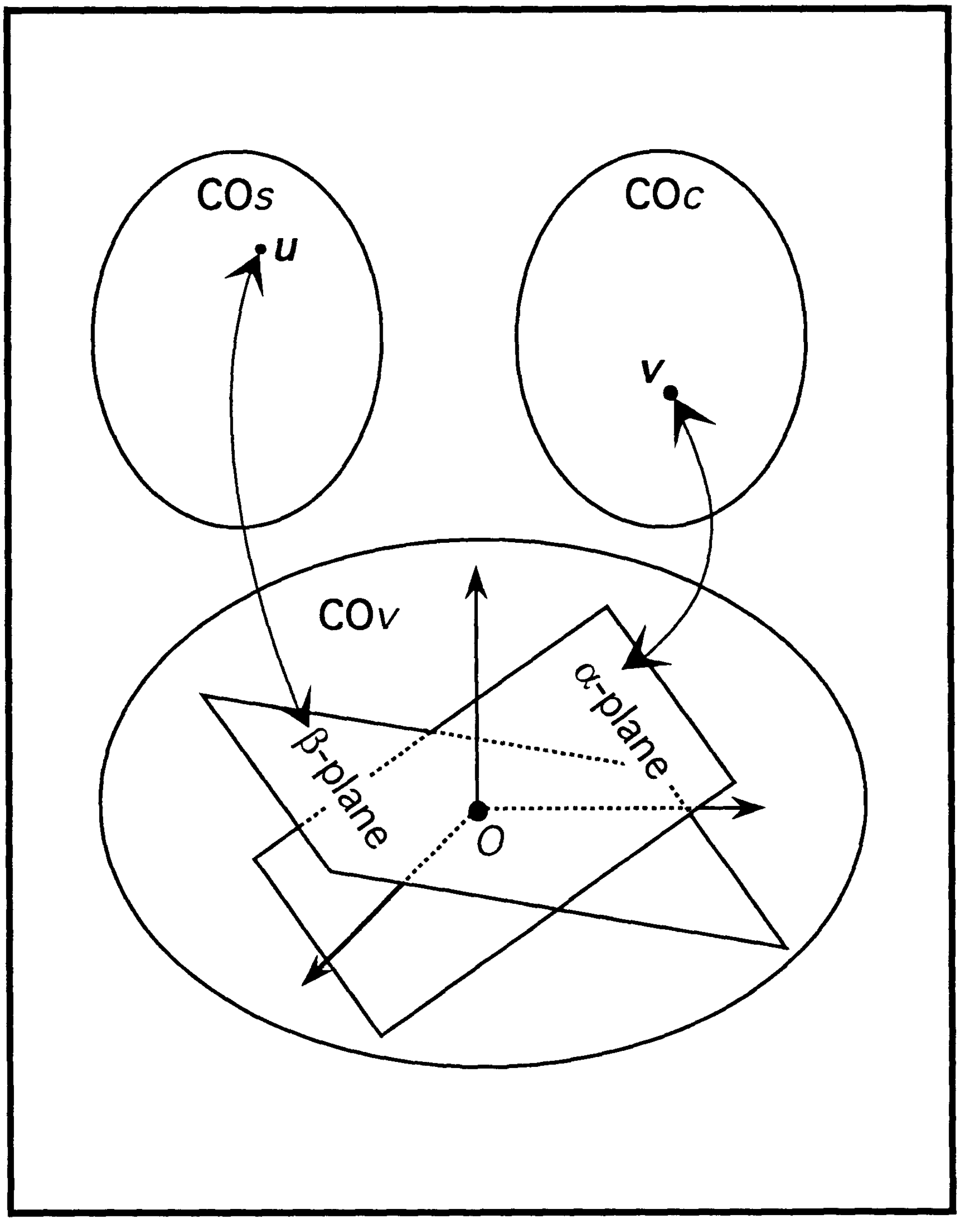

Fig. 1 An α-plane and a β-plane in eight-dimensional vector space.

Therefore we have

$$2\boldsymbol{u}(\Gamma\cdot\boldsymbol{v})-(\boldsymbol{uv})\bar{\Gamma}=0.$$

Multiplying this equation by Γ from the right, we obtain

$$2\boldsymbol{z}(\Gamma\cdot\boldsymbol{v})-\boldsymbol{uv}N(\Gamma)=0,$$

$$\therefore \boldsymbol{z}=\tau\boldsymbol{uv},\quad \tau:=\frac{N(\Gamma)}{2\Gamma\cdot\boldsymbol{v}}\in\mathbf{C}.$$

This is clearly an equation of null line, with a complex parameter τ, passing through

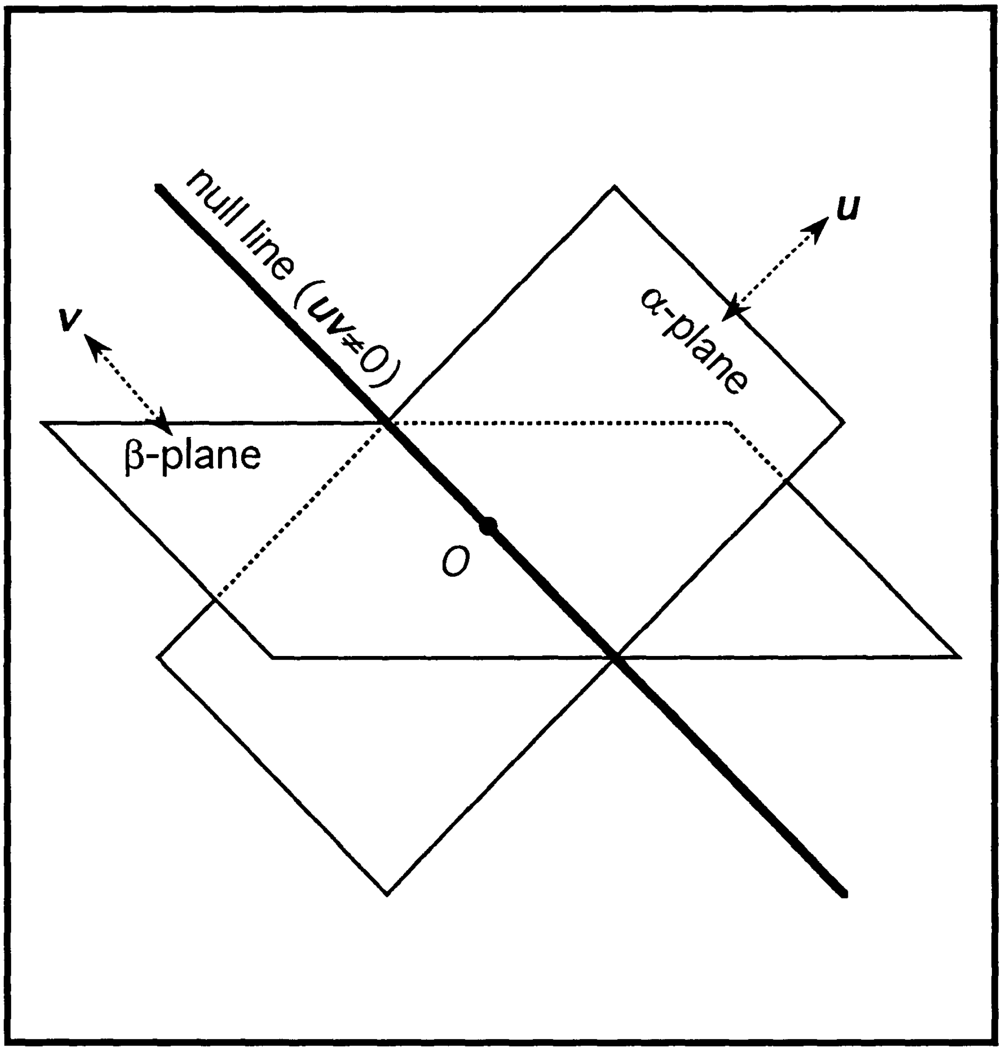

Fig. 2 The intersection of an α-plane and a β-plane, where $\boldsymbol{uv} \neq 0$.

the origin. Thus this proposition is true. ■

PROPOSITION 12. *Let* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *be null octonions being elements of* $\mathbf{CO}_S$ *and* $\mathbf{CO}_C$, *respectively. Suppose that there exists an intersection of an* α*-plane and a* β*-plane given by such null octonions. Then the intersection is a totally null three-plane passing through the origin if and only if* $\boldsymbol{uv} = 0$. (See Figure 3.)

Proof. Similarly to the proof of PROPOSITION 11, we have

$$2\boldsymbol{z}(\Gamma \cdot \boldsymbol{v}) - \boldsymbol{uv}N(\Gamma) = 0.$$

Since $\boldsymbol{uv} = 0$ and $\boldsymbol{z} \neq 0$, we obtain from above the equation

$$\Gamma \cdot \boldsymbol{v} = 0.$$

Therefore, intersection obtained by the assumption $\boldsymbol{uv} = 0$ can be represented by the equation (38) in which parameters have above linear constraint. Thus this proposition is true. ■

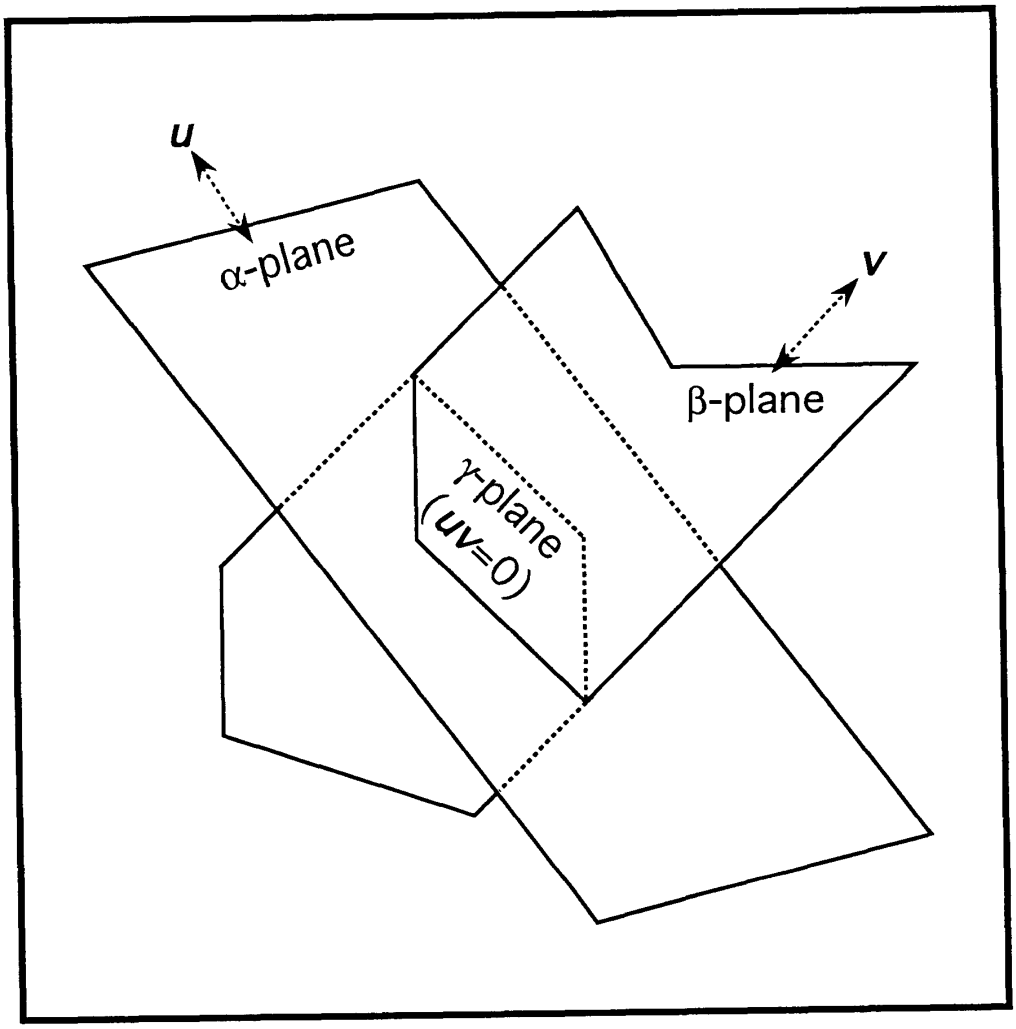

Fig. 3 The intersection of an α-plane and a β-plane, where $\boldsymbol{uv}=0$.

We call totally null three-planes obtained by intersections of α-planes and β-planes *γ-planes*.

PROPOSITION 13. *Let* $\boldsymbol{u}$ *be a null octonion being an element of* $\mathbf{CO}_S$ *and* $\boldsymbol{p}$ *is a point of* $\mathbf{CO}_V$*. Then there exists a unique β-plane passing through* $\boldsymbol{p}$ *with property that it intersects the α-plase given by* $\boldsymbol{u}$ *in a γ-plane.*

Proof. From (34b) and (18), we have

$$\boldsymbol{v} := \bar{\boldsymbol{u}}\boldsymbol{p} \in \mathbf{CO}_C, \quad N(\boldsymbol{v}) = 0.$$

The null octonion $\boldsymbol{v}$ gives clearly a β-plane passing through $\boldsymbol{p}$. We have from the above expression and (19)

$$\boldsymbol{uv} = \boldsymbol{u}(\bar{\boldsymbol{u}}\boldsymbol{p}) = N(\boldsymbol{u})\boldsymbol{p} = 0.$$

From this and PROPOSITION 12, the intersection of the α-plane given by $\boldsymbol{u}$ and the

β-plane given by $\boldsymbol{v}$ is a γ-plane. Thus this proposition is true. ■

Note that, from the principle of triality, statements such that $\mathbf{CO}_S$ is replaced to $\mathbf{CO}_C$ and the α-plane in exchanged with the β-plane in PROPOSITION 13 are also true.

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References

1) P. A. M. Dirac, *Proc. Roy. Soc. London* **A117**, 610 (1928).
2) É. Cartan, *The theory of spinors,* Dover, New York (1981).
3) R. Penrose and W. Rindler, *Spinor and space-time vol. 2,* Cambridge Univ. Press, Cambridge (1986).
4) S. Giler, P. Kosinski, J. Rembielinski and P. Maslanka, *Acta Physica Polonica* **B18**, 713 (1987).
5) I. M. Benn and R. W. Tucker, *An introduction to spinors and geometry with applications in physics,* Adam Higer, Bristol (1987).
6) J. Dadok and F. R. Harvey, *Acta Math.* **170**, 83 (1983).
7) J. F. Adams, in *Superspace and Supergravity*, eds. S. W. Hawking and M. Rocek, Cambridge Univ. Press, Cambridge, 435 (1981).
8) L. P. Hughston, in *Grvitation and Geometry*, eds. W. Rindler and A. Trautman, Bibliopolis, Naples, 243 (1987).
9) A. Cayley, *Philos. Mag*. **26,** 208 (1845).
10) A. Gamba, *J. Math. Phys*. **8,** 755 (1967).
11) A. Sudbery, *J. Phys*. **A17**, 939 (1984).
12) A. J. Davies and G. C. Joshi, *J. Math. Phys*. **27**, 3036 (1986).
13) K. W. Chung and A. Sudbery, *Phys. Lett*. **198B**, 161 (1987).
14) L. M. Evans, *Nucl. Phys*. **B298**, 92 (1988).
15) H. Tachibana and K. Imaeda, *Nuovo Cimento* **104B**, 91 (1989).
16) H. Tachibana, *Bull. Okayama Univ. Science* **24A**, 181 (1989).
17) H. Tachibana, Ph. D. Thesis, Okayama Univ. Science (1990).
18) H. Tachibana, *J. Kibi International Univ.*, (to be published).