# Subordinate Biharmonic Spaces

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## 1. Introduction

Let $X$ be a locally connected and locally compact Hausdorff space with a countable basis and $(X, \boldsymbol{H})$ be an elliptic biharmonic space in the sence of Smyrnêlis [7]. We denoto by $(X, \boldsymbol{H}_j)$ $(j=1, 2)$ the Brelot's harmonic spaces associated with $(X, \boldsymbol{H})$ and suppose that $(X, \boldsymbol{H}_1)$ has a consistent system $\{G_1^\omega(x, y)\}$ of $\boldsymbol{H}_1$-Green's functions. For an $\boldsymbol{H}$-regular set $\omega$, we denote by $(\mu_x^\omega, \nu_x^\omega, \lambda_x^\omega)$ the system of biharmonic measures of $(X, \boldsymbol{H})$, then the harmonic measures with respected to $(X, \boldsymbol{H}_1)$ (resp. $(X, \boldsymbol{H}_2)$) are given by $(\mu_x^\omega)$ (resp. $(\lambda_x^\omega)$). It is shown in [11] that there exists a unique positive Radon measure $\alpha$ on $X$ satisfing the relation $\nu_x^\omega = \int G_1^\omega(x, y)\lambda_y^\omega \, d\alpha(y)$ for any $\boldsymbol{H}$-regular set $\omega$ and $x \in \omega$. This measure $\alpha$ is called the composing measure of $(X, \boldsymbol{H})$. Let $(X, \tilde{\boldsymbol{H}})$ be an another biharmonic space with the same space $X$. We say that this biharmonic space $(X, \tilde{\boldsymbol{H}})$ is subordinate to $(X, \boldsymbol{H})$ if every $\boldsymbol{H}$-regular sets are $\tilde{\boldsymbol{H}}$-regular and for any open set $U \subset X$, $\boldsymbol{H}^*(U)^+ \subset \tilde{\boldsymbol{H}}^*(U)^+$, where $\boldsymbol{H}^*(U)^+$ (resp. $\tilde{\boldsymbol{H}}^*(U)^+$) is the set of all non-negative $\boldsymbol{H}$-hyperharmonic (resp. $\tilde{\boldsymbol{H}}$-hyperharmonic) couples on $U$. In this note we shall give a characterization of a subordinate biharmonic space as follows. Supposing that the associated harmonic space $(X, \boldsymbol{H}_1)$ has a consistent system of symmetric $\boldsymbol{H}_1$-Green's functions and every points in $X$ are $\boldsymbol{H}_1$-polar, a biharmonic space $(X, \tilde{\boldsymbol{H}})$ is subordinate to a biharmonic space $(X, \boldsymbol{H})$ if and only if the associated harmonic spaces $(X, \tilde{\boldsymbol{H}}_j)$ $(j=1, 2)$ are subordinate to $(X, \boldsymbol{H}_j)$ respectively and there exists a unique composing measure $\tilde{\alpha}$ of $(X, \tilde{\boldsymbol{H}})$ satisfing $\tilde{\alpha} \leqq \alpha$.

## 2. Biharmonic spaces

Let $X$ be a locally connected and locally compact Hausdorff space with countable basis. For an open set $U \neq \phi$ in $X$, we denote by $\boldsymbol{C}(U)$ the real vector space of finite continuous functions on $U$. An element $(h_1, h_2)$ in $\boldsymbol{C}(U) \times \boldsymbol{C}(U)$ is called compatible if $h_1 = 0$ on an open subset $U'$ of $U$ implies $h_2 = 0$ on $U'$. Let $\boldsymbol{H}$ be an application $U \rightarrow \boldsymbol{H}(U)$, where $\boldsymbol{H}(U)$ is a real vector subspace of compatible couples in $\boldsymbol{C}(U) \times \boldsymbol{C}(U)$. An ellement in $\boldsymbol{H}(U)$ is called $\boldsymbol{H}$-harmonic in $U$.

A relatively compact open set $\omega$ in $X$ is called $\boldsymbol{H}$-regular if for any couple $(f_1, f_2)$ of finite continuous functions on the boundary $\partial\omega$ of $\omega$, there exists a unique $(h_1, h_2) \in$

$\boldsymbol{H}(\omega)$ such that :

(i) $\lim_{x\to a} h_j(x) = f_j(a)$ for any $a \in \partial\omega$ $(j=1, 2)$ ;

(ii) $f_j \geqq 0$ $(j=1, 2)$ implies $h_1 \geqq 0$ and $f_2 \geqq 0$ implies $h_2 \geqq 0$.

For an $\boldsymbol{H}$-regular set $\omega$, there exists a unique system $(\mu_x^\omega, \nu_x^\omega, \lambda_x^\omega)$ of positive Radon measures on $\partial\omega$ such that

$$h_1(x) = \int f_1\, d\mu_x^\omega + \int f_2\, d\nu_x^\omega, \quad h_2(x) = \int f_2\, d\lambda_x^\omega.$$

This system $(\mu_x^\omega, \nu_x^\omega, \lambda_x^\omega)$ is called the system of biharmonic measures of $(X, \boldsymbol{H})$.

We say that $(X, \boldsymbol{H})$ is an elliptic biharmonic space in the sence of Smyrnélis [7] if it satisfies the following four axioms.

**Axiom** I. $\boldsymbol{H}$ is a sheaf on $X$.

**Axiom** II. The $\boldsymbol{H}$-regular open sets form a basis of $X$.

Let $U$ be an open set in $X$. A couple $(v_1, v_2)$ of functions on $U$ is called $\boldsymbol{H}$-hyperharmonic on $U$ if

(i) $v_j$ is lower semi-continuous and $> -\infty$ on $U$ $(j=1, 2)$,

(ii) $v_1(x) \geqq \int v_1\, d\mu_x^\omega + \int v_2\, d\nu_x^\omega$ and $v_2(x) \geqq \int v_2\, d\lambda_x^\omega$ for any $\boldsymbol{H}$-regular neighborhood $\omega$ of x with $\bar{\omega} \subset U$.

The set of all $\boldsymbol{H}$-hyperharmonic couples on $U$ is denoted by $\boldsymbol{H}^*(U)$. A couple $(s_1, s_2) \in \boldsymbol{H}^*(U)$ is called $\boldsymbol{H}$-superharmonic on $U$ if $s_j$ is not identically $+\infty$ on any connected component of $U$ $(j=1, 2)$ and an $\boldsymbol{H}$-superharmonic couple $(p_1, p_2)$ on $U$ is called $\boldsymbol{H}$-potential on $U$ if $p_j \geqq 0$ and, for any $(h_1, h_2) \in \boldsymbol{H}(U)$, $h_j = 0$ so far as $0 \leqq h_j \leqq p_j$ $(j=1, 2)$. The set of all $\boldsymbol{H}$-superharmonic couples (resp. $\boldsymbol{H}$-potentials) on $U$ is denoted by $\boldsymbol{S}(U)$ (resp. $\boldsymbol{P}(U)$). For an open set $U$, we put $\boldsymbol{H}_1^*(U) = \{v_1 : (v_1, 0) \in \boldsymbol{H}^*(U)\}$, $\boldsymbol{H}_2^*(U) = \{v_2 : (v_1, v_2) \in \boldsymbol{H}^*(U)$ for some $v_1\}$, and $\boldsymbol{H}_j(U) = \boldsymbol{H}_j^*(U) \cap [-\boldsymbol{H}_j^*(U)]$ $(j=1, 2)$.

**Axiom** III. (i) $\boldsymbol{H}_j^*(X)$ separates the points of $X$ linearly $(j=1, 2)$.

(ii) On each relatively compact open set $U$ there exists a strictly positive $h_j \in \boldsymbol{H}_j(U)$ $(j=1, 2)$.

**Axiom** IV. If $U$ is a domain in $X$ and $\{h_j^{(n)}\}_n$ is an increasing sequence of functions in $\boldsymbol{H}_j(U)$, then either $\sup_n h_j^{(n)} = +\infty$ or $\sup_n h_j^{(n)} \in \boldsymbol{H}_j(U)$ $(j=1, 2)$.

Set $\boldsymbol{H}_j = \{\boldsymbol{H}_j(U) : U$ is open set in $X\}$. It is shown by Theorem 1.29 in [7] that $(X, \boldsymbol{H}_j)$ $(j=1, 2)$ is a Brelot's harmonic space. We call $(X, \boldsymbol{H}_j)$ $(j=1, 2)$ the Brelot's harmonic space associated with $(X, \boldsymbol{H})$. The set of all $\boldsymbol{H}_j$-superharmonic functions (resp. $\boldsymbol{H}_j$-potentials) on $U$ is denoted by $\boldsymbol{S}_j(U)$ (resp. $\boldsymbol{P}_j(U)$) $(j=1, 2)$. Let $\boldsymbol{\Omega}$ be the set of all $\boldsymbol{H}$-regular sets in $X$ and $\boldsymbol{\Omega}_x = \{\omega \in \boldsymbol{\Omega} : x \in \omega\}$.

Let $(X, \boldsymbol{H})$ be an ellptic biharmosic space and $(X, \boldsymbol{H}_j)$ $(j=1, 2)$ be the Brelot's harmonic space associated with $(X, \boldsymbol{H})$. We say that $(X, \boldsymbol{H}_1)$ has a consistent system $\{G_1^{\omega}(x, y) : \omega \in \boldsymbol{\Omega}\}$ of $\boldsymbol{H}_1$-Green's functions if to each $\omega \in \boldsymbol{\Omega}$ there corrosponds a function $G_1^{\omega}(x, y)$ on $\omega \times \omega$ having the following properties :

(i) for each $y \in \omega$, $G_1^{\omega}(\cdot, y)$ is an $\boldsymbol{H}_1$-potential on $\omega$ and $\boldsymbol{H}_1$-harmonic on $\omega - \{y\}$,

(ii) if $\omega' \subset \omega$, $\omega' \in \boldsymbol{\Omega}$ and $y \in \omega'$ then the function $G_1^{\omega}(x, y) - G_1^{\omega'}(x, y)$ of $x$ is $\boldsymbol{H}_1$-harmonic on $\omega'$ ;

(iii) for each $\boldsymbol{H}_1$-potential $p$ on $\omega$, there exists a unique positive Radon measure $\beta$ on $\omega$ such that $p(x) = \int G_1^{\omega}(x, y)\, d\beta(y)$.

By Theorem 9 in [11] we have

**Lemma 1.** *Let $(X, \boldsymbol{H})$ be an elliptic biharmonic space. Suppose that $(X, \boldsymbol{H}_1)$ has a consistent system $\{G_1^{\omega}(x, y) : \omega \in \boldsymbol{\Omega}\}$ of $\boldsymbol{H}_1$-Green's functions. Then there exists a unique positive Radon measure $\alpha$ on $X$ such that*

$$\nu_x^{\omega} = \int G_1^{\omega}(x, y)\, \lambda_y^{\omega}\, d\alpha(y)$$

*for any $\omega \in \boldsymbol{\Omega}$ and any $x \in \omega$, that is for any finite continuous function $f$ on $X$*

$$\int f\, d\nu_x^{\omega} = \int G_1^{\omega}(x, y) \left( \int f\, d\lambda_y^{\omega} \right) d\alpha(y).$$

This positive Radon measure $\alpha$ is called the composing measure of $(X, \boldsymbol{H})$. By virtue of the compatibility of biharmonic couples, this measure $\alpha$ is everywhere dense in $X$.

**Lemma 2.** *For any $x \in X$, we have*

(i) $\displaystyle \lim_{\substack{\omega \to x \\ \omega \in \Omega_x}} \int d\mu_x^{\omega} = 1,$

(ii) $\displaystyle \lim_{\substack{\omega \to x \\ \omega \in \Omega_x}} \int d\nu_x^{\omega} = 0,$

(iii) $\displaystyle \lim_{\substack{\omega \to x \\ \omega \in \Omega_x}} \int d\lambda_x^{\omega} = 1.$

*Proof.* For any $x \in X$, we take a relatively compact open set $U$ such that $x \in U$. By Axiom III (ii), there exists a strictly positive $h_1 \in \boldsymbol{H}_1(U)$ with $h_1(x) = 1$. For any $\varepsilon$ with $0 < \varepsilon < 1$, $V = \{y \in U : |h_1(y) - 1| < \varepsilon\}$ being open neighborhood of $x$, there exists $\omega \in \boldsymbol{\Omega}_x$ with $\omega \subset \bar{\omega} \subset V$. Then,

$$\left| \int d\mu_x^{\omega} - 1 \right| = \left| \int (1 - h_1)\, d\mu_x^{\omega} \right| \leqq \varepsilon \int d\mu_x^{\omega}.$$

Hence

$$\frac{1}{1+\varepsilon} \leqq \int d\mu_x^\omega \leqq \frac{1}{1-\varepsilon}$$

and we have ( i ). To show (ii), we take $\omega' \in \boldsymbol{\Omega}_x$ with $\omega' \subset \bar{\omega}' \subset U$. By Axiom III (ii), there exists a strictly positive function $h_2 \in \boldsymbol{H}_2(U)$, and so $\left(\int h_2\, d\nu_\cdot^{\omega'}, h_2\right)$ is a strictly positive couple in $\boldsymbol{H}(\omega')$. Since $V_1 = \{y \in \omega' : |h_2(y) - h_2(x)| < \varepsilon\}$ is an open neiborhood of $x$ for any $\varepsilon$ with $0 < \varepsilon < h_2(x)$, there exists $\omega \in \boldsymbol{\Omega}_x$ with $\omega \subset \bar{\omega} \subset V_1$. Hence

$$0 \leqq (h_2(x) - \varepsilon) \int d\nu_x^\omega \leqq \int h_2\, d\nu_x^\omega = \int h_2\, d\nu_x^{\omega'} - \int\left(\int h_2\, d\nu_y^{\omega'}\right) d\mu_x^\omega(y).$$

Letting $\omega \to x$ ($\omega \in \boldsymbol{\Omega}_x$), $\int h_2\, d\nu_x^{\omega'} - \int(\int h_2\, d\nu_y^{\omega'})\, d\mu_y^\omega \to 0$ by ( i ), we have (ii). Similarly to ( i ), we have (iii).

**3. Subordinate biharmonic space**

Let $(X, \boldsymbol{H})$ be an elliptic biharmonic space and $(X, \tilde{\boldsymbol{H}})$ be an another biharmonic space with the same space $X$ in the sence of Smyrnélis [7]. Let $\boldsymbol{H}^*(U)^+$ (resp. $\tilde{\boldsymbol{H}}^*(U)^+$) be the set of all nonnegative couples in $\boldsymbol{H}^*(U)$ (resp. $\tilde{\boldsymbol{H}}^*(U)$). We shall give the definition of a subordinate biharmonic space as follows.

**Definition.** *We say that a biharmonic space $(X, \tilde{\boldsymbol{H}})$ is subordinate to $(X, \boldsymbol{H})$ if every $\boldsymbol{H}$-regular sets are $\tilde{\boldsymbol{H}}$-regular and $\boldsymbol{H}^*(U)^+ \subset \tilde{\boldsymbol{H}}^*(U)^+$ for any open set $U \subset X$.*

**Lemma 3.** *If $(X, \tilde{\boldsymbol{H}})$ is subordinate to $(X, \boldsymbol{H})$, then for any $\omega \in \boldsymbol{\Omega}$ and for any nonnegative continuous function $f$ on $\partial\omega$, we have*

( i ) $\int f\, d\tilde{\mu}_x^\omega \leqq \int f\, d\mu_x^\omega$ *on* $\omega$,

(ii) $\int f d\tilde{\nu}_x^\omega \leqq \int f\, d\nu_x^\omega$ *on* $\omega$,

(iii) $\int f\, d\tilde{\lambda}_x^\omega \leqq \int f\, d\lambda_x^\omega$ *on* $\omega$,

*where $(\tilde{\mu}_x^\omega, \tilde{\nu}_x^\omega, \tilde{\lambda}_x^\omega)$ is the system of biharmonic measures of $(X, \tilde{\boldsymbol{H}})$.*

*Proof.* Let $(X, \tilde{\boldsymbol{H}}_j)$ $(j=1, 2)$ be the Brelot's harmonic space associated with $(X, \tilde{\boldsymbol{H}})$. Then, trivially $\boldsymbol{H}_j^*(U)^+ \subset \tilde{\boldsymbol{H}}_j^*(U)^+$ $(j=1, 2)$ for any open set $U \subset X$, where $\boldsymbol{H}_j^*(U)^+$ (resp. $\tilde{\boldsymbol{H}}_j^*(U)^+$) is the set of nonnegative functions in $\boldsymbol{H}_j^*(U)$ (resp. $\tilde{\boldsymbol{H}}_j^*(U)$). That is, harmonic spaces $(X, \tilde{\boldsymbol{H}}_j)$ $(j=1, 2)$ are subordinate to harmonic spaces $(X, \boldsymbol{H}_j)$ respectively. Hence $\int f\, d\mu_x^\omega$ is $\tilde{\boldsymbol{H}}_1$-superharmonic on $\omega$. Therefore $\int f\, d\mu_x^\omega - \int f\, d\tilde{\mu}_x^\omega$ is $\tilde{\boldsymbol{H}}_1$-superharmonic on $\omega$ and equal to 0 on $\partial\omega$ and it is nonnegative by the minimum principle. Similarly we have (iii). To show (ii), let $\omega'$ be any $\boldsymbol{H}$-regular set with $\bar{\omega}' \subset \omega$. Since $(\int f\, d\nu_\cdot^\omega, \int f\, d\lambda_\cdot^\omega)$ is $\tilde{\boldsymbol{H}}$-superharmonic on $\omega$,

$$\int f\,d\nu_x^{\omega}-\int\Bigl(\int f\,d\nu_y^{\omega}\Bigr)d\tilde{\mu}_x^{\omega'}\geqq\int\Bigl(\int f\,d\lambda_y^{\omega}\Bigr)d\tilde{\nu}_x^{\omega'}(y)\geqq\int\Bigl(\int f\,d\tilde{\lambda}_y^{\omega}\Bigr)d\tilde{\nu}_x^{\omega'}(y)$$

$$=\int f\,d\tilde{\nu}_x^{\omega}-\int\Bigl(\int f\,d\tilde{\nu}_y^{\omega}\Bigr)d\tilde{\mu}_x^{\omega'}(y)$$

for any $x\in\omega'$. Hence for any $x\in\omega'$,

$$\int f\,d\nu_x^{\omega}-\int f\,d\tilde{\nu}_x^{\omega}\geqq\int\Bigl(\int f\,d\nu_y^{\omega}-\int f\,d\tilde{\nu}_y^{\omega}\Bigr)d\tilde{\mu}_x^{\omega'}(y).$$

Therefore $\int f\,d\nu_x^{\omega}-\int f\,d\tilde{\nu}_x^{\omega}$ is $\tilde{\boldsymbol{H}}_1$-superharmonic on $\omega$ and equal to 0 on $\partial\omega$ and so it is nonnegative by the minimum principle.

**Lemma 4.** *If $(X,\tilde{\boldsymbol{H}})$ is subordinate to $(X,\boldsymbol{H})$, then for any $\omega\in\boldsymbol{\Omega}$ and for any nonnegative continuous function $f$ on $\partial\omega$, $\int f\,d\nu_x^{\omega}-\int f\,d\tilde{\nu}_x^{\omega}$ is an $\tilde{\boldsymbol{H}}_1$-potential on $\omega$.*

*Proof.* By the above lemma (ii), $\int f\,d\nu_x{}^{\omega}-\int f\,d\tilde{\nu}_x^{\omega}$ is a nonnegative $\tilde{\boldsymbol{H}}_1$-superharmonic function on $\omega$. Let $\tilde{h}_1$ be a nonnegative $\tilde{\boldsymbol{H}}_1$-harmonic function on $\omega$ such that $\tilde{h}_1\leqq\int f\,d\nu_x^{\omega}-\int f\,d\tilde{\nu}_x^{\omega}$. Then by the above lemma (i), $-\tilde{h}_1\in\boldsymbol{S}_1(\omega)$ and $\int f\,d\nu_x^{\omega}-\tilde{h}_1\geqq 0$. Since $\int f\,d\nu_x^{\omega}$ is an $\boldsymbol{H}_1$-potential on $\omega$ by Lemma 6.15 in [7], we have $-\tilde{h}_1\geqq 0$ and so $\tilde{h}_1=0$. Hence $\int f\,d\nu_x^{\omega}-\int f\,d\tilde{\nu}_x^{\omega}$ is an $\tilde{\boldsymbol{H}}_1$-potential on $\omega$.

If $(X,\tilde{\boldsymbol{H}})$ is subordinate to $(X,\boldsymbol{H})$, then associated harmonic spaces $(X,\tilde{\boldsymbol{H}}_j)$ $(j=1,2)$ are subordinate to harmonic spaces $(X,\boldsymbol{H}_j)$ respectively. In this case, by Theorem 3.7 in [6], $(X,\tilde{\boldsymbol{H}}_j)$ $(j=1,2)$ are obtained by perturbations of $(X,\boldsymbol{H}_j)$ and these perturbations are unique. If $(X,\boldsymbol{H}_1)$ has a consistent system $\{G_1^{\omega}(x,y):\omega\in\boldsymbol{\Omega}\}$ of $\boldsymbol{H}_1$-Green's functions, by the theory in [1], $(X,\tilde{\boldsymbol{H}}_1)$ is obtained by a perturbation of $(X,\boldsymbol{H}_1)$ as follows. That is, this perturbation is characterized by the unique existence of a positive Radon measure $m$ on $X$ satisfing the equality

$$\int f\,d\mu_x^{\omega}=\int f\,d\tilde{\mu}_x^{\omega}+\int G_1^{\omega}(\mathrm{x},y)\Bigl(\int f\,d\tilde{\mu}_y^{\omega}\Bigr)dm(y)$$

on $\omega$ for any $\omega\in\boldsymbol{\Omega}$ and any finite continuous function $f$ on $\partial\omega$. Further if the consistent system $\{G_1^{\omega}(x,y):\omega\in\boldsymbol{\Omega}\}$ of $\boldsymbol{H}_1$-Green's functions is symmetric and every points in $X$ are $\boldsymbol{H}_1$-polar, then there exists a consistent system $\{\tilde{G}_1^{\omega}(x,y):\omega\in\boldsymbol{\Omega}\}$of $\tilde{\boldsymbol{H}}_1$-Green's functions satisfing the resolvent equation

$$(*):\ G_1^{\omega}(x,y)=\tilde{G}_1^{\omega}(x,y)+\int\tilde{G}_1^{\omega}(x,z)G_1^{\omega}(z,y)\,dm(z).$$

From these results, we have

**Lemma 5.** *If $(X,\tilde{\boldsymbol{H}})$ is subordinate to $(X,\boldsymbol{H})$ and $(X,\boldsymbol{H}_1)$ has a consistent system $\{G_1^{\omega}(x,y):\omega\in\boldsymbol{\Omega}\}$ of symmetric $\boldsymbol{H}_1$-Green's functions and every points in $X$ are $\boldsymbol{H}_1$-polar, then there exists a unique composing measure $\tilde{a}$ of $(X,\tilde{\boldsymbol{H}})$ such that*

$$\int f\,d\tilde{\nu}_x^{\omega}=\int\tilde{G}_1^{\omega}(x,y)\Bigl(\int f\,d\tilde{\lambda}_y^{\omega}\Bigr)d\tilde{a}(y)$$

*on $\omega$ for any $\omega \in \boldsymbol{\Omega}$ and any finite continuous function $f$ on $X$.*

Now we shall give a characterization of a subordinate biharmonic space as follows.

**Theorem.** *Let $(X, \boldsymbol{H})$ be an elliptic biharmonic space. Suppose that the associated harmonic space $(X, \boldsymbol{H}_1)$ has a consistent system $\{G_1^{\omega}(x, y) : \omega \in \boldsymbol{\Omega}\}$ of symmetric $\boldsymbol{H}_1$-Green's functions and every points in $X$ are $\boldsymbol{H}_1$-polar. Then a biharmonic space $(X, \tilde{\boldsymbol{H}})$ is subordinate to $(X, \boldsymbol{H})$ if and only if the associated harmonic spaces $(X, \tilde{\boldsymbol{H}}_j)$ $(j=1, 2)$ are subordinate to $(X, \boldsymbol{H}_j)$ respectively and there exists a unique composing measure $\tilde{a}$ of $(X, \tilde{\boldsymbol{H}})$ satisfing $\tilde{a} \leqq a$*
*(i.e. $\int f\, d\tilde{a} \leqq \int f\, da$ for any nonnegativve continuous function $f$ on $X$ with a compact support).*

*Proof.* Assume that $(X, \tilde{\boldsymbol{H}})$ is subordinate to $(X, \boldsymbol{H})$. Then by Lemmas 1 and 5,

$$\int d\nu_x^{\omega} = \int G_1^{\omega}(x, y)\left(\int d\lambda_y^{\omega}\right) da(y),$$

$$\int d\tilde{\nu}_x^{\omega} = \int \tilde{G}_1^{\omega}(x, y)\left(\int d\tilde{\lambda}_y^{\omega}\right) d\tilde{a}(y),$$

on $\omega$ for any $\omega \in \boldsymbol{\Omega}$. By the resolvent equation $(*)$ we have

$$\int d\nu_x^{\omega} - \int d\tilde{\nu}_x^{\omega}$$

$$= \int G_1^{\omega}(x, y)\left(\int d\lambda_y^{\omega}\right) da(y) - \int \tilde{G}_1^{\omega}(x, y)\left(\int d\tilde{\lambda}_y^{\omega}\right) d\tilde{a}(y)$$

$$= \int \tilde{G}_1^{\omega}(x, y)\left(\int d\lambda_y^{\omega}\right) da(y) + \int\left[\left(\int \tilde{G}_1^{\omega}(x, z) G_1^{\omega}(z, y)\, dm(z)\right)\left(\int d\lambda_y^{\omega}\right)\right] da(y)$$

$$- \int \tilde{G}_1^{\omega}(x, y)\left(\int d\tilde{\lambda}_y^{\omega}\right) d\tilde{a}(y)$$

$$= \int \tilde{G}_1^{\omega}(x, y)\left(\int d\lambda_y^{\omega}\right) da(y) + \int \tilde{G}_1^{\omega}(x, z)\left(\int d\nu_z^{\omega}\right) dm(z)$$

$$- \int \tilde{G}_1^{\omega}(x, y)\left(\int d\tilde{\lambda}_y^{\omega}\right) d\tilde{a}(y).$$

By lemma 4, $\int d\nu_x^{\omega} - \int d\tilde{\nu}_x^{\omega}$ being an $\tilde{\boldsymbol{H}}_1$-potential on $\omega$, $(\int d\lambda_{\cdot}^{\omega})a + (\int d\nu_{\cdot}^{\omega})m - (\int d\tilde{\lambda}_{\cdot}^{\omega})\tilde{a}$ is a positive Radon measure on $\omega$ for any $\omega \in \boldsymbol{\Omega}$. Hence for any nonnegative continuous function $g$ on $\omega$ with a compact support, we have

$$(**) : \int g(x)\left(\int d\lambda_x^{\omega}\right) da(x) + \int g(x)\left(\int d\nu_x^{\omega}\right) dm(x) \geqq \int g(x)\left(\int d\tilde{\lambda}_x^{\omega}\right) d\tilde{a}(x).$$

Let $f$ be any nonnegative continuous function on $X$ with a compact support $K$. By Lemma 2, for any $\varepsilon > 0$ and any $x \in K$, there exists $\omega_{x,\varepsilon} \in \boldsymbol{\Omega}_x$ such that

$$(***) : 1-\varepsilon < \int d\lambda_{\cdot}^{\omega(x,\varepsilon)} < 1+\varepsilon, \quad 1-\varepsilon < \int d\tilde{\lambda}_{\cdot}^{\omega(x,\varepsilon)} < 1+\varepsilon, \quad 0 < \int d\nu_{\cdot}^{\omega(x,\varepsilon)} < \varepsilon$$

on $\omega(x, \varepsilon)$. Since $\{\omega(x, \varepsilon) : x \in K\}$ is an open covering of $K$, there exists a finite subcovering $\{\omega(x_i, \varepsilon) : 1 \leqq i \leqq k\}$ of $K$. Let $\{\Psi_i : 1 \leqq i \leqq k\}$ be the partition of unity subordinated to this finite subcovering $\{\omega(x_i, \varepsilon) : 1 \leqq i \leqq k\}$. We put $g_i = f \cdot \Psi_i$ and $\omega(i) = \omega(x_i, \varepsilon)$ $(1 \leqq i \leqq k)$. Then by $(**)$

$$\int g_i(x)\Big(\int d\lambda_x^{\omega(i)}\Big)\, d\alpha(x) + \int g_i(x)\Big(\int d\nu_x^{\omega(i)}\Big)\, dm(x) \geqq \int g_i(x)\Big(\int d\tilde{\lambda}_x^{\omega(i)}\Big)\, d\tilde{\alpha}(x).$$

By the inequalities $(***)$, we have

$$\int g_i(x)\, d\tilde{\alpha}(x) - \int g_i(x)\, d\alpha(x) \leqq \varepsilon\Big\{\int g_i(x)\, d\alpha(x) + \int g_i(x)\, d\tilde{\alpha}(x) + \int g_i(x)\, dm(x)\Big\}.$$

Since $f = \sum_{i=1}^{k} g_i$,

$$\begin{aligned}\int f(x)\, d\tilde{\alpha}(x) - \int f(x)\, d\alpha(x) &= \sum_{i=1}^{k}\Big\{\int g_i(x)\, d\tilde{\alpha}(x) - \int g_i(x)\, d\alpha(x)\Big\} \\ &\leqq \varepsilon \sum_{i=1}^{k}\Big\{\int g_i\, d\alpha + \int g_i\, d\tilde{\alpha} + \int g_i\, dm\Big\} \\ &= \varepsilon\Big\{\int f\, d\alpha + \int f\, d\tilde{\alpha} + \int f\, dm\Big\}\end{aligned}$$

Letting $\varepsilon \to 0$, we have $\int f\, d\tilde{\alpha} \leqq \int f\, d\alpha$ and so $\tilde{\alpha} \leqq \alpha$.

Conversely let $(X, \tilde{\boldsymbol{H}})$ be a biharmonic space such that the associated harmonic spaces $(X, \tilde{\boldsymbol{H}}_j)$ $(j = 1, 2)$ are subordinate to $(X, \boldsymbol{H}_j)$ respectively and its composing measure $\tilde{\alpha}$ satisfies $\tilde{\alpha} \leqq \alpha$. Then $(X, \tilde{\boldsymbol{H}}_j)$ $(j = 1, 2)$ are obtained by perturbations of $(X, \boldsymbol{H}_j)$. Hence their harmonic measures $(\tilde{\mu}_x^{\omega})$ (resp. $(\tilde{\lambda}_x^{\omega})$) are defined for any $\omega \in \boldsymbol{\Omega}$ and satify $\tilde{\mu}_x^{\omega} \leqq \mu_x^{\omega}$ (resp. $\tilde{\lambda}_x^{\omega} \leqq \lambda_x^{\omega}$). Since

$$\tilde{\nu}_x^{\omega} = \int \tilde{G}_1^{\omega}(x, y)\, \tilde{\lambda}_y^{\omega}\, d\tilde{\alpha}(y) \leqq \int G_1^{\omega}(x, y) \lambda_y^{\omega}\, d\alpha(y) = \nu_x^{\omega}$$

for any $\omega \in \boldsymbol{\Omega}$, we know that $(X, \tilde{\boldsymbol{H}})$ is subordinate to $(X, \boldsymbol{H})$. This completes the proof.

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