On Pseudo-Affine Domains

Ken-ichi YOSHIDA and Susumu ODA*
Department of Applied Mathematics,
Okayama University of Science,
Ridai-cho, Okayama 700, Japan
* *Uji-Yamada High School,*
Uraguchi, Ise, Mie 516, Japan
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In what follows, all rings considered are commutative with identity.

We say that a ring A is a *Hilbert ring* if each prime ideal of A is an intersection of maximal ideals of R. It is known that a k-affine domain over a field k is a Hilbert ring ([G, (31.11)]).

We say that a ring A is a *catenary* ring if the following condition is satisfied: for any prime ideals p and q of A with $p \subseteq q$, then exists a saturated chain of prime ideals starting from p and ending at q, and all such chains have the same (finite) length. We say that a ring A is a *universally catenary* ring if A is Noetherian and every finitely generated A-algebra is catenary.

Let k be a field and R a K-affine domain. Then R is Noetherian, Hilbert and catenary. Moreover $\dim R_m = Tr.deg_k R < +\infty$ for each maximal ideal m of R.

Our objective in this paper is to investigate integral domains having these properties.

Throughout this paper, k denotes a field and R an integral domain containing k and $K(R)$ denotes the quotient field of R unless otherwise specified. Any unexplained terminology is standard, as in [M], [N].

Definition 1. An integral domain R is called a *preudo-affine domain over* k (**PAD**(k) *for short*) *if the following conditions are satisfied:*

(i) R *is Noetherian;*

(ii) R *is Hilbert and catenary;*

(iii) $\dim R_m = Tr.deg_k R < +\infty$ for each maximal ideal m of R.

Remark 2. It is known that a k-affine domain is a PAD(k) ([M, (5.6)]). A field K containing k is a PAD(k) if and only if K is algebraic over k.

The following Lemma 3 is shown in [O].

Lemma 3. *Let R be an integral domain containing a field k. Let*

$$(0) = P_0 \subset P_1 \subset \cdots \subset P_r$$

be a strict ascending chain of prime ideals of R and let $a_i \in P_i \backslash P_{i-1}$ $(1 \leq i \leq r)$. Then

$a_1, \dots, a_r$ are algebraically independent over k.

Proof. Suppose that there exists a non-trivial polynomial $F(X_1, \dots, X_r)$ in a polynomial ring $k[X_1, \dots, X_r]$ such that $F(a_1, \dots, a_r)=0$. We can assume that deg $F(X_1, \dots, X_r)$ is minimal among such polynomials. Write:

$$0=F(a_1, \dots, a_r)=F_0(a_2, \dots, a_r)a_1+ \cdots +F_n(a_2, \dots, a_r)a_1^n.$$

Let $a_2', \dots, a_r'$ denote the images in R/P_1. By induction on r, we may assume that $a_2', \dots, a'_r$ are algebraically independent over k. Thus since $F(a_1', \dots, a'_r)=F_0(a_2', \dots, a_r')=0$, we have $F_0(X_2, \dots, X_r)=0$ in $k[X_2, \dots, X_r]$. We have:

$$F(a_1, \dots, a_r)=a_1(F_1(a_2, \dots, a_r)+ \cdots +F_n(a_2, \dots, a_r)a_1^{n-1})=0,$$

and hence

$$F_1(a_2, \dots, a_r)+ \cdots +F_n(a_2, \dots, a_r)a_1^{n-1}=0.$$

By the minimality of deg $F(X_1, \dots, Xr)$, we conclude that:

$$F_1(X_2, \dots, X_r)+ \cdots +F_n(X_2, \dots, X_r)X_1^{n-1}=0$$

in $k[X_1 \dots, X_r]=0$ in $k[X1, \dots, X_r]$, a contradiction. □

Proposition 4. *$\dim R \leq Tr.deg_k R$.*

Proof. This follows Lemma 3 immediately. □

Corollary 4.1. *Let R be a* PAD(k) *and let $p \in Ht_1(R)$. Then*
(i) dim $R/p=\dim R-1$;
(ii) $Tr.deg_k R/p=Tr.deg_k R-1$.

Proof. Since R is catenary, dim $R-1=\dim R/p$. By definithon, $Tr.deg_k R/p \leq Tr.deg_k R/p$ by Proposition 4. Thus $Tr.deg_k R/p=Tr.deg_k R-1=\dim R-1=\dim R/p$. □

Proposition 5. *Let R be a* PAD(k) *and let p is a prime ideal of R. Then R/p is also a* PAD(k).

Proof. Since R is Hilbert (resp. Noetherian), so is R/p. Corollary 4.1 repeatedly, dim $R/p=\dim R-ht(p)=Tr.deg_k R/p$. □

Corollary 5.1. *An integral domain which is a homomorphic image of a PAD(k) is also a PAD(k).*

Proof. Let p be a prime ideal of $ht(p)=1$. Then it is clear that R/p is a Hilbert ring. Hence R/p is a PAD(k) by Corollary 4.1. So we get our conclusion by induction on dim R. □

Proposition 6. *Assume that R is a PAD(k). Then $ht(p)=\dim R-\dim R/p=Tr.deg_k R-Tr.deg_k R/p$ for each $p \in Spec(R)$.*

Proof. This follows from the proof of Corollary 4.1 and $\dim R = Tr.deg_k R$ and $\dim R/p = Tr.deg_k R/p$ by definition. □

Lemma 7 ([G, (31.18)]). *The following condictions are equivalent:*
(1) R is a Hilbert ring;
(2) For each maximal ideal M of a polynomial ring $R[X_1, \ldots, X_n]$, $M \cap R$ is a maximal ideal of R;
(3) A polynomial ring $R[X_1, \ldots, X_n]$ is a Hilbert ring;
(4) R/I is a Hilbert ring for each proper ideal I of R.

Example. Let k be a field and $k[t]$ a polynomial ring. Put $R = k[t]_{(t)}$. Then $R[X]/(tX-1) \cong k(t)$ and $(tX-1)$ is a maximal ideal of $R[X]$ with $R \cap (tX-1) = (0)$. So $R[X]$ is a Hilbert ring but is not a PAD(k).

Lemma 8 ([G, (31.9)]). *If R is a Hilbert ring and if M is a maximal ideal of a polynomial ring $R[X_1, \ldots, X_n]$, then $R[X_1, \ldots, X_n]/M$ is algebraic over $R/M \cap R$.*

L. J. Ratliff shows the following result:

Lemma 9 (cf. [M, p. 31]) *Let (A, m) be a Noetherian local domain. Then A is catenary if and only if $ht(p) + \dim A/p = \dim A$ for each $p \in Spec(A)$.*

Lemma 10. *Let R be a PAD(k), let $R[X]$ be a polynomial ring and let P be a prime ideal of $R[X]$ such that $(P \cap R)R[X] \neq P$. Then* $\dim R[X]/P = Tr.deg_k R[X]/P$.

Proof. Since R is a PAD(k), we have $\dim R/P \cap R = Tr.deg_k R/P \cap R$ by Proposition 5. Since $(P \cap R)R[X] \neq P$, it follows that $\dim R/P \cap P = \dim R[X]/P$. By the same reason, $R[X]/P$ is algebraic over $R/P \cap R$. Thus we have $Tr.\ deg_k R/P \cap R = Tr.deg_k R[X]/P$. Hence $\dim R[X]/P = \dim R/P \cap R = Tr.deg_k R/P \cap R = Tr.deg_k R[X]/P$.
□

Proposition 11. *A PAD(k) is universally catenary.*

Proof. We have only to prove a polynomial ring $R[X]$ is catenary. Take $P \in Spec(R[X])$. First assume that $P = pR[X]$ for some $p \in Spec(R)$ i.e., $(P \cap R)[X] = P$. Then $ht(P) = ht(p] = (\dim R+1) - (\dim R/p+1) = \dim R[X] - \dim R[X]/pR[X]$ by Lemma 9. Second, assume that $(P \cap R) \neq P$. Then $ht(p) - ht(P \cap R) = 1$. Hence $ht(p) = ht(P \cap R) + 1 = \dim R - \dim R/P \cap R + 1 = \dim R[X] - Tr.deg_k R/P \cap R = \dim R[X] - Tr.deg_k R/P \cap R = \dim R[X] - Tr.deg_k R/P \cap R \geq \dim R[X] - Tr.deg_k R[X]/P = \dim R[X] - \dim R[X]/P$, where the last equality follows from Lemma 10. But we know that $ht(P) \leq \dim R[X] - \dim R[X]/P$. Thus we get $ht(P) = \dim R[X] - \dim R[X]/P$. Therefore by Lemma 9, we conclude that $R[X]$ is catenary.

Let A be a Noetherian domain and B a finitely generated extension domain. We say that the *dimension formala* holds between A and B if

$$htP = htp + Tr.deg_A B - Tr.deg_{k(p)} k(P)$$

for every $P \in Spec(R)$, where $p = P \cap A$.

Corollary 11.1. *Assume that R is a $PAD(k)$. Then dimension formula holds between R/p and B for every prime ideal p of R and every finitely generated domain B of R/p.*

Proof. Since R is universally catenary by Proposition 11, the conclusion follows from [M, (15.6)]. □

Theorem 12. *The following conditions are equivalent:*
(i) R is a $PAD(k)$;
(ii) A polynomial ring $R[X_1, \dots, X_n]$ is a $PAD(k)$;
(iii) Every integral domain containing R which is finitely generated over R is a $PAD(k)$.

Proof. $(i) \Rightarrow (ii)$ follows from Lemmas 7 and 8 because $\dim R[X_1, \dots, X_n] = \dim R + n = Tr.deg_k R + n = Tr.deg_k R[X_1, \dots, X_n]$. By Proposition 11, $R[X_1, \dots, X_n]$ is catenary. $(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (i)$ are immediately verified by Proposition 5. □

Proposition 13. *Let R be a normal $PAD(k)$ and A a Noetherian domain which is integral over R. Then A is a $PAD(k)$.*

Proof. Let p be a prime ideal of A with $\dim A/p = 1$. Since A is integral and since B is a Hilbert ring, p is contained in infinitely many maximal ideals by Lying-Over Theorom. So by [G, (31, Ex. 22)], A is a Hilbert ring. Let M be a maximal ideal of A and put $m \cap R$. Then m is a maximal ideal of R. Note that A/M is algebraic over R/m. Moreover $\dim A_M = \dim R_m$ by Going-Down Theorom. Since $Tr.deg_k A = Tr.deg_k R$ and $\dim R_m = Tr.deg_k R$, we have $\dim A_M = Tr.deg_k A$. It is easy to see that A is catenary. Thus A is PAD(k). □

Proposition 14. *Let A be a Noetherian domain containing R with $K(A)$ algebraic over $K(R)$. If A is faithfully flat over R and R is a $PAD(k)$, then A is a $PAD(k)$.*

Proof. Since a canonical morphism $Spec(A) \rightarrow \mathrm{Spec}(R)$ is surjective. Let M be a maximal ideal of A and put $m = \mathrm{M} \cap R$. Then $Tr.deg_k A = Tr.deg_k R = \dim R_m$ and $\dim R_m = \dim A_M$ by Going-Down Theorem. By the same way as the proof of Proposition 13, we can show that A is Hilbert. Since A is faithfully flat over the catenary ring R, A is lso catenary. Thus A is a PAD(k). □

Proposition 15. *Assume that R be a $PAD(k)$ and let m be a maximal ideal of R. Then R/m is algebraic over k.*

Proof. The field R/m is a PAD(k) by Proposition 5. So by the fact stated in Remark 2, R/m is algebraic over k.

Proposition 16. *Assume that R is a normal $PAD(k)$. Let L be a finite separable field extension of the quotient field $K(R)$ of R. Let B be intermediate ring between R and L which is integral over R. Then B is a $PAD(k)$.*

Proof. By [N, (10.16)], the integral closure R_L of R in L is a finite R-module. Hence B is a finite R-module. So B is PAD(k) by Theorom 12. □

Corollary 16.1. *Assume that R is a PAD(k) whose derived normal ring $\bar{R}$ is Noetherian. Let L be finite separable extension field of the quotient field $K(R)$. Then the integral closure R_L of R in L is a PAD(k).*

Proof. By Proposition 13, $\bar{R}$ is a *PAD*(k). Note that R_L is integral closure of $\bar{R}$ in L. Since is Noetherian, R_L is a PAD(k) by Proposition 16. □

Proposition 17. *Let $\bar{R}$ denote the derived normal domain of a Noetherian domain R. If $\bar{R}$ is a PAD(k), then so is R.*

Proof. The domain R is Hilbert by Lying-Over Theorem, which is seen in the same maners of the proof of Proposition 13 because $\bar{R}$ is Hilbert and catenary (Proposition 11). Moreover $Tr.deg_k R_m = Tr.deg_k \bar{R}_M = \dim \bar{R}_M \leq \dim \mathrm{R}_m \leq Tr.deg_k R_m$, where M is maximal ideal of $\bar{R}$ lying over a maximal ideal m of R. Hence R is a PAD(k). □

Let A be a ring and I an ideal of A. We recall that J is called a *reduction* of I if $J \subseteq I$ and $JI^r = I^{r+1}$ for at least one positive integer r ([L], [O]). It is easy to see that $\sqrt{J} = \sqrt{I}$ and $ht(J) = ht(I)$.

Proposition 18. *Assume that a Noetherian domain R satisfies the condition :* $\dim R = Tr.deg_k R := n$ *and let I be an ideal of R. Then I has a reduction generated by $(n+1)$-elements.*

Proof. This follows from [L] or [O, (3.4)]. □

Corollary 18.1. *Assume that R is a PAD(k) with* $\dim R = n$. *Then each ideal I of R has a reduction J generated by $(n+1)$-elements.*

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