Quaternions and Null Curves in Eight-Dimensional Space

Hirohisa TACHIBANA

Department of Industrial Sociology, Kibi International University, 8 Iga-mach, Takahashi-City, Okayama 716, Japan

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This paper analyzes a null curve in complex eight-dimensional space. A general solution of the null curve is presented. The solution is given by freely speciafiable three complex quaternions.

1 Introduction

Eight-dimensional complex flat space Ω^8, compactified and with the usual conformal structure, can be embedded in ten-dimensional complex Minkowski space with the metric $g_{ab} = g^{ab} = \text{diag}\ (1, -1, -1, \ldots, -1)$, where the indices $a, b, \ldots$ take the values of 0, 1, 2,...9. Then a point z^a in Ω^8 can be expressed by a homogeneous coorinates of CP^9 satisfying the null quadratic condition

$$z^a z_a = 0. \tag{1}$$

A null curve in Ω^8 is a world-line $z^a(s)$ satisfying following differential equation;

$$z'^a z'_a = 0, \tag{2}$$

where s is a real affine parameter and the primed symbol indicates the differentiation with respect to s. Although it seem easy to solve the equation (2), it is difficult to solve the equation since it is a *nonlinear* differential equation.

Recently, Hughston and Shaw[1] showed that Cartan's pure spinors[2] in ten dimensions provide the solution of (1) and (2). However, since any ten-dimensional pure spinor has five *nonlinear* algebraic constraints with respect to its components, it is difficult to solve their constraints as well as differential equation (2). The purpose of this paper is to find the general solution of the equations (1) and (2), and show that it has *at most single condition.*

In section, 2, we describe briefly Hughston and Shaw's theorem for the general null curve in eight dimensions. Section 3 is devoted to solve pure spinor constraints in ten dimensions by making use of compex octonions and complex quaternions. In section 4, we give again, in terms of three complex quaternions, the general null curve in eight dimensions. Then we show that this curve has at most single constraint.

2 Hughston and Shaw's theorem

Let $\Gamma_{AB}^{a}$and Γ^{aAB} be the ten-dimensional reduced gamma matrices, where the spinor indices A, B,... take the values of 0, 1, 2,..., 15. These gamma matrices satisfy the Clifford equations

$$\Gamma^{(a|AC}\Gamma_{BC}^{|b)} = g^{ab}\ \delta_{B}^{A}, \tag{3}$$

and the symmetry properties

$$\Gamma_{AB}^{a} = \Gamma_{BA}^{a}, \qquad \Gamma^{aAB} = \Gamma^{aBA}. \tag{4}$$

Furthermore, we have the particular quadratic relations

$$\Gamma^{a(AB}\ \Gamma_{a}^{C)D} = 0, \quad \Gamma_{(AB|}^{a}\ \Gamma_{a|C)D} = 0. \tag{5}$$

A ten-dimensional pure spinor is given as the reduced spinor ξ^{A} satisfying following conditions[3];

$$\Gamma_{AB}^{a}\xi^{A}\ \xi^{B} = 0. \tag{6}$$

This is called the *purity conditions* for the ten-dimensional spinor. Althoght the purity conditions (6) are in appearance ten conditions, only five conditions of them are really independent. Therefore a ten-dimensional pure spinor has eleven independent components[4]. This comes from the symmetric properties (5) of the ten-dimensional reduced gamma matrices.

THEOREM (Hughston and Shaw[1]). *Let ξ^{A} (s) be a pure spinor with a real parameter s. Then a general null curve $z^{a}(s)$ in Ω^{8} is given by*

$$z^{a} = \Gamma_{AB}^{a}\ \xi'^{A}\ \xi'^{B}. \tag{7}$$

Since the proof of this theorem is explicitly given by Hughston and Shaw in their paper[1], we do not repeat it in this paper. By the equations (7), we have no restrictions on the derivatives. However, we have still five nonlinear algebraic constraints (6), that is, the purity constraints.

3 Ten-dimensional pure spinors represented by three complex quaternions

Any ten-dimensional spinor can be expressed by a pair of complex octonions. This can be carried out by following processes. Let i_j be octonion imaginary units, where the induces j, k,... take the values of 1, 2, ..., 7. We exchange any ten-dimensional reduced spinor ξ^{A} as follows;

$$\xi^{A} \rightarrow \xi^{\alpha} := \begin{pmatrix} \xi^{0} + i_j\ \xi^{j} \\ \xi^{8} + i_j\ \xi^{j+8} \end{pmatrix}, \tag{8}$$

where the indices α, β,... take the values of 1 and 2. ξ^{α} is called an *octonionic spinor* whose algebraic, geometric and transformation properties are completely investigated by many authors[5]. Uning such an octonionic spinor formulation, we can express the pure spinor constraints in ten dimensions as follows[6];

$$\xi^{\alpha}\bar{\xi}^{\beta} = 0, \tag{9}$$

where the symbol bar indicates the octonion conjugate.

Now we should note that any octonion is defined as a pair of any quaternions by the *Cayley-Dickson process*[7]. Therefore, we can write the complex octonionic spinor ξ^{α} as follows;

$$\xi^{\alpha} = \begin{pmatrix} (\Pi,\ \Phi) \\ (\bar{\Omega},\ -\Psi) \end{pmatrix}, \tag{10}$$

where Π, Φ, Ω and Ψ are any complex quaternions and the symbol bar indicates the quaternion conjugate. By (10) and the Cayley-Dickson product, pure spinor constraints (9) can be rewritten as following complex quaternion equations;

$$\begin{aligned} &N(\Pi)+N(\Phi) = 0, \\ &N(\Omega)+N(\Psi) = 0. \\ &\Pi\Omega - \bar{\Psi}\Phi = 0, \\ &\Phi\bar{\Omega} + \Psi\Pi = 0, \end{aligned} \tag{11}$$

where N gives the norm of any quaternion. Since the equations (11) are pure spinor constraints in ten dimensions, only five complex equations of these are independent. Introducing a new complex quaternion W, we can easily solve the equations (11) as follows;

$$\Omega = \bar{\Pi}W,\ \Psi = -\Phi\bar{W}. \tag{12}$$

Consequently, we obtain

$$\xi^{\alpha} = \begin{pmatrix} (\Pi,\ \Phi) \\ (\bar{W}\Pi,\ \Phi\bar{W}) \end{pmatrix}, \tag{13}$$

where

$$N(\Pi)+N(\Phi) = 0. \tag{14}$$

Thus we can express any pure spinor ξ^{A} with any three complex quaternions Π, Φ and W. Moreover, there are at most single constraint (14) between these three complex quaternions[8].

4 Quaternion formulation of eight-dimensional null curve

According to Hughston and Shaw's theorem, we can obtain the octonionic spinor form of an eight-dimensional null curve. From expression (7), we obtain

$$z^{\alpha\beta} = \xi'^{\alpha}\ \bar{\xi}'^{\beta}, \tag{15}$$

where $\xi^\alpha = \xi^\alpha(s)$ is an octonionic pure spinor satisfying (9) and $z^{\alpha\beta} = z^{\alpha\beta}(s)$ is the $2\times$ 2-octonionic matrix given by

$$z^{\alpha\beta} = \begin{pmatrix} z^+ & \zeta \\ \bar{\zeta} & z^- \end{pmatrix} \tag{16}$$

and

$$z^+ = z^0 + z^9,\ z^- = z^0 - z^9,\ \zeta = z^1 + i_1 z^2 + \ldots + i_7 z^8. \tag{17}$$

In convenience, we decompose the complex octonion ζ into a pair of complex quaternions $Z_1(s)$ and $Z_2(s)$ as follows;

$$\zeta = (Z_1,\ Z_2). \tag{18}$$

According to (13), the octonionic pure spinor $\xi^\alpha = \xi^\alpha(s)$ can be constructed by three complex quaternions $\Pi = \Pi(s)$, $\Phi = \Phi(s)$ and $W = W(s)$, where Π and Φ satisfy the condition (14). Differentiating ξ^α with respect to s, we have

$$\xi'^\alpha = \begin{pmatrix} (\Pi',\ \Phi') \\ (\bar{W}'\Pi + \bar{W}\Pi',\ \Phi'\bar{W} + \Phi\bar{W}') \end{pmatrix}. \tag{19}$$

Substituting (19) into (15), comparing it with (16) and using (14) and (18), we obtain

$$\begin{aligned} z^+ &= N(\Pi') + N(\Phi'), \\ z^- &= N(W)(N(\Pi') + N(\Phi')) + W\cdot(\Pi'\bar{\Pi}W' + W'\bar{\Phi}\Phi'), \\ Z_1 &= (N(\Pi') + N(\Phi'))W' + \Pi'\bar{\Pi}W' + W'\bar{\Phi}\Phi', \\ Z_2 &= \Phi'\bar{W}'\Pi + \Phi\bar{W}'\Pi', \end{aligned} \tag{20}$$

where the symbol dot indicates the inner product between two quaternions.

In conclusion, we can express a general null curve in eight dimensions by making use of any three quaternions $\Pi(s)$, $\Phi(s)$ and $W(s)$ under at most single constraint (14) between these quaternions.

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