Exact Solutions with Two Gauge Charges in Poincaré Gauge Theory

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In this paper we consider the Poincaré gauge theory with one linear and some quadratic Lagrangians, and investigate the possibility of some exact solutions which can be derived from the Kerr-Newman solution in Einstein-Maxwell theory. As a result, we find a series of solutions with two kinds of gauge charges.

1. Introduction

In this paper we consider a Poincaré gauge theory (PGT), which has been first formulated by Utiyama[1] and Kibble[2], and later extended by Hayashi[3]. We here adopt the Lagrangians proposed by Hayashi.

PGT has two gauge fields, a translational gauge field $c_k{}^\mu$ and a Lorentz gauge field $A_{km\mu}$. We here treat these two fields as independent and fundamental fields. For, in this treatment the Lorentz gauge field can be looked upon as a gauge field in an *internal* gauge theory based on the localized Lorentz transformations of the tetrads $b_k{}^\mu$ $(= \delta_k{}^\mu + c_k{}^\mu)$. From this viewpoint, we recently[4,5] have shown a possibility such that the equations for the gauge fields can be reduced to the Einstein-Maxwell (EM) equations via complex Einstein-Yang-Mills (CEYM) equations. Now we here review this briefly (see also Appendices).

First we introduce the various complex quantities in order to rewrite the equations for our purpose*: Such as

$$\vec{\mathfrak{A}}_\mu = \vec{v}_\mu + i\vec{a}_\mu \tag{1.1}$$

$$\vec{\mathfrak{F}}_{\mu\nu} = \vec{F}^P{}_{\mu\nu} + i\vec{F}^A{}_{\mu\nu}, \tag{1.2}$$

where $\vec{v}_\mu$, $\vec{a}_\mu$, $\vec{\mathfrak{F}}^P{}_{\mu\nu}$ and $\vec{\mathfrak{F}}^A{}_{\mu\nu}$ are defined as

$$\vec{v}_\mu:\ v_{(a)\mu} \overset{\text{def}}{=} A_{0a\mu}, \qquad \vec{a}_\mu:\ a_{(a)\mu} \overset{\text{def}}{=} \frac{1}{2}\varepsilon_{abc}A_{bc\mu}, \tag{1.3}$$

* Throughout this paper we use the same notation as those of Ref. [5].

$$\vec{F}^{P}{}_{\mu\nu}:\ F^{P}{}_{(a)\mu\nu} \stackrel{\text{def}}{=} F_{0a\mu\nu}, \qquad \vec{F}^{A}{}_{\mu\nu}:\ F^{A}{}_{(a)\mu\nu} \stackrel{\text{def}}{=} \frac{1}{2}\varepsilon_{abc}F_{bc\mu\nu}. \tag{1.4}$$

Here lower-case latin letters $a,\ b,\ c\ \cdots\cdot$ mean the *internal* degrees of freedom and range over the three values, 1, 2, 3, and ε_{abc} is the three-dimensional Levi-Civita symbol with $\varepsilon_{123} = 1$.

Using above and the analogous quantities, the following two complex equations can be derived for the Lorentz gauge field (see Appendix A):

$$\vec{\mathfrak{T}}^{\mu\nu}{}_{;\nu} - i\vec{\mathfrak{A}}_{\nu} \times \vec{\mathfrak{T}}^{\mu\nu} + \vec{\mathfrak{J}}^{\mu} = \vec{\mathfrak{S}}^{\mu}{}_{(M)}, \tag{1.5}$$

$$\vec{\mathfrak{F}}^{\dagger\mu\nu}{}_{;\nu} - i\vec{\mathfrak{A}} \times \vec{\mathfrak{F}}^{\dagger\mu\nu} = 0. \tag{1.6}$$

Here note the "covariant operator" $(;\mu)$ is equal to the usual one with the Christoffel connections, omitting the operation on the internal indices $a,\ b,\ c\ \cdots\cdot$.

The first equations for our purpose can be obtained from above ones if the following three conditions are imposed on them:†

$$[\ \mathrm{I}\ ] \quad \vec{\mathfrak{A}}_{\mu} = \vec{\beta}\mathfrak{A}_{\mu}$$

$$[\ \mathrm{II}\ ] \quad \vec{\mathfrak{T}}_{\mu\nu} = \frac{1}{\mathrm{g}}\vec{\mathfrak{F}}_{\mu\nu}$$

$$[\mathrm{III}] \quad C_1 = C_2 = C_3 = 0$$

where g and $\vec{\beta}$ are a certain complex constant and a vector, respectively.

When using conditions [II] and [III], the equations (1.5) and (A.1), together with (1.6), are reduced at once to the CEYM equations

$$2a\hat{G}^{\mu\nu} = T_{(M)}{}^{\mu\nu} + T_{(M)}{}^{\mu\nu}, \tag{1.7}$$

$$\vec{\mathfrak{F}}^{\mu\nu}{}_{;\nu} - i\vec{\mathfrak{A}}_{\nu} \times \vec{\mathfrak{F}}^{\mu\nu} = \mathrm{g}\vec{\mathfrak{S}}_{(M)}{}^{\mu}, \tag{1.8}$$

where

$$\vec{\mathfrak{F}}_{\mu\nu} = \vec{\mathfrak{A}}_{\nu,\mu} - \vec{\mathfrak{A}}_{\mu,\nu} - i\vec{\mathfrak{A}}_{\mu} \times \vec{\mathfrak{A}}_{\nu}. \tag{1.9}$$

Further, when taking account of [I], we can reach the *complex* EM equations:

$$\mathfrak{F}^{\mu\nu}{}_{;\nu} = \mathrm{g}\vec{\beta}^{-1}\cdot\vec{\mathfrak{S}}_{(M)}{}^{\mu} \tag{1.10}$$

$$\mathfrak{F}^{\dagger\mu\nu}{}_{;\nu} = 0 \tag{1.11}$$

where $\vec{\mathfrak{F}}^{\mu\nu} = \vec{\beta}\mathfrak{F}_{\mu\nu}$ with $\mathfrak{F}_{\mu\nu} = \mathfrak{A}_{\nu,\mu} - \mathfrak{A}_{\mu,\nu}$.

Furthermore, if either of the following (1) or (2) is satisfied, then we are led to the *real* or ordinary EM equations:

† Note we are here adopting the condition [III] instead of the condition $\vec{\mathfrak{T}}_{\mu} = 0$ of Ref. [4]. Because the latter is not a sufficient condition for the translational equation (A.1) to reduce to the Einstein equation (see Appendix B for details).

$$[\mathrm{IV}]\begin{cases}(1)\ \mathfrak{A}_\mu = A_\mu\ (\text{real})\ \text{and}\ \mathrm{Re}\left[\dfrac{\vec{\beta}^2}{\mathfrak{g}}\right] = -\dfrac{1}{2},\\ (2)\ \mathfrak{A}_\mu = iB_\mu\ (\text{pure imaginary})\ \text{and}\ \mathrm{Re}\left[\dfrac{\vec{\beta}^2}{\mathfrak{g}}\right] = \dfrac{1}{2}.\end{cases}$$

The first of (1) or (2) could restrict considerably the variety of matters, since they force on $\vec{\mathfrak{S}}_{(M)}{}^\mu$ such conditions as

$$\mathrm{Im}[\mathfrak{g}\vec{\beta}^{-1}\cdot\vec{\mathfrak{S}}_{(M)}{}^\mu] = 0 \text{ or } \mathrm{Re}[\mathfrak{g}\vec{\beta}^{-1}\cdot\vec{\mathfrak{S}}_{(M)}{}^\mu] = 0, \tag{1.12}$$

respectively. We, however, do not have worry about this. In this paper we shall consider the source-free case only. The second is a requirement, owing to which the energy-momentum tensor $T_{(L)}{}^{\mu\nu}$ of the Lorentz gauge field coincides with one of the electromagnetic field.

Thus we have got a method, by which a special series of solutions in PGT can be obtained. The procedure is summarized as follows:

1. *First, find any solution for the real or ordinary source-free Einstein-Maxwell equations and identify it with either* A_μ *or* B_μ,
2. *next, form the combinations* $\vec{\beta}\mathfrak{A}_\mu(=\vec{\mathfrak{A}}_\mu)$ *with any complex constant vectors* $\vec{\beta}$ *belonging to the constraints* [IV]-(1) *or* (2), *which give us the linear solutions (called abelian) of the complex Einstein-Yang-Mills equations,*
3. *then perform any local Lorentz-transformations, which are the local rotations of the tetrads* $b^{k'\mu} = \Lambda^k{}_m b^{m\mu}$, *in order to get a series of nonlinear solutions for the complex Einstein-Yang-Mills equations, and*
4. *finally, check the condition* [II].

In the following sections we treat a concrete example of above procedures. In the next section we derive Abelian solutions with two kinds of gauge charges $\vec{Q}_1$, $\vec{Q}_2$ of the CEYM equations from the well-known Kerr-Newman solution. In section 3 those solutions are classified in terms of the invariants S_1, S_2 made from the gauge charges. In section 4 we consider the nonlinear solutions. There we take up only three typical cases, namely a null case $S_1 = S_2 = 0$ and two cases: (1) $\vec{Q}_1 = 0$ and (2) $\vec{Q}_2 = 0$ in general case. The last section is devoted to the concluding remarks.

2. Abelian solutions

In this section we investigate Abelian solutions to be derived from ones known as Kerr-Newman solutions in EM theory. It is well-known that Kerr-Newman solutions are given in the Boyer-Lindquist coordinates as[6]

$$\begin{aligned} ds^2 &= \rho^{-2}\Delta\,[dt - \ell\sin^2\theta\, d\phi]^2 - \rho^2\Delta^{-1}dr^2 \\ &\quad - \rho^2 d\theta^2 - \rho^{-2}\sin^2\theta[(r^2+\ell^2)d\phi - \ell dt]^2 \end{aligned} \tag{2.1}$$

$$A = Q\cos\alpha\,\rho^{-2}r[dt - \ell\sin^2\theta\, d\phi] + Q\sin\alpha\,\rho^{-2}\cos\theta[(r^2+\ell^2)d\phi - \ell dt] \tag{2.2}$$

where $\Delta = r^2 - 2Mr + \ell^2 + Q^2$, $\rho^2 = r^2 + \ell^2\cos^2\theta$, and M, ℓ, Q and α are the mass, angular momentum per unit mass, electromagnetic charge, and complexion of a source, respectively.

Following the procedure mentioned in the previous section we can now get two types of Abelian solutions, corresponding to each case of [IV]-(1), (2). In fact, by making the replacement of Q by either $\vec{\mathfrak{Q}}_{[1]} = \vec{\beta}Q$ or $\vec{\mathfrak{Q}}_{[2]} = i\vec{\beta}Q$ and noting a relation

$$Q^2 = -2\mathrm{Re}\left[\frac{\vec{\mathfrak{Q}}_{[1,2]}{}^2}{\mathfrak{g}}\right] = -2g_1(\vec{Q}_{1[1,2]}{}^2 - \vec{Q}_{2[1,2]}{}^2) + 4g_2\vec{Q}_{1[1,2]}\cdot\vec{Q}_{2[1,2]}, \tag{2.3}$$

we can obtain explicitly the solutions for each case of (1), (2) as

$$\begin{aligned} ds^2{}_{[1,2]} &= \rho^{-2}\Delta_{[1,2]}[dt - \ell\sin^2\theta\, d\phi]^2 \\ &\quad - \rho^2\Delta_{[1,2]}{}^{-1}dr^2 - \rho^2 d\theta^2 - \rho^{-2}\sin^2\theta[(r^2+\ell^2)d\phi - \ell dt]^2 \end{aligned} \tag{2.4}$$

$$\begin{aligned} \vec{\mathfrak{A}}_{[1,2]} &= \vec{\mathfrak{Q}}_{[1,2]}\cos\alpha\,\rho^{-2}r[dt - \ell\sin^2\theta\, d\phi] \\ &\quad + \vec{\mathfrak{Q}}_{[1,2]}\sin\alpha\,\rho^{-2}\cos\theta[(r^2+\ell^2)d\phi - \ell dt]. \end{aligned} \tag{2.5}$$

Further, noting a relation $\vec{\mathfrak{A}}_{[1,2]} = \vec{v}_{[1,2]} + i\vec{a}_{[1,2]}$ one obtains from the latter

$$\begin{aligned} \vec{v}_{[1,2]} &= \vec{Q}_{1[1,2]}\cos\alpha\,\rho^{-2}r[dt - \ell\sin^2\theta\, d\phi] \\ &\quad + \vec{Q}_{1[1,2]}\sin\alpha\,\rho^{-2}\cos\theta[(r^2+\ell^2)d\phi - \ell dt], \end{aligned} \tag{2.6}$$

$$\begin{aligned} \vec{a}_{[1,2]} &= \vec{Q}_{2[1,2]}\cos\alpha\,\rho^{-2}r[dt - \ell\sin^2\theta\, d\phi] \\ &\quad + \vec{Q}_{2[1,2]}\sin\alpha\,\rho^{-2}\cos\theta[(r^2+\ell^2)d\phi - \ell dt]. \end{aligned} \tag{2.7}$$

Here note that the following substitutions have been made for $k = 1, 2$

$$\vec{\mathfrak{Q}}_{[k]} = \vec{Q}_{1[k]} + i\vec{Q}_{2[k]} \tag{2.8}$$

$$\Delta_{[k]} = r^2 - 2Mr + \ell^2 - 2\mathrm{Re}\left[\frac{\vec{\mathfrak{Q}}_{[k]}{}^2}{\mathfrak{g}}\right]. \tag{2.9}$$

For later discussion it is important to note here the following facts (The indices [1, 2] will be suppressed hereafter.):

- We have now two kinds of gauge charges $\vec{Q}_1$, $\vec{Q}_2$ in each case of [IV]-(1) and (2).
- If $|\vec{Q}| = |\vec{Q}_2|$ and $\vec{Q}_1\cdot\vec{Q}_2 = 0$ for each case of (1), (2), then two fields $\vec{v}$, $\vec{a}$ are equal in magnitude and mutually orthogonal, and then the structure of spacetime is Kerr-like, in spite of the presence of the gauge charges.

3. Gauge transformations

Let us now consider the local proper Lorentz transformations. We regard here them as the local rotations of a vier-bein not associated with any coordinate transformations. Accordingly, the transformations are defined by $\Lambda^k{}_m(x)$ satisfying the relations

$$\begin{aligned} b^{k'\mu} &= \Lambda^k{}_m b^{m\mu} \\ \eta_{km}\Lambda^k{}_n\Lambda^m{}_l &= \eta_{nl}, \end{aligned} \tag{3.1}$$

where the latter can be derived from the invariance of the metric $g_{\mu\nu} = b_{k\mu}b^k{}_\nu$.

Under these transformations the gauge charges $\vec{Q}_1$, and $\vec{Q}_2$ ought to transform as

$$\begin{cases} \vec{Q}'_1 = \dfrac{1}{2}\varepsilon_{abc}(\vec{\Lambda}_b \times \vec{\Lambda}_c)Q_{1(a)} - \vec{\Lambda}_a(\vec{U} \times \vec{Q}_2)_a, \\ \vec{Q}'_2 = \dfrac{1}{2}\varepsilon_{abc}(\vec{\Lambda}_b \times \vec{\Lambda}_c)Q_{2(a)} + \vec{\Lambda}_a(\vec{U} \times \vec{Q}_1)_a, \end{cases} \tag{3.2}$$

because of relation (2.4). However, the fields $\vec{v}$, $\vec{a}$ are changed by the same transformations to

$$\begin{aligned} \vec{v}' &= \frac{1}{2}\varepsilon_{abc}(\vec{\Lambda}_b \times \vec{\Lambda}_c)v_{(a)} - \vec{\Lambda}_a(\vec{U} \times \vec{a})_a + (\Lambda_{00}\vec{V}_{,\mu} - U_a\vec{\Lambda}_{a,\mu})dx^\mu, \\ \vec{a}' &= \frac{1}{2}\varepsilon_{abc}(\vec{\Lambda}_b \times \vec{\Lambda}_c)a_{(a)} + \vec{\Lambda}_a(\vec{U} \times \vec{v})_a + \frac{1}{2}(\vec{V} \times \vec{V}_{,\mu} - \vec{\Lambda}_a \times \vec{\Lambda}_{a,\mu})dx^\mu, \end{aligned} \tag{3.3}$$

since the Lorentz gauge field $A_{km\mu}$ must transform as

$$A'_{km\mu} = \Lambda_k{}^l\Lambda_m{}^n A_{ln\mu} + \Lambda_{kl}\Lambda_m{}^l{}_{,\mu}.$$

Here note the following notations have been used:

$$(\Lambda^k{}_m) = \left(\begin{array}{c|c} \Lambda_{00} & \Lambda_{0b} \\ \hline -\Lambda_{a0} & -\Lambda_{ab} \end{array}\right) = \left(\begin{array}{c|c} \Lambda_{00} & (\vec{U})_b \\ \hline -(\vec{V})_a & -(\vec{\Lambda}_b)_a \end{array}\right), \tag{3.4}$$

where the quantities $\vec{U}$, $\vec{V}$ and $\vec{\Lambda}_a$ belong to the relations

$$\begin{cases} \vec{\Lambda}_a\cdot\vec{\Lambda}_b = U_aU_b + \delta_{ab} \\ \vec{\Lambda}_a \times \vec{\Lambda}_b = \varepsilon_{abc}(U_c\vec{V} - \Lambda_{00}\vec{\Lambda}_c) \\ \vec{V} = \dfrac{U_a\vec{\Lambda}_a}{\Lambda_{00}},\ \vec{U}^2 = \Lambda_{00}{}^2 - 1 \end{cases} \tag{3.5}$$

Above the difference on the transformations of charges and fields shows us that the Lorentz gauge field $\vec{\mathfrak{A}}$ can not maintain its original form $\vec{\beta}\mathfrak{A}$ after the transformations. This is a reason why we can obtain the nonlinear solutions for the complex Einstein-Yang-Mills equations by any local transformations, starting from any solutions for the Einstein-Maxwell equations. It should be also remarked that the metric dose not change under any above transformations.

Before going forward, we would like further to remark here a few interesting facts:

(a) $\vec{Q}_1^2 - \vec{Q}_2^2$ ($\equiv S_1$) and $\vec{Q}_1\cdot\vec{Q}_2$ ($\equiv S_2$) are invariant-quantities of the transformations.

(b) We can use the scalar S_1 and S_2 to classify the fields in terms of the gauge charges. To do so, it is enough for us to consider the following two cases only:

i. the "null" case: a case where both scalars S_1 and S_2 vanish. In this case the gravitational field is equal to a field which is generated by a body without any charges, in spite of the presence of the fields $\vec{v}$ and $\vec{a}$ being created by the charges.

ii. the general case: a case where S_1 and /or S_2 dose not vanish. In this case we can always choose the charges such as $p\vec{Q}_1 = q\vec{Q}_2$ at any point in spacetime, where p and q are certain scalars. We shall call this "charge wrench", corresponding to the electromagnetic one[7]. We have two interesting cases for the wrench. Those are such cases that either of the charges Q_1 or Q_2 vanishes and therefore $S_2 = 0$. The fields are then classified in accordance with $S_1 > 0$ or $S_2 < 0$.

In the next section we shall investigate a few solutions obtained by certain special transformations in both null and general cases.

4. Nonlinear solutions

In this section we investigate nonlinear solutions generated by a spatial rotation and a boost transformation.

4.1. Solutions generated by a spatial rotation

We know from (3.3) that the fields $\vec{a}$, $\vec{v}$ are generally transformed by any spatial rotations as

$$\begin{cases} \vec{v}' = -\vec{\Lambda}_a v_{(a)}, \\ \vec{a}' = -\vec{\Lambda}_a a_{(a)} - \dfrac{1}{2}\vec{\Lambda}_a \times \vec{\Lambda}_{a,\mu}\, dx^\mu \end{cases} \tag{4.1}$$

In this paper we, however, consider exclusively special one of them, i.e., $\vec{\Lambda}_a = (\vec{e}_\phi, \vec{e}_\theta, \vec{e}_r)$, where $\vec{e}_r$, $\vec{e}_\theta$, $\vec{e}_\phi$ are unit vectors along with the directions of r, θ and ϕ. This rotation is performed concretely according to the following procedure:

- We first set up the 3rd axes of the local frames along with the direction of one of gauge charges $\vec{Q}_1$, $\vec{Q}_2$. (This is always achieved by a global rotation, because of the constancy of the charges.)
- We then rotate locally frames so that their 3rd axes coincide with the radial direction at any points in the spacetime.

1. null case

In this case $|\vec{Q}_1| = |\vec{Q}_2|$ and $\vec{Q}_1 \cdot \vec{Q}_2 = 0$. Therefore, we can put generally as $\vec{Q}_1 = (Q_{11}, Q_{12}, 0)$ and $\vec{Q}_2 = (0, 0, Q_2)$, where Q_{11}, Q_{12} and Q_2 are any constants belonging to a relation $Q_{11}^2 + Q_{12}^2 = Q_2^2$.

A little calculation leads us to the following results:

$$\begin{aligned} ds^2 &= \rho^{-2}\Delta\,[dt - \ell \sin^2\theta\, d\phi]^2 - \rho^2\Delta^{-1}dr^2 - \rho^2 d\theta^2 \\ &\quad - \rho^{-2}\sin^2\theta[(r^2 + \ell^2)d\phi - \ell dt]^2, \end{aligned} \tag{4.2}$$

with chargeless $\Delta = r^2 - 2Mr + \ell^2$, and

$$\begin{aligned} \vec{v}' &= -(\vec{v}\cdot\vec{e}_1)\vec{e}_\phi - (\vec{v}\cdot\vec{e}_2)\vec{e}_\theta, \\ \vec{a}' &= -(\vec{a}\cdot\vec{e}_3)\vec{e}_r - \vec{e}_\phi d\theta - \vec{e}_3 d\phi, \end{aligned} \tag{4.3}$$

where we put $\vec{e}_1 = (1, 0, 0)$, $\vec{e}_2 = (0, 1, 0)$ and $\vec{e}_3 = (0, 0, 1)$. As was expected, $\vec{v}' \cdot \vec{a}' = (\vec{v}\cdot\vec{e}_1)d\theta - \sin\theta(\vec{v}\cdot\vec{e}_2)d\phi \neq 0$.

2. general case
   (a) the case $\vec{Q}_1 = 0$ and $\vec{Q}_2 = (0, 0, Q_2)$.
   In this case the metric is equal to (4.2) with $\Delta = r^2 - 2Mr + \ell^2 + 2g_1 Q_2^2$, and

$$\vec{v}' = 0,$$
$$\vec{a}' = -(\vec{a}\cdot\vec{e}_3)\vec{e}_r - \vec{e}_\phi d\theta - \vec{e}_3 d\phi, \tag{4.4}$$

   (b) the case $\vec{Q}_2 = 0$ and $\vec{Q}_1 = (0, 0, Q_1)$.
   The metric is equal to (4.2) with $\Delta = r^2 - 2Mr + \ell^2 - 2g_1 Q_1^2$, and

$$\vec{v}' = -(\vec{v}\cdot\vec{e}_3)\vec{e}_r,$$
$$\vec{a}' = -\vec{e}_\phi d\theta - \vec{e}_3 d\phi. \tag{4.5}$$

4.2 Solutions generated by a boost transformation

We here consider only a boost transformation in the $\vec{e}_3$ direction, which is given by

$$(\Lambda^k{}_m) = \begin{pmatrix} \cosh\eta & 0 & 0 & -\sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix} \tag{4.6}$$

Here η is a parameter of the transformation which can depend generally on the coordinates (t, r, θ, ϕ).

After a simple calculation we shall get the following results:

1. null case
   As we did so before, we put again $\vec{Q}_1 = (Q_{11}, Q_{12}, 0)$ and $\vec{Q}_2 = (0, 0, Q_2)$ and get

$$\vec{v}' = \begin{cases} v'_{(1)} = \cosh\eta\; v_{(1)}, \\ v'_{(2)} = \cosh\eta\; v_{(2)}, \\ v'_{(3)} = \eta_{,\mu}\, dx^\mu, \end{cases} \qquad \vec{a}' = \begin{cases} a'_{(1)} = -\sinh\eta\; v_{(2)}, \\ a'_{(1)} = \sinh\eta\; v_{(2)}, \\ a'_{(3)} = a_{(3)}. \end{cases} \tag{4.7}$$

2. general case
   (a) the case $\vec{Q}_1 = 0$ and $\vec{Q}_2 = (0, 0, Q_2)$.

$$\vec{v}' = \begin{cases} v'_{(1)} = 0, \\ v'_{(2)} = 0, \\ v'_{(3)} = \eta_{,\mu}\, dx^\mu, \end{cases} \qquad \vec{a}' = \begin{cases} a'_{(1)} = 0, \\ a'_{(2)} = 0, \\ a'_{(3)} = a_{(3)}. \end{cases} \tag{4.8}$$

   (b) the case $\vec{Q}_2 = 0$ and $\vec{Q}_1 = (0, 0, Q_1)$.

$$\vec{v}' = \begin{cases} v'_{(1)} = 0, \\ v'_{(2)} = 0, \\ v'_{(3)} = v_{(3)} + \eta_{,\mu}\, dx^\mu, \end{cases} \qquad \vec{a}' = \begin{cases} a'_{(1)} = 0, \\ a'_{(2)} = 0, \\ a'_{(3)} = 0. \end{cases} \tag{4.9}$$

Finally, we must check the condition [II]. However, we already know that all above solutions can always exist in a model of PGT with the parameters $C_1 = C_2 = C_3 = 0$ and (B.4). But it is also easily conceivable that some rare solutions of them could be in other models. In this paper, however, we do not consider this possibility, though it will be discussed in other chance.

5. Concluding remarks

In this paper we have presented some special solutions of Poincaré gauge theory being compatible with Einstein equation. These solutions have been created from the Kerr-Newman solution of Einstein-Maxwell equations by means of the local Lorentz transformations. It is a feature that they have two kinds of gauge charges, in terms of which they can be classified. We find also an interesting fact. It is a fact that our solutions are divided into three categories, according as the properties of the associated-space-times. They are as follows:

- The space-time is Kerr-like, i.e., it is equal to one generated by a body with no charges, in spite of the presence of the Lorentz gauge field (which is created by the charges).
- Gravity has a post-Newtonian limit such that the redundant force is repulsive.
- Gravity has a post-Newtonian limit such that the redundant force is attractive.

Appendix

A. The complex Einstein-Yang-Mills equations

We are adopting the Lagrangian proposed by Hayashi[3], from which we can get the following two equations and Bianchi identity in PGT:‡

$$2a\widehat{G}^{km} = T_{(M)}{}^{km} + T_{(L)}{}^{km} + T_{(C)}{}^{km}, \tag{A.1}$$

$$\nabla_p H^{kmnp} + K^k{}_{rp} H^{rmnp} + K^m{}_{rp} H^{krnp} + I^{[km]n} = S_{(M)}{}^{kmn}, \tag{A.2}$$

$$\nabla_p F^{\dagger kmnp} + K^k{}_{rp} F^{\dagger rmnp} + K^m{}_{rp} F^{\dagger krnp} = 0 \tag{A.3}$$

Let us now show how we can obtain the equations (1.5) and (1.6) from the equations (A.2) and (A.3). To do so, we must first note that we assume $\nabla_\mu b_k{}^\nu = 0$. Then noting $\Delta^k{}_{r\nu} = K^k{}_{r\nu} - A^k{}_{r\nu}$, we can get at once the following equations from (A.2) and (A.3):

$$H^{km\mu\nu}{}_{;\nu} + A^k{}_{r\nu} H^{rm\mu\nu} + A^m{}_{r\nu} H^{kr\mu\nu} + I^{[km]\mu} = S_{(M)}{}^{km\mu}, \tag{A.4}$$

$$F^{\dagger km\mu\nu}{}_{;\nu} + A^k{}_{r\nu} F^{\dagger rm\mu\nu} + A^m{}_{r\nu} F^{\dagger kr\mu\nu} = 0. \tag{A.5}$$

Here, in place of ∇_μ we have used an operator $(;\mu)$. This operator does not operate on the *internal* indices k, m, $\cdots$ and just in this point $(;\mu)$ differs from ∇_μ. After all, $(;\mu)$ is essentially equal to General-Relativistic covariant operator with the Christoffel connection.

In order to rewrite above equations for our purpose, we must introduce the complex quantities $\vec{\mathfrak{H}}^{\mu\nu}$, $\vec{\mathfrak{F}}^{\dagger\mu\nu}$, $\vec{\mathfrak{I}}^{\mu}$ and $\vec{\mathfrak{S}}_{(M)}{}^{\mu}$ corresponding to $H^{km\mu\nu}$, $F^{\dagger km\mu\nu}$, $I^{[km]\mu}$ and $S^{km\mu}$, respectively. These quantities are made from the corresponding ones in the same way

‡ We are here adopting the notation of Appendix B of Ref. [5].

as $\vec{\mathfrak{F}}^{\mu\nu}$ is done from $F^{km\mu\nu}$. Using these quantities, above equations can be written as

$$\vec{\mathfrak{H}}^{\mu\nu}{}_{;\nu} - i\vec{\mathfrak{A}}_{\nu} \times \vec{\mathfrak{H}}^{\mu\nu} + \vec{\mathfrak{J}}^{\mu} = \vec{\mathfrak{S}}_{(M)}{}^{\mu}, \tag{A.6}$$
$$\vec{\mathfrak{F}}^{\dagger\mu\nu}{}_{;\nu} - i\vec{\mathfrak{A}}_{\nu} \times \vec{\mathfrak{F}}^{\dagger\mu\nu} = 0. \tag{A.7}$$

These equations imply that we may adopt the following equation in place of the first of above pair:

$$\vec{\mathfrak{T}}^{\mu\nu}{}_{;\nu} - i\vec{\mathfrak{A}}_{\nu} \times \vec{\mathfrak{T}}^{\mu\nu} + \vec{\mathfrak{J}}^{\mu} = \vec{\mathfrak{S}}_{(M)}{}^{\mu}, \tag{A.8}$$

where

$$\vec{\mathfrak{T}}^{\mu\nu} = \vec{\mathfrak{H}}^{\mu\nu} - (h_1 + ih_2)\vec{\mathfrak{F}}^{\dagger\mu\nu} \tag{A.9}$$

with h_1 and h_2 being real any constants.

If we now assume a condition with a complex constant $\mathfrak{g}$

$$\vec{\mathfrak{T}}^{\mu\nu} = \frac{1}{\mathfrak{g}}\vec{\mathfrak{F}}^{\mu\nu}, \tag{A.10}$$

which is just the condition [II], and one more condition

$$\vec{\mathfrak{J}}^{\mu} = 0, \tag{A.11}$$

then we can get from (A.17) and (A.7) the complex Yang-Mills equations

$$\vec{\mathfrak{F}}^{\mu\nu}{}_{;\nu} - i\vec{\mathfrak{A}}_{\nu} \times \vec{\mathfrak{F}}^{\mu\nu} = \mathfrak{g}\vec{\mathfrak{S}}_{(M)}{}^{\mu}, \tag{A.12}$$
$$\vec{\mathfrak{F}}^{\dagger\mu\nu}{}_{;\nu} - i\vec{\mathfrak{A}}_{\nu} \times \vec{\mathfrak{F}}^{\dagger\mu\nu} = 0 \tag{A.13}$$

with

$$\vec{\mathfrak{F}}^{\mu\nu} = \vec{\mathfrak{A}}_{\nu,\mu} - \vec{\mathfrak{A}}_{\mu,\nu} - i\vec{\mathfrak{A}}_{\mu} \times \vec{\mathfrak{A}}_{\nu}. \tag{A.14}$$

However, it should be remarked that above equations can not yet be thought of as complex Einstein-Yang-Mills equations. Because the condition (A.20) is equivalent to $I^{[km]n} = 0$ rather than $I^{kmn} = 0$, so that the equation (A.1) can not be identified with the Einstein equation. Thus we here, instead of (A.20), adopt more stringent conditions

$$C_1 = C_2 = C_3 = 0, \tag{A.15}$$

which result from (A.20) when it must be satisfied by any fields. In this case we have $I^{kmn} = 0$ and therefore $T_{(C)}{}^{km} = 0$. In addition, we can see from (A.4) that the antisymmetric part of $T_{(L)}{}^{km}$ vanishes automatically when the condition [II] is fullfilled. Thus we can reach the complex Einstein-Yang-Mills equations, whenever the condition [II] and [III] are satisfied.

B. The supplementary conditions

B.1. The condition [II]

We first note that our condition [II] can be written in the ordinary form as

$$H_{kmnp} = g_1 F_{kmnp} + g_2 {}^{\dagger}F_{kmnp} + h_1 F^{\dagger}{}_{kmnp} + h_2 {}^{\dagger}F^{\dagger}{}_{kmnp} \tag{B.1}$$

with

$$\begin{cases} {}^{\dagger}F_{kmnp} = \frac{1}{2}\varepsilon_{km}{}^{qr}F_{qrnp}, \\ F^{\dagger}{}_{kmnp} = \frac{1}{2}F_{km}{}^{qr}\varepsilon_{qrnp}, \\ {}^{\dagger}F^{\dagger}{}_{kmnp} = \frac{1}{2}\varepsilon_{km}{}^{qr}F^{\dagger}{}_{qrnp}, \\ \frac{1}{g} = g_1 + ig_2 \ (g_1,\ g_2 = \text{real any constants}). \end{cases}$$

It would be useful for a later calculation to write these conditions in terms of the irreducible spinors Ψ_{ABCD}, Φ_{AB}, $X_{\dot{A}\dot{B}CD}$ and Λ, by which the corresponding spinor $F_{\dot{A}B\dot{C}D\dot{E}F\dot{G}H}$ of F_{kmnp} is defined as[8]

$$\begin{aligned} F_{\dot{A}B\dot{C}D\dot{E}F\dot{G}H} = & \{\Psi_{BDFH} + \Phi_{BH}\varepsilon_{DF} + \Phi_{DF}\varepsilon_{BH} + \Lambda(\varepsilon_{BF}\varepsilon_{DH} + \varepsilon_{BH}\varepsilon_{DF})\}\varepsilon_{\dot{A}\dot{C}}\varepsilon_{\dot{E}\dot{G}} \\ & + \{\Psi_{\dot{A}\dot{C}\dot{E}\dot{G}} + \Phi_{\dot{A}\dot{G}}\varepsilon_{\dot{C}\dot{E}} + \Phi_{\dot{C}\dot{E}}\varepsilon_{\dot{A}\dot{G}} + \Lambda^{*}(\varepsilon_{\dot{A}\dot{E}}\varepsilon_{\dot{C}\dot{G}} + \varepsilon_{\dot{A}\dot{G}}\varepsilon_{\dot{C}\dot{E}})\}\varepsilon_{BD}\varepsilon_{FH} \\ & + X_{\dot{A}\dot{C}FH}\varepsilon_{BD}\varepsilon_{\dot{E}\dot{G}} + X_{BD\dot{E}\dot{G}}\varepsilon_{\dot{A}\dot{C}}\varepsilon_{FH}. \end{aligned} \tag{B.2}$$

After lengthy calculation we shall find the followings:

$$\begin{cases} \{3a_2 - g_1 + h_2 + i(g_2 + h_1)\}\Psi_{ABCD} = 0, \\ \{2a_4 - g_1 + h_2 + i(g_2 + h_1)\}\Phi_{AB} = 0, \\ \{2(a_1 + 6a_6) - g_1 + h_2 + i(g_2 + h_1)\}\Lambda - 2(a_1 - 6a_6)\bar{\Lambda} = 0, \\ \{2a_3 + a_5 - g_1 - h_2 - i(g_2 - h_1)\}X_{\dot{A}\dot{B}CD} - (2a_3 - a_5)X_{CD\dot{A}\dot{B}} = 0 \end{cases} \tag{B.3}$$

In particular, when above conditions must be satisfied by any fields, we are led to the relations

$$\begin{cases} g_2 = 0, \quad h_1 = 0 \\ g_1 = 2(a_1 + a_3), \quad h_2 = 2a_3 - a_4, \\ 4a_1 = 3a_2 = 2a_4 = 24a_6 = 0, \\ 2a_3 = a_5. \end{cases} \tag{B.4}$$

Additionally, it may be fruitful to compare our conditions with the well-known double duality ansatz of E.W. Mielke et al. Following the strategy of Belavin et al.[9] in the search for instanton solutions of Yang-Mills equations, E.W. Mielke and his collaborators (called hereafter Germany group) have introduced their double ansatz and its modified versions (which can be reviewed in Refs. [11,12]) as a means of reducing the differential order.

The generalized double duality ansatz can be written in our notations for the analogous models to ours, i.e., quasi-linear PG models without the constant term ($\sim\Lambda$), as follows:

$$
\begin{cases}
(a_2 - \ \ 8\hat{\xi})\Psi_{ABCD} = 0, \\
(a_4 - 12\hat{\xi})\Phi_{AB} = 0, \\
(a_3 + \ \ 6\hat{\xi})(X_{\dot{A}\dot{B}CD} - X_{CD\dot{A}\dot{B}}) = 0, \\
(a_5 + 12\hat{\xi})(X_{\dot{A}\dot{B}CD} + X_{CD\dot{A}\dot{B}}) = 0, \\
(a_1 - \ \ 6\hat{\xi})(\Lambda - \bar{\Lambda}) = 0, \\
48\ell_0^2(a_6 - \ \hat{\xi})(\Lambda + \bar{\Lambda}) = 2\gamma - \dfrac{1}{\chi}
\end{cases}
\tag{B.5}
$$

Here $\hat{\xi} = \xi/24\varkappa$, and ℓ_0, $\varkappa$, ξ, γ and χ are the notations of Germany group. Our parameters α, β, γ and a, a_i $(i = 1\sim 6)$ correspond to their parameters a_i, b_j $(i = 1\sim 3,\ j = 1\sim 6)$ in the following manner:

$$
\begin{cases}
\alpha \Longleftrightarrow \dfrac{a_1}{3\ell_0^2}, & \beta \Longleftrightarrow \dfrac{a_2}{6\ell_0^2}, & \gamma \Longleftrightarrow -\dfrac{3a_3}{2\ell_0^2}, \\
a_1 \Longleftrightarrow -\dfrac{b_3}{4\varkappa}, & a_2 \Longleftrightarrow -\dfrac{b_1}{3\varkappa}, & a_3 \Longleftrightarrow -\dfrac{b_2}{4\varkappa}, \\
a_4 \Longleftrightarrow -\dfrac{b_5}{2\varkappa}, & a_5 \Longleftrightarrow -\dfrac{b_4}{2\varkappa}, & a_6 \Longleftrightarrow -\dfrac{b_6}{24\varkappa}, \\
a \Longleftrightarrow -\dfrac{1}{2\chi\ell_0^2}.
\end{cases}
\tag{B.6}
$$

B.2. The condition [III]

First of all, we consider a more general condition $\overrightarrow{\mathfrak{J}}_\mu = -(M^2/G)\overrightarrow{\mathfrak{K}}_\mu$ which has been adopted in Ref. [4]. This condition is equal to

$$
I_{[km]n} = -\frac{M^2}{G}K_{kmn}, \tag{B.7}
$$

where M and G are arbitrary real constants. Using the relations

$$
I_{[km]n} = -2\{C_1{}^T\mathfrak{C}_{n[km]} + C_2\eta_{n[k}{}^V\mathfrak{C}_{m]} + \frac{1}{3}C_3\varepsilon_{kmnp}{}^A\mathfrak{C}^p\} \tag{B.8}
$$

and

$$
K_{kmn} = -\frac{4}{3}{}^T\mathfrak{C}_{n[km]} - \frac{2}{3}\eta_{n[k}{}^V\mathfrak{C}_{m]} + \frac{1}{2}\varepsilon_{kmnp}{}^A\mathfrak{C}^p, \tag{B.9}
$$

we can see easily that above conditions are equivalent to the followings:

$$
(3C_1 + \frac{2M^2}{G})^T\mathfrak{C}_{k[mn]} = 0, \tag{B.10}
$$

$$
(3C_2 + \frac{M^2}{G})^V\mathfrak{C}_k = 0, \tag{B.11}
$$

$$
(4C_3 - \frac{3M^2}{G})^A\mathfrak{C}_k = 0. \tag{B.12}
$$

Here we note that these conditions can be satisfied by any fields if

$$9C_1 = 18C_2 = -8C_3 = -\frac{6M^2}{G}. \tag{B.13}$$

Our condition [III] is just a special case of them, i.e., a case $M = 0$. In this case $T_{(c)}{}^{km}$ vanishes automatically because of $I^{kmn} = 0$. Accordingly, the equation (A.1) can be identified with the Einstein equation having as the sources the symmetric energy-momentum tensors of a matter and the Lorentz gauge field[12].

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