Charge Simulation Solutions for Eigenvalue Problems

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Abstract

The charge simulation and the boundary matrix methods for solving eigenvalue problems for the Laplace operator are formulated in this paper. The numerical methods are based on a nonlinear representation of the eigenvalue problem on the boundary. The nonlinear eigenvalue problems are solved by using the Newton interation method. Numerical examples for simple models by the present methods are shown. From the numerical solutions the present methods give us accurate numerical eigen modes even for high eigenfrequencies.
Key words: Helmholtz equation, Eigenvalue problem, Charge simulation method.

1 Introduction

Numerical methods for solving the eigenvalue problem in the form of

$$-\triangle u(x) = \lambda^2 u(x) \text{in} \Omega \tag{1}$$

where Ω denotes a bounded region in $R^k(\kappa = 1, 2, 3)$, with the boundary condition :

$$u(x) = g(x) \text{on} \partial\Omega \tag{2}$$

have been studied by using the finite difference method, the finite element method and the boundary element method, where $\partial\Omega$ denotes the boundary of the domain Ω. If we seek approximate solutions of high eigenfrequencies with the finite difference and finite element methods it is necessary to take fine mesh and element discretizatio. If the sizes of the finite difference mesh and the finite element are not enough small to approximate the eigenfunction of the problem, the ghost solution (the inaccurate solution) is occurred. In order to avoid the difficulty the Petrov Galerkin finite element method was presented by Sawami et al., Ikeuchi et al. and Niki et al.([1] [2] [3]). Applying those methods to the problem we have linear algebraic eigenvalue problems. The boundary element approach is different from those methods since it is necessary to use the fundamental solution of a differential operator to formulate the boundary integral equation. If we take the fundamental solution for the Laplace operator we obtain an integral equation formulation. In this case we also obtain the linear algebraic eigenvalue problem (see Kitahra [5]), The searching method with boundary

integral equation method with the fundamental solution for the Helmholz operator was presennted by Niwa et al. [4] . For plate problems with the boundary integral equation method we refer to the text book by Kitahara [5] . The searching method gives numerical eigen frequencies which satisfy the determinat free condition of the matrix generated by the boundary element discretization. Since the variation of the determinant near eigenvalues is steep and sensitive it is difficult to determin eigenvalues accurately. On the other hand the boundary element approach has two advantages:

1. It is unnecessary to discretize the interior of the given demain.
2. When we use fundamental solutions of the Helmholtz operators, we can avoid ghost modes since numerical eigenfunctions are expressed with the fundamental solution with the eigenparameter.

Because of those advantages, the author presented a numerical method by using the fundamental solution of Helmholtz operator [6] . For one-dimensional problems the boundary matrix method was also formulated by author [7] . For two-dimensional problems the charge simulation method is examined in this paper for a problem defined on the rectanglar domain. Taking acconut of the normalizing condition of weight coefficients for approximation, nonlinear algebraic eigenvalue problems on the given boundary are induced by applying the present methods. By using the Newton method we obtain numerical solutions. From numerical experiments we show that the present methods give accurate numerical solutions with small unknowns. Moreover it is shown that the boundary matrix method has approximately uniform accuracy with respect to the frequency of vibration.

2 Charge Simulation Method

In this section we consider an extension of the boundary martrix method to two-dimensional eigenvalue problems with the charge simulation method. The charge simulation method is well known as an efficient numerical solution for static electric problems. For the case fundamental solution for the Laplace operator is used to derive a discrete system. The approximate solution of a given problems expressed as a linear combination of fundamental solutions whose soure points are distributed around the given bounded domain. The fundamental solutions for the Helmholtz operator:

$$-\triangle E(r)-\lambda^2 E(r) = \delta(r) \tag{3}$$

where $\triangle$ denotes the Laplacian in two-dimension, is

$$\frac{i}{4} H_0^{(1)}(i\lambda r_2), \tag{4}$$

respectively, where $r^2 = \sum_{i=1}^{2}(x_i-y_j)^2$ in which y and x are the source point and the observation point, respectively, and $\delta(r)$ is the Dirac's delta function. A linear combination of functions $E(r(x, y_i))$

$$U(x) = \sum_{i=1}^{n} c_i E(r(x, y_i)) \tag{5}$$

also satisfies the Helmholtz equation (1). If coefficiens c_i are determined to approximate the boundary condition, we obtain an approximate solution of the problem (1)-(2) when we know the exact eigenvalue for the given demain Ω. A method to decide the coefficients c_i is well known the charge simulation (CS) method. As is shown in Fig. 1 we set n source points around the outside of the given domain Ω and the n observation points on the boundary $\partial\Omega$. From the boundary condition and those collocation points we obtain the following equation:

$$A[\lambda]C = G \tag{6}$$

where each element of the matrix $A[\lambda]$ and the vector G are given as follows:

$$a_{ij}[\lambda] = E(r(x_j, y_i)), \quad C = \{c_1,..., c_n\}^t \quad G = \{g(x_1),...g(x_n)\}^t \tag{7}$$

Let us condider normalization in the form of

$$\|C\|^2 = \sum_{i=1}^{n} c_i^2 = 1. \tag{8}$$

For the equation system (12) and (14) we obtain the following iteration procedure to seek an approximate eigenvalue and eigenvector:

$$\begin{Bmatrix} C^{(m+1)} \\ \lambda_{(m+1)} \end{Bmatrix} = \begin{Bmatrix} C^{(m)} \\ \lambda_{(m)} \end{Bmatrix} - F(A[\lambda^{(m)}], \|C^{(m)}\|^2)^{-1} f(A[\lambda^{(m)}], \|C^{(m)}\|^2), \tag{9}$$

where

$$f(A[\lambda^{(m)}], \|C^{(m)}\|^2) = \begin{Bmatrix} A[\lambda^{(m)}]C^{(m)} - G \\ \|C^{(m)}\|^2 - 1 \end{Bmatrix} \tag{10}$$

and

$$F(A[\lambda^{(m)}], \|C^{(m)}\|^2) = \begin{bmatrix} A[\lambda^{(m)}] & \frac{\partial}{\partial\lambda} A[\lambda^{(m)}]C^{(m)} \\ \{2C_i^{(m)}\} & 0 \end{bmatrix}. \tag{11}$$

Throughout the process we get an approximate solution for (1)-(2).

3 Numerical Experiments

Numerical results for a two-dimensional problem is shown in this section. Exact solutions are all real values. Numerical solutions have quite small imaginary part as calculating errors. Let us consider the two-dimensional eigenvalue problem which is defined on the domain illustrated in the figure 1, for the case that L = 1. We set the unite rectagular of which the center is the origin of the coordinate of the plane.

The region is illustrated with the solid line. The collocation points (observation points), which are denoted by small circles on the solid line, are set on the sides of the rectangular regularly. In order to put the source points we larger rectangular, which

Table 1 Numerical solutions and Relative error for the two-dimensional problem.

d	$sol.\ (\lambda)$	$\lvert Re[sol.]-\sqrt{2\pi}\rvert/\sqrt{2\pi}\rvert\times 100$
1.1	$4.32272+0.0134257_s$	2.7046%
1.2	$4.41153+0.00884192_t$	0.7056%
1.3	$4.43471+0.00356681_t$	0.1839%
1.4	$4.44104+0.00113445_t$	0.0414%
1.5	$4.44264+0.00021394_t$	0.0054%

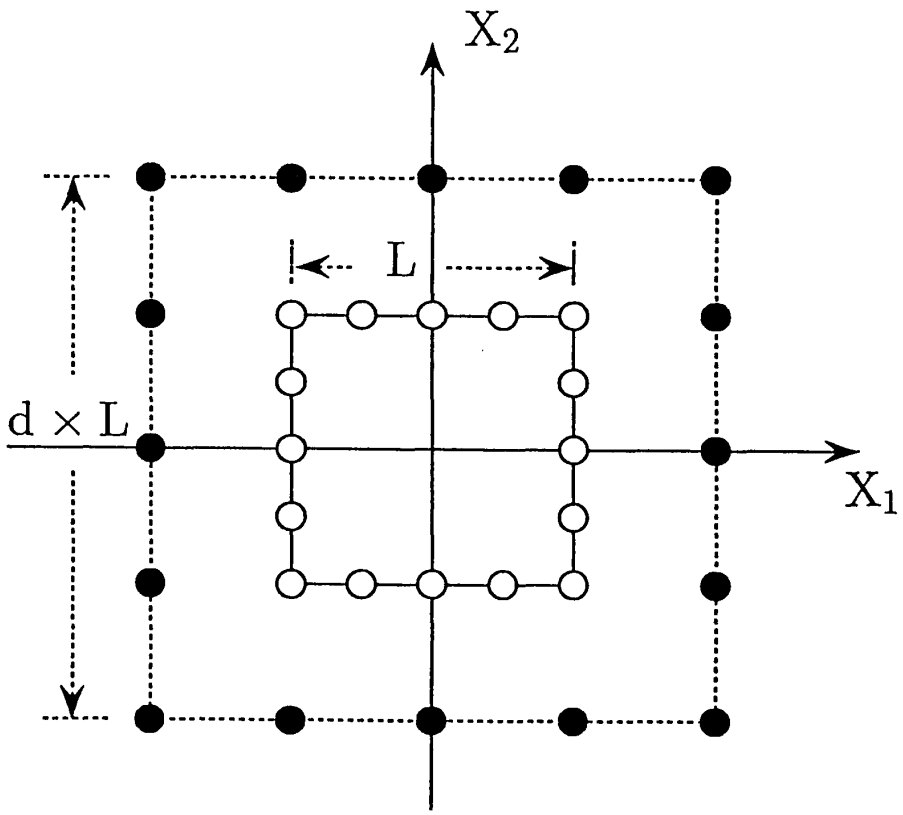

Fig. 1 Illustration of model domain

is illustrated with the broken line. The source points are distributed on the sides regularly. Here d denotes the ratio of the length of the side for the small rectangular and the large rectangular, Numerical solutions in table 1 is for the problem with the boundary condition $g(x) = 0.16$ collocation and source points are taken for the caclulation. The Newton iteration are carried out with initial values:

$$\lambda^0 = 3.0 \qquad c_i^0 = 1\ (i = 1,\ldots, 16) \tag{12}$$

5 Conclusion

The charge simulation method for solving eigenvalue problems is formulated and examined in this paper. In the present formulation the eigenparamter is involved

nonliearly in the discretized system. Therefore the Newton method is available to calculate approximate solutions of the system. Since we apply the fundamental solution of the Helmholz operator the discretization is carried only on the boundary for the given domain. From numerical experiments we observe the following consequences :

1. The present methods are accurate numerical methods for solving the eigenvalue probelem.
2. Accuracy of the present methods is independence of the frequency of vibration.
3. For the case of charge simulation method the location of source points affects accuracy of numerical solutions.
4. The charge simulation solutions for the two-dimensional problem depend on the location of souce points.

It is realized that the present charge simulation methods gives quite accurate numerical solution even for we take only 16 unknowns to constract approximate eigenfunction. The second result is very important property of the present method since the result implies that we can avoid the ghost solution which is appeared in the finite difference method and the finite element method.

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