On Exclusive Extensions of Noetherian Domains

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Let R be a Noetherian integral domain and $R[X]$ a polynomial ring. Let γ be an element of a field extension L of the quotient field K of R. If $\gamma \in L$ is transcendental over R, then $R[\gamma] \cap K = R$, and if $\gamma \in K$ then $R[\gamma] \cap K = R[\gamma]$. So we consider an element algebraic over R.

We study the following problem:

Problem Let R be a Noetherian integral domain, let K denote the quotient field of R and let α be an algebraic element over R of degree d. When does $R[\alpha] \cap K = R$?

In what follows, we use the following notations unless otherwise specified.

R : a Noetherian integral domain,
$K := K(R)$: the quotient field of R,
$L =$ an algebraic field extension of K,
α : a non-zero element of L,
$d = [K(\alpha) : K]$,
$\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$, the minimal polynomial of α over K.
$I_{[\alpha]} := \bigcap_{i=1}^{d} (R :_R \eta_i)$, which is an ideal of R.
$I_a := R :_R aR$ for $a \in K$.
It is clear that for $a \in K$, $I_{[a]} = I_a$ by definition.

We also use the following standard notation:
$Dp_1(R) := \{p \in Spec(R) | depth\ R_p = 1\}$.
Throughout this paper, all fields, rings and algebras are assumed to be commutative with unity. Our special notations are indicated above, and our general reference for unexplained technical terms is [M].

We start with the following definition.

Definition 1. When $R[\alpha]\cap K = R$, we say that α is an *exclusive* element over R and that $R[\alpha]$ is an *exclusive* extension of R.

Proposition 1. *If $R[\alpha]\cap K = R$, then $\cap_{i=1}^{d-1} I_{\eta_i} \subseteq I_{\eta_d}$.*

Proof. Note that $\alpha^d + \eta_1\alpha^{d-1} + \cdots + \eta_d = 0$. Take $a \in \cap_{i-1}^{d-1} I_{\eta_i}$. Then $a\eta_d = -(a\alpha^d + a\eta_1\alpha^{d-1} + \cdots + a\eta_{d-1}\alpha) \in R[\alpha]\cap K = R$. Thus $a \in I_{\eta_d}$, which means that $\cap_{i=1}^{d-1} I_{\eta_d}$. □

Let $\pi : R[X] \to R[\alpha]$ be the R-algebra homomorphism sending X to α. The element α is called an *anti-integral* element of degree d over R if $Ker\ \pi = I_{[\alpha]}\varphi_\alpha(X)R(X)$. When α is an anti-integral ělement over R, $R[\alpha]$ is called an *anti-integral extension* of R. (See [OSY] for details) For $f(X) \in R[X]$, let $C(f(X))$ denote the ideal generated by the coefficients of $f(X)$, that is, the content ideal of $f(X)$. Let $J_{[\alpha]} := I_{[\alpha]}C(\varphi_\alpha(X))$, which is an ideal of R and contains $I_{[\alpha]}$. The element α is called a *super-primitive* element of degree d over R if $J_{[\alpha]} \nsubseteq p$ for all primes p of depth one. It is known that a super-primitive element is an anti-integral element (See [OSY] for details).

Corollary 1.1. *Assume that α is super primitive over R. When $\cap_{i=1}^{d-1} I_{\eta_i} + I_{\eta_d} = R$, the following statements are equivalent:*
(i) *α is exclusive over R i.e., $R[\alpha]\cap K = R$,*
(ii) *$\cap_{i-1}^{d-1} I_{\eta_i} \subseteq I_{\eta_d}$.*

Proof. (ii) ⇒ (i) : By [OSuY, Proposition 13], we have $R[\alpha]\cap K = R[(\cap_{i=1}^{d-1} I_{\eta_i})\eta_d] = R$ because $(\cap_{i=1}^{d-1} I_{\eta_i})\eta_d \subseteq R$.
(i) ⇒ (ii) follows from Proposition 1. □

By [OSuy, Proposition 9], we have the following result:

Corollary 1.2. *If $I_{[\alpha]} = R$, then $R[\alpha]\cap K = R$. Moreover if α is super-primitive over R, then $I_{\eta_d} = R$ implies $R[\alpha]\cap K = R$.*

The following lemma is very elementary, but we give a proof for convenience.

Lemma 2. *Let $f(X) \in R[X]$ be a monic polynomial and let $g(X) \in K[X]$. If $f(X)g(X)$ is a monic polynomial in $R[X]$, then $g(X) \in R[X]$.*

Proof. Put $f(X) = X^n + a_1X^{n-1} + \cdots + a_n$ with $a_i \in R$ and $g(X) = \zeta_0 X^m + \cdots + \zeta_m$ with $\zeta_i \in K$. Since $f(X)g(X)$ monic, we have $\zeta_0 = 1$. Assume that $\zeta_0 = 1, \zeta_2, \cdots, \zeta_{\ell-1} \in R$ and $\zeta_\ell \notin R$. Then the coefficient of the degree $(\ell + n)$-term is $\zeta_\ell + a_1\zeta_{\ell-1} + \cdots + a_\ell\zeta_0 \in R$. Hence $\zeta_\ell \in R$, a contradiction. Thus $\zeta_i \in R$ for all i. So we have $g(X) \in R$. □

Proposition 3. (cf. [OSY, (2.2)]) Assume that R is normal. Then α is integral over R if and only if $I_{[\alpha]} = R$.

Proof. ($\Rightarrow$): Since R is normal, $\varphi_\alpha(X) \in R[X]$ by [M,(9.2)]. By [OSY, (1.13)], α is super-primitive and hence anti-integral because R is a Krull domain. Hence $I_{[\alpha]} = R$ follows from [OSY,(2.2)].
($\Leftarrow$): Since $I_{[\alpha]} = R$, we have $Ker\ \pi = I_{[\alpha]}\varphi_\alpha(X)R[X] = \varphi_\alpha(X)R[X]$. Thus $\varphi_\alpha(\alpha) = 0$ gives rise to an integral dependence. So α is integral over R. □

Corollary 3.1. *Assume that R is normal, If α is integral over R, then α is an exclusive element over R.*

Proof. By Proposition 3, $I_{[\alpha]} = R$ and hence $R[\alpha] \cap K = R$ by Corollary 1.2. □

Proposition 4. *If $\bigcap_{i=1}^{d-1} I_{\eta_i} = R$ and α is exclusive over R i.e., $R[\alpha] \cap K = R$, then $R[\alpha]$ is integral over R.*

Proof. By Proposition 1, $\bigcap_{i=1}^{d-1} I_{\eta_i} \subseteq I_{\eta_d}$ and hence $I_{[\alpha]} = \bigcap_{i=1}^{d} I_{\eta_i} = R$. Thus $\eta_1, \cdots, \eta_d \in R$, which means $\varphi_\alpha(X) \in R[X]$ and $\varphi_\alpha(\alpha) = 0$. This yields that α is integral over R. □

In [OY,(1.3)], we see that if $\alpha \in K$ is both integral and anti-integral over R then $\alpha \in R$. For the case $d \geqq 0$ we have a similar result as follows.

Theorem 5. *If α is both anti-integral and integral over R, then α is exclusive over R, i.e., $R[\alpha] \cap K = R$.*

Proof. Since α is integral over R, there exists a monic polynomial $f(X) \in R[X]$ such that $f(\alpha) = 0$. Since α is anti-integral over R, we have $Ker\ \pi = I_{[\alpha]}\varphi_\alpha(X)R[X]$. Hence $f(X) \in Ker\ \pi = I_{[\alpha]}\varphi_\alpha(X)R[X]$. So there exists a polynomial $g(X) \in I_{[\alpha]}R[X]$ such that $f(X) = \varphi_\alpha(X)g(X)$. Since $f(X)$ and $\varphi_\alpha(X)$ is monic, $g(X)$ is monic. Thus $\varphi_\alpha(X) \in R[X]$ by Lemma 2, which showes that $I_{[\alpha]} = R$. Therefore $R[\alpha] \cap K = R$ by Corollary1.2. □

Proposition 6. *There exists a non-zero element $a \in R \backslash \{0\}$ such that $a\alpha$ is exclusive over R.*

Proof. Take $a \in I_{[\alpha]} \backslash \{0\}$. Then $\varphi_{a\alpha}(X) = a^d \varphi_\alpha(X) = (aX)^d + \cdots + a^d \eta_d \in R[X]$. Hence $I_{[a^d\alpha]} = R$ implies that $R[a\alpha] \cap K = R$ by Corollary 1.2. □

Lemma 7. *Let a be a non-zero element in R. If α is exclusive over R, then α is also exclusive over $R[1/a]$.*

Proof. Since $R[1/a]$ is flat over R, we have:

$$(R[\alpha]\cap K)\otimes_R R[1/a] = R\otimes_R R[1/a],$$
$$R[\alpha]\otimes_R R[1/a]\cap K\otimes_R R[1/a] = R[1/a],$$
$$R[1/a][\alpha]\cap K = R[1/a].$$

□

Definition 2. Let

$$E(\alpha) = \{a \in R | R[1/a][\alpha]\cap K = R[1/a]\}\cup\{0\}.$$

As is seen in the next theorem, $E(\alpha)$ represents the obstruction ideal of exclusiveness of α.

Theorem 8. *$E(\alpha)$ is a radical ideal of R.*

Proof. Let $b \in R$ and $a \in E(\alpha)$. Note here that $R[1/(ab)] = R[1/a,1/b]$. Hence $R[1/(ab)][\alpha]\cap K = R[1/a][1/b][\alpha]\cap K$. Since $R[1/a][\alpha]\cap K = R[1/a]$, we obtain $R[1/a][1/b][\alpha]\cap K = R[1/a][1/b]$ by Lemma 8. Hence $R[1/(ab)[\alpha]\cap K = R[1/a, 1/b][\alpha]\cap K = R[1/a, 1/b] = R[1/(ab)]$, which shows $ab \in E(\alpha)$. Next take $a,b \in E(\alpha)$. Let $\zeta \in R[1/(a+b)][\alpha]\cap K$. Then for sufficiently large integer ℓ, we have $(a+b)^\ell\zeta \in R[\alpha]\cap K \subseteq \mathrm{R}[1/a][\alpha]\cap K = R[1/a]$ and similarly $(a+b)^\ell\zeta \in R[\alpha]\cap K \subseteq R[1/b][\alpha]\cap K = R[1/b]$. Thus we have $R[1/(a+b)][\alpha] \subseteq R[1/a]\cap R[1/b]$. Now we must show that $R[1/a]\cap R[1/b] \subseteq R[1/(a+b)]$. Take $\xi \in R[1/a]\cap R[1/b]$. Then $\xi = c/a^n = d/b^m$ for some $c,d \in R$ and for some integres n,m. Take $\ell \gg 0$, e.g. $\ell > 2Max\{n,m\}$. Then $(a+b)^\ell\xi = a^\ell(c/a^n) + \cdots + b^\ell(d/b^m) \in R$. Hene $\xi \in R[1/a(a+b)]$. Therefore $R[1/(a+b)][\alpha]\cap K \subseteq R[1/a]\cap R[1/b] \subseteq R[1/(a+b)]$, which means $R[1/(a+b)][\alpha]\cap K \subseteq R[1/(a+b)]$. So we conclude that $a+b \in E(\alpha)$. We have shown that $E(\alpha)$ is an ideal of R. It is clear that $E(\alpha)$ is a radical ideal of R. □

The next Corollary 8.1 follows from the definition immediately.

Corollary 8.1. *The following statements are equivalent:*
(i) *$E(\alpha) = R$,*
(ii) *α is exclusive over R.*

Theorem 9. *The following inclusions hold:*

$$I_{[\alpha]} \subseteq E(\alpha) \subseteq \sqrt{I_{\eta_d} : \cap_{i=1}^{d-1} I_{\eta_i}}.$$

Moreover if α is super-primitive over R, then $I_{\eta_d} \subseteq E(\alpha)$.

Proof. Take $p \in Spec(R)$ with $I_{[\alpha]} \nsubseteq p$. Then $R_p[\alpha]\cap K = R_p$ by Corollary 1.2.

Hence $E(\alpha)_p = R_p$, which means $E(\alpha) \nsubseteq p$. Since $E(\alpha)$ is a radical ideal, we have $I_{[\alpha]} \subseteq E(\alpha)$. Next take $p \in Spec\,(R)$ with $E(\alpha) \nsubseteq p$. Then $R[\alpha]\cap K = R_p$. By Proposition 1, we have $(\bigcap_{i=1}^{d-1} I_{\eta_i})_p \subseteq (I_{\eta_d})_p$. Hence $(I_{\eta_d} : \bigcap_{i=1}^{d-1} I_{\eta_i})_p = R_p$, which yields that $I_{\eta_d} : \bigcap_{i=1}^{d-1} I_{\eta_i} \nsubseteq p$. Thus $E(\alpha) \subseteq \sqrt{I_{\eta_d} : \bigcap_{i=1}^{d-1} I_{\eta_i}}$. The second part follows from Corollary 1.1. □

Corollary 9.1. *If $E(\alpha) \neq R$, then there exists $p \in Dp_1(R)$ such that $E(\alpha) \subseteq p$.*

Proof. Suppose that $E(\alpha) \nsubseteq p$ for all $p \in Dp_1(R)$. Then $R_p[\alpha]\cap K = R_p$ and hence $R[\alpha]\cap K \subseteq \bigcap_{p \in Dp1(R)} R_p[\alpha]\cap K = \bigcap_{p \in Dp1(R)} R_p = R$. Thus $R[\alpha]\cap K = R$, which shows $E(\alpha) = R$. □

The following corollary is seen immediately from Proposition 9.

Corollary 9.2. *If α is exclusive over R_p for each $p \in Dp_1(R)$, then α is exclusive over R.*

Corollary 9.3. *α is exclusive over $R[1/a]$ for any $a \in I_{[\alpha]} \setminus \{0\}$, i.e., $R[1/a][\alpha]\cap K = R[1/a]$.*

Proof. By Theorem 9, $I_{[\alpha]} \subseteq E(\alpha)$, so that $R[1/a][\alpha]\cap K = R[1/a]$. □

Concerning the converse of Proposition 1, we have the following special result.

Proposition 10. *Assume that α is a super-primitive element of degree 2. Then $R[\alpha]\cap K = R$ if and only if $I_{\eta_1} \subseteq I_{\eta_2}$.*

Proof. We must prove the if-part by Proposition 1. By Corollary 9.2, we may assume that R is a local domain (R,m) of depth one. When $I_{\eta_2} \nsubseteq m$, $\eta_2 \in R$.

Since α is super-primitive over R, we have $R[\alpha]\cap K = R$ by Theroem 9. Next we consider the case $I_{\eta_2} \subseteq m$. Then $I_{[\alpha]} = I_{\eta_1} \cap I_{\eta_2} = I_{\eta_1}$ by the assumption. Since α is super-primitive over R, $I_{[\alpha]}$ is a principal ideal by [OSY,(1,12)]. So $I_{[\alpha]} = aR = I_{\eta_1}$ for some $a \in R$. In this case, $\eta_1 = b/a$ and $b \notin m$ because if $b \in m$ then I_{η_1} contains aR properly. Hence we may assume that $b = 1$. Since $a \in I_{\eta_1} \subset I_{\eta_2}$ yields $\varphi_\alpha(X) = X^2 + (1/a)X + (c/a)$ for some $c \in R$. Hence by [OSuY, Remark in § 3], we have $R[\alpha]\cap K = R$. □

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