Notes on Asymptotic Stability Property for Second Order Linear Evolution Equations

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In this article, we will be concerned with the following second order linear evolution equation

$$\ddot{u}(t) + p(t)A^{\alpha}\dot{u}(t) + Au(t) = 0, \tag{1}$$

where $\alpha = 0$ or $\alpha = 1$, and investigate the asymptotic behavior of solutions of (1) as $t \to \infty$. In a special case, Equation (1) represents a linear wave equation with damping term in which the case $\alpha = 0$ and the case $\alpha = 1$ respectively correspond to the first order damping and the third order damping (cf. [3]).

In what follows, we impose the following conditions on (1):

(H1) $p : R^+ \to R^+$, $R^+ := [0, \infty)$, is a continuous function.

(H2) A is a linear operator in a real separable Hilbert space H with dense domain $D(A)$ which is self-adjoint and positive definite with discrete spectrum $\{\lambda_n\}_{n=1}^{\infty}$; here $0 < \lambda_1 \leq \lambda_2 \leq \cdots\cdots \leq \lambda_n \leq \cdots\cdots$ and $\lim_{n\to\infty} \lambda_n = \infty$.

By (H2), there exists a complete orthonormal system $\{e_n\}$ in H such that $Ae_n = \lambda_n e_n$ for $n = 1, 2, \cdots\cdots$. For each $u \in H$, we have an expansion

$$u = \sum_{n=1}^{\infty} a_n e_n, \quad a_n = (u, e_n)_H$$

with $\|u\|_H^2 = \sum_{n=1}^{\infty} a_n^2$, where $(\ ,\ )_H$ denotes an inner product of H. For each $\beta \geq 0$, we define the operator $A^{\beta} : D(A^{\beta}) \to H$ by

$$D(A^{\beta}) = \{\sum_{n=1}^{\infty} a_n e_n : \sum_{n=1}^{\infty} (\lambda_n^{\beta} a_n)^2 < \infty\}$$

$$A^{\beta}u = \sum_{n=1}^{\infty} \lambda_n^{\beta} a_n e_n, \quad u = \sum_{n=1}^{\infty} a_n e_n \in D(A^{\beta}).$$

A^{β} is a closed linear operator in H, and $D(A^{\beta})$ equipped with the norm $\|u\|_{\beta} := \|A^{\beta}u\|_H$ is a real Banach space. When $u \in D(A^{\beta})$ with $u = \sum_{n=1}^{\infty} a_n e_n$, we get

$$\|u\|_{\beta} = \{\sum_{n=1}^{\infty} (\lambda_n^{\beta} a_n)^2\}^{1/2}. \tag{2}$$

If $\beta \leq \gamma$, then $D(A^\gamma) \subset D(A^\beta)$ and the canonical inclusion map from $D(A^\gamma)$ into $D(A^\beta)$ is continuous. For any nonnegative integer k and any Banach space X, we denote by $C^k(R^+; X)$ the set of all X-valued functions which are continuously differentiable up to order k on R^+.

Definition. *An H-valued function $u(t)$ on R^+ is said to be a solution of* (1), *if*

$$u \in C(R^+; D(A^{1+\alpha})) \cap C^1(R^+; D(A^{(1/2)+\alpha})) \cap C^2(R^+; H)$$

and $u(t)$ satisfies (1) *in H for $t \in R^+$.*

In what follows, we shall study the asymptotic behavior of solutions of (1) in connection with the asymptotic stability property of an ordinary differential equation

$$\ddot{y}(t) + \lambda_n^\alpha p(t)\dot{y}(t) + \lambda_n y(t) = 0 \tag{1_n}$$

for $n = 1, 2, \cdots$. The zero solution of (1_n) is said to be asymptotically stable, if each solution $y(t)$ of (1_n) satisfies $\lim_{t\to\infty}[|y(t)| + |\dot{y}(t)|] = 0$.

Theorem. *Assume Conditions* (H1) *and* (H2). *Then the following statements are equivalent:*

(i) *The solution $u(t)$ of* (1) *with $u(0) \in D(A^{1+\alpha})$ and $\dot{u}(0) \in D(A^{(1/2)+\alpha})$ satisfies $\|u(t)\|_{(1/2)+\alpha} + \|\dot{u}(t)\|_\alpha \to 0$ as $t \to \infty$.*

(ii) *For each $n = 1, 2, \cdots$, the zero solution of* (1_n) *is asymptotically stable.*

In order to establish the theorem, we need the following proposition.

Proposition. *Assume Conditions* (H1) *and* (H2). *If $u : R^+ \to H$ is a solution of* (1) *with $u(0) \in D(A^{1+\alpha})$ and $\dot{u}(0) \in D(A^{(1/2)+\alpha})$, then the coefficients $a_n(t)$, $n = 1, 2, \cdots$, in the expansion*

$$u(t) = \sum_{n=1}^{\infty} a_n(t)e_n \tag{3}$$

satisfy the following two conditions:

(C1) $\sum_{n=1}^{\infty}\{(\lambda_n^{1+\alpha}a_n(0))^2 + (\lambda_n^{(1/2)+\alpha}\dot{a}_n(0))^2\} < \infty$.

(C2) *$a_n(t)$ is a solution of (E_n) for each $n = 1, 2, \cdots$.*

Conversely, if $a_n(t)$, $n = 1, 2, \cdots$, satisfy (C1) *and* (C2), *then the function u defined by* (3) *is a solution of* (1).

Proof of Proposistion. Let $u : R^+ \to H$ be a solution of (1), and consider the expansion $\dot{u}(t) = \sum_{n=1}^{\infty} b_n(t)e_n$. Then $b_n(t) = (\dot{u}(t), e_n)_H = (d/dt)(u(t), e_n)_H = \dot{a}_n(t)$, and hence $\dot{u}(t) = \sum_{n=1}^{\infty} \dot{a}_n(t)e_n$. Similarly, we can get $\ddot{u}(t) = \sum_{n=1}^{\infty} \ddot{a}_n(t)e_n$. Then (C1) follows from (2) because of $u(0) \in D(A^{1+\alpha})$ and $\dot{u}(0) \in D(A^{(1/2)+\alpha})$. Moreover, we obtain that

$$\begin{aligned}\ddot{a}_n(t) + \lambda_n^\alpha p(t)\dot{a}_n(t) + \lambda_n a_n(t) &= (\ddot{u}(t), e_n)_H + p(t)(\dot{u}(t), A^\alpha e_n)_H + (u(t), Ae_n)_H \\ &= (\ddot{u}(t) + p(t)A^\alpha \dot{u}(t) + Au(t), e_n)_H = 0,\end{aligned}$$

which shows (C2).

Conversely, assume that $a_n(t)$, $n = 1, 2, \cdots$, satisfy (C1) and (C2), and consider the function $u(t)$ defined by (3). Since $p(t) \geq 0$, we have $(d/dt)\{(\lambda_n^{1/2}a_n(t))^2 + (\dot{a}_n(t))^2\} = -2\lambda_n^\alpha p(t)(\dot{a}_n(t))^2 \leq 0$, and hence

$$(\lambda_n^{1/2}a_n(t))^2 + (\dot{a}_n(t))^2 \leq (\lambda_n^{1/2}a_n(0))^2 + (\dot{a}_n(0))^2, \ t \geq 0\,;\ 1, 2, \cdots. \tag{4}$$

Then $\sum_{n=1}^{\infty}\{(\lambda_n^{1+\alpha}a_n(t))^2 + (\lambda_n^{(1/2)+\alpha}\dot{a}_n(t))^2\} \leq \sum_{n=1}^{\infty}\{(\lambda_n^{1+\alpha}a_n(0))^2 + (\lambda_n^{(1/2)+\alpha}\dot{a}_n(0))^2\} < \infty$ by (C1), and consequently $u(t) = \sum_{n=1}^{\infty} a_n(t)e_n \in D(A^{1+\alpha})$ and $v(t) := \sum_{n=1}^{\infty} \dot{a}_n(t)e_n \in D(A^{(1/2)+\alpha})$ for $t \geq 0$.

Claim 1. *$u \in C(R^+; D(A^{1+\alpha}))$ and $v \in C(R^+; D(A^{(1/2)+\alpha}))$.*
For any $\varepsilon > 0$, select an integer $N > 0$ so large that

$$\sum_{n=N+1}^{\infty}\{(\lambda_n^{1+\alpha}a_n(0))^2 + (\lambda_n^{(1/2)+\alpha}\dot{a}_n(0))^2\} < \varepsilon/4, \tag{5}$$

which is possible by (C1). Then

$$\begin{aligned}\|u(t) - u(\bar{t})\|_{1+\alpha}^2 &= \sum_{n=1}^{\infty}\{\lambda_n^{1+\alpha}(a_n(t) - a_n(\bar{t}))\}^2 \\ &\leq \sum_{n=1}^{N}\{\lambda_n^{1+\alpha}(a_n(t) - a_n(\bar{t}))\}^2 \\ &\quad + 2\sum_{n=N+1}^{\infty}[\{\lambda_n^{1+\alpha}a_n(t)\}^2 + \{\lambda_n^{1+\alpha}a_n(\bar{t})\}^2] \\ &\leq \sum_{n=1}^{N}\{\lambda_n^{1+\alpha}(a_n(t) - a_n(\bar{t}))\}^2 \\ &\quad + 4\sum_{n=N+1}^{\infty}\{(\lambda_n{}^{1+\alpha}a_n(0))^2 + (\lambda_n^{(1/2)+\alpha}\dot{a}_n(0))^2\} \\ &\leq \sum_{n=1}^{N}\{\lambda_n^{1+\alpha}(a_n(t) - a_n(\bar{t}))^2\} + \varepsilon\end{aligned}$$

by (4) and (5). Thus $\limsup_{t\to\bar{t}}\|u(t) - u(\bar{t})\|_{1+\alpha}^2 \leq \varepsilon$. Because ε is an arbitrary positive number, one gets $\lim_{t\to\bar{t}}\|u(t) - u(\bar{t})\|_{1+\alpha} = 0$, and hence $u \in C(R^+; D(A^{1+\alpha}))$. In a similar way, one can get $v \in C(R^+; D(A^{(1/2)+\alpha}))$.

Claim 2. $\frac{d}{dt}u(t) = v(t)$ *in $D(A^{(1/2)+\alpha})$; that is,*

$$\left\| \frac{u(t+h)-u(t)}{h} - v(t) \right\|^2_{(1/2)+\alpha} \to 0 \text{ as } h \to 0. \tag{6}$$

For any $\varepsilon > 0$, let N be the integer selected in the proof of Claim 1. Then the left hand side of (6) is evaluated as:

$$\begin{aligned}
&\frac{1}{h^2}\sum_{n=1}^{\infty} \lambda_n^{1+2\alpha}\{a_n(t+h) - a_n(t) - h\dot{a}_n(t)\}^2 \\
&\quad = \frac{1}{h^2}\sum_{n=1}^{\infty}\lambda_n^{1+2\alpha}[\int_t^{t+h}\{\dot{a}_n(\theta) - \dot{a}_n(t)\}d\theta]^2 \\
&\quad \leq |\frac{1}{h}\int_t^{t+h}\sum_{n=1}^{\infty}\lambda_n^{1+2\alpha}|\dot{a}_n(\theta) - \dot{a}_n(t)|^2 d\theta| \\
&\quad \leq |\frac{1}{h}\int_t^{t+h}\sum_{n=1}^{N}\lambda_n^{1+2\alpha}|\dot{a}_n(\theta) - \dot{a}_n(t)|^2 d\theta| \\
&\qquad + |\frac{2}{h}\int_t^{t+h}\sum_{n=N+1}^{\infty}\lambda_n^{1+2\alpha}\{|\dot{a}_n(\theta)|^2 + |\dot{a}_n(t)|^2\}d\theta| \\
&\quad \leq \max\{\sum_{n=1}^{N}\lambda_n^{1+2\alpha}|\dot{a}_n(\theta) - \dot{a}_n(t)|^2 : |t-\theta| \leq |h|\} \\
&\qquad + 4\sum_{n=N+1}^{\infty}[(\lambda_n^{1+\alpha}a_n(0))^2 + (\lambda_n^{(1/2)+\alpha}\dot{a}_n(0))^2] \\
&\quad \leq \max\{\sum_{n=1}^{N}\lambda_n^{1+2\alpha}|\dot{a}_n(\theta) - \dot{a}_n(t)|^2 : |t-\theta| \leq |h|\} + \varepsilon,
\end{aligned}$$

and hence (6) follows from the above inequality and the continuity of $\dot{a}_n(t)$.

From Claims 1 and 2, we see that $u \in C(R^+; D(A^{1+\alpha})) \cap C^1(R^+; D(A^{(1/2)+\alpha}))$. It remains only to show that $(d/dt)v(t) = -Au(t) - p(t)A^\alpha v(t)$ in H; that is,

$$\left\| \frac{v(t+h)-v(t)}{h} + Au(t) + p(t)A^\alpha v(t) \right\|^2_H \to 0 \text{ as } h \to 0. \tag{7}$$

Let ε and N be the same ones as in the proof of Claim 2. We can assume $\lambda_N \geq 1$. The left hand side of (7) is evaluated as:

$$\begin{aligned}
&\frac{1}{h^2}\sum_{n=1}^{\infty}[\dot{a}_n(t+h) - \dot{a}_n(t) + h\lambda_n a_n(t) + h\lambda_n^\alpha p(t)\dot{a}_n(t)]^2 \\
&\quad = \frac{1}{h^2}\sum_{n=1}^{\infty}\left[\int_t^{t+h}\{\ddot{a}_n(\theta) + \lambda_n a_n(t) + \lambda_n^\alpha p(t)\dot{a}_n(t)\}d\theta\right]^2 \\
&\quad = \frac{1}{h^2}\sum_{n=1}^{\infty}\left[\int_t^{t+h}\{\lambda_n(a_n(t) - a_n(\theta)) + \lambda_n^\alpha(p(t)\dot{a}(t) - p(\theta)\dot{a}_n(\theta))\}d\theta\right]^2 \\
&\quad \leq \left|\frac{1}{h}\sum_{n=1}^{\infty}\int_t^{t+h}|\lambda_n(a_n(t) - a_n(\theta)) + \lambda_n^\alpha(p(t)\dot{a}(t) - p(\theta)\dot{a}_n(\theta))|^2 d\theta\right| \\
&\quad \leq \left|\frac{2}{h}\int_t^{t+h}\sum_{n=1}^{\infty}[\lambda_n(a_n(t) - a_n(\theta))]^2 d\theta\right| \\
&\qquad + \left|\frac{2}{h}\int_t^{t+h}\sum_{n=1}^{N}|\lambda_n^\alpha(p(t)\dot{a}_n(t) - p(\theta)\dot{a}_n(\theta))|^2 d\theta\right| \\
&\qquad + \left|\frac{4}{h}\int_t^{t+h}\sum_{n=N+1}^{\infty}[(\lambda_n^\alpha p(t)\dot{a}_n(t))^2 + (\lambda_n^\alpha p(\theta)\dot{a}_n(\theta))^2]d\theta\right| \\
&\quad =: I_1^{(h)} + I_2^{(h)} + I_3^{(h)} \ (=: I^{(h)}).
\end{aligned}$$

By Claim 1, $u \in C(R^+; D(A^{1+\alpha})) \subset C(R^+; D(A))$, and hence $I_1^{(h)} \to 0$ as $h \to 0$, because of the inequality

$$I_1^{(h)} \le \left|\frac{2}{h}\int_t^{t+h}\|u(t) - u(\theta)\|_1^2 d\theta\right| \le 2\max\{\|u(t) - u(\theta)\|_1^2 : |t-\theta| \le |h|\}.$$

Moreover, since $p(t)\dot{a}_n(t)$ is continuous in $t \ge 0$, it is easy to see that $I_2^{(h)} \to 0$ as $h \to 0$. On the other hand, since $1 \le \lambda_N \le \lambda_n$ for $n \ge N$, we get

$$\begin{aligned} I_3^{(h)} &\le \left|\frac{4}{h}\int_t^{t+h}\{p(t)^2 + p(\theta)^2\}d\theta\right| \sum_{n=N+1}^{\infty}[(\lambda_n^{1+\alpha}a_n(0))^2 + (\lambda_n^{(1/2)+\alpha}\dot{a}_n(0))^2] \\ &< \varepsilon \max\{p(t)^2 + p(\theta)^2 : |t-\theta| \le |h|\}. \end{aligned}$$

Hence $\limsup_{h\to 0} I^{(h)} \le \varepsilon \max\{p(t)^2 + p(\theta)^2 : |t-\theta| \le 1\}$, which shows $\lim_{h\to 0} I_3^{(h)} = 0$. This completes the proof.

Now we prove the theorem.

[(i) $\longrightarrow$ (ii)]. Take any natural number n, and let $a_n(t)$ be any solution of (1_n). Then $u(t) := a_n(t)e_n$ is a solution of (E) with $u(0) \in D(A^{1+\alpha})$ and $\dot{u}(0) \in D(A^{(1/2)+\alpha})$ by Proposition. Then $|u(t)|_{(1/2)+\alpha} + |\dot{u}(t)|_\alpha \to 0$ or $|a_n(t)| + |\dot{a}_n(t)| \to 0$ as $t \to \infty$, which shows that the zero solution of (1_n) is asymptotically stable.

[(ii) $\longrightarrow$ (i)]. Let $u(t) = \sum_{n=1}^{\infty} a_n(t)e_n$ be any solution of (1) with $u(0) \in D(A^{1+\alpha})$ and $\dot{u}(0) \in D(A^{(1/2)+\alpha})$. By virtue of Proposition, $a_n(t)$ is a solution of (1_n) for each $n = 1, 2, \cdots\cdot$. Since the zero solution of (1_n) is asymptotically stable, one gets

$$|a_n(t)| + |\dot{a}_n(t)| \to 0 \text{ as } t \to \infty \tag{8}$$

for each $n = 1, 2, \cdots\cdot$. For any $\varepsilon > 0$, select a positive integer N so large that $\lambda_N \ge 1$ and (5) holds. By virture of (4), we obtain

$$\begin{aligned} |u(t)|_{(1/2)+\alpha}^2 + |\dot{u}(t)|_\alpha^2 &= \sum_{n=1}^{\infty}[\lambda_n^{(1/2)+\alpha}a_n(t))^2 + (\lambda_n^\alpha \dot{a}_n(t))^2] \\ &\le \sum_{n=1}^{N}[(\lambda_n^{(1/2)+\alpha}a_n(t))^2 + (\lambda_n^\alpha \dot{a}_n(t))^2] \\ &\quad + \sum_{n=N+1}^{\infty}[(\lambda_n^{1+\alpha}a_n(0)^2 + (\lambda_n^{(1/2)+\alpha}\dot{a}_n(0))^2] \\ &\le \sum_{n=1}^{N}[(\lambda_n^{(1/2)+\alpha}a_n(t))^2 + (\lambda_n^\alpha \dot{a}_n(t))^2] + \frac{\varepsilon}{4}, \end{aligned}$$

and hence $\limsup_{t\to\infty}[|u(t)|^2_{(1/2)+\alpha} + |\dot{u}(t)|_\alpha^2] \le \varepsilon/4$ by (8). Since ε is an arbitrary positive number, we thus obtain $\lim_{t\to\infty}[|u(t)|_{(1/2)+\alpha}^2 + |\dot{u}(t)|_\alpha^2] = 0$, completing the proof.

Hatvani and Totik [4] have recently generalized a classical result due to Smith [7] which provides a necessary and sufficient condition for the zero solution of (1_n) to be asymptotically stable. Combining our theorem with Hatvani and Totik's result, one can obtain the following result which has recently been proved by Zhang [8] in a different manner in case of $\alpha = 0$, $H = L^2(\Omega)$ (Ω is a bounded domain in R^n with the smooth boundary) and $-A = \sum_{i=1}^{n} \partial^2/\partial x_i^2 + k$ (k is a constant) with the Dirichlet boundary condition.

Corollary 1. *Assume Conditions* (H1) *and* (H2), *and moreover assume that*

(H3) $\liminf_{t\to\infty} \int_t^{t+\delta} p(s)\,ds > 0$ *for each* $\delta > 0$.

Then the condition (i) *of Theorem is equivalent to the following condition:*

(H4) $\int_0^\infty \exp\{-\lambda_n^\alpha P(t)\}\left(\int_0^t \exp\{\lambda_n^\alpha P(s)\}\,ds\right) = \infty,\ n = 1, 2, \cdots,$

where $P(t) = \int_0^t p(s)\,ds$.

While it seems to be rather difficult to check the condition (H4) in practical cases, the following result provides a flexible criterion for the condition (i) of Theorem.

Corollary 2. *Let* $p(t) \geq \varepsilon > 0$, p *continuous, and assume Condition* (H2) *and one of the following conditions;*

(H5) $1/p(t)$ *is of bounded variation.*

(H6) $p(t)$ *is differentiable and* $-\lambda_1^\alpha < \dot{p}(t)/p^2(t) < K$ *(K; a constant).*

Then the condition (i) *of Theorem is equivalent to* $\int^\infty dt/p(t) = \infty$.

In fact, applying [2, Theorem 3] one can easily see that the condition (H4) is equivalent to $\int^\infty dt/p(t) = \infty$ under the assumptions of the corollary. Then the corollary is a direct consequence of Corollary 1.

Finally, combining Theorem with a result in [5] or [6], we obtain the following result which provides a sufficient condition for the condition (i) of Theorem to hold true. In fact, it is an almost best possible (computable) result because the condition (H7) cited below holds true when $0 < p(t) = O(t \cdot \log t)$ as $t \to \infty$, while $1 + (t+1)^{-\varepsilon}$ is a solution of (1_n) with $p(t) = \{(\varepsilon + 1)/(t + 1) + \lambda_n(t + 1 + (t + 1)^{\varepsilon+1})/\varepsilon\}/\lambda_n^\alpha$ for $\varepsilon > 0$ (cf. [1]).

Corollary 3. *In addition to* (H1) *through* (H3), *assume the following condition;*

(H7) *there exist a sequence of positive numbers* $\{s_n\}$ *and a positive constant* d *such that* $s_{n+1} - s_n \leq d$, $n = 1, 2, \cdots$, *and that* $\int_{s_n}^{s_n+d} p(s)\,ds > 0$ *for all* n *and*

$\sum_{n=1}^{\infty}\left[\int_{s_n}^{s_n+d} p(s)\,ds\right]^{-1} = \infty.$

Then the condition (i) of Theorem holds true.

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