Notes on metrics and counting functions on $B^n$

Dedicated to Professor Kenji Nakagawa on his 70th birthday

Shigeyasu KAMIYA

Department of Mechanical Science
Okayama University of Science
1-1 Ridai-cho Okayama 700 Japan
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0. Introduction and preliminaries. Let $C$ be the field of the complex numbers. Let $V = V^{1,n}(C)$ ($n \geq 1$) denote the vector space $C^{n+1}$, together with the unitary structure defined by the Hermitian form

$$\Phi(z^*, w^*) = -\sum_{k=1}^{n} z_k^* w_k^* + \sum_{k=1}^{n} z_k^* w_{k*},$$

where $z^* = (z_0^*, z_1^*, ..., z_n^*)$ and $w^* = (w_0^*, w_1^*, ..., w_n^*)$ in $V$. An automorphism $g$ of $V$ will be called a unitary transformation. ($g$ must be linear and $\Phi(g(z^*), g(w^*)) = \Phi(z^*, w^*)$ for all $z^*, w^* \in V$.) We denote the group of all unitary transformations by $U(1, n; C)$.

Let $V_0 = \{ z^* \in V \mid \Phi(z^*, z^*) = 0 \}$ and $V_- = \{ z^* \in V \mid \Phi(z^*, z^*) < 0 \}$. Obviously $V_0$ and $V_-$ are invariant under $U(1, n; C)$. Let $\pi(V)$ be the projective space obtained from $V$. This is defined, as usual, by using the equivalence relation in $V - \{ 0 \} : u^* \sim v^*$ if there exists $\lambda \in C - \{ 0 \}$ such that $u^* = \lambda v^*$. Let $\pi : V - \{ 0 \} \to \pi(V)$ denote the projection map. We define $H^n(C) = \pi(V_-)$. Let $\overline{H^n(C)}$ denote the closure of $H^n(C)$ in the projective space $\pi(V)$. An element $g \in U(1, n; C)$ operates in $\pi(V)$, leaving $\overline{H^n(C)}$ invariant. If $(z^*_0, z^*_1, ..., z^*_n) \in V_-$, the condition $-|z_0^*|^2 + \sum_{k=1}^{n} |z_k^*|^2 < 0$ implies that $z_0^* \neq 0$. Therefore we may define a set of coordinates $z = (z_1, z_2, ..., z_n)$ in $\overline{H^n(C)}$ by $z_k(\pi(z^*)) = z_k^* z_0^{*^{-1}}$. In this way $\overline{H^n(C)}$ becomes identified with the complex unit ball $B^n = B^n(C) = \{ z = (z_1, z_2, ..., z_n) \in C^n \mid \|z\|^2 = \sum_{k=1}^{n} |z_k|^2 < 1 \}$. A unitary transformation is regarded as a transformation operating on $B^n$. We use the same symbol $U(1, n; C)$ to denote the group of all these transformations. Throughout this paper $G$ will denote a discrete subgroup of $U(1, n; C)$. We assume that the stabilizer $G_0$ of 0 consists only of the identity.

In this paper we shall show the properties of metrics on $B^n$ in Section 1 and discuss a counting function $n(r, z)$ in Section 2.

1. Metrics $d$, $\delta$ and $\delta_\omega$. Let $d(\cdot)$ be the distance which is induced from the metric
\[ g_\delta(z) = \delta_\delta(1 - \|z\|^2)^{-1 + \frac{1}{2}} z_i(1 - \|z\|^2)^{-2}, \]

where \( z = (z_1, z_2, \ldots, z_n) \in B^n \). We recall that \( d(z, w) \) is expressed as

\[ d(z, w) = \cosh^{-1} \left( \| \Phi(z^*, w^*) \| \Phi(z^*, z^*) \Phi(w^*, w^*) \right)^{-1/2}, \]

where \( z^* \in \pi^{-1}(z) \) and \( w^* \in \pi^{-1}(w) \). Set

\[ \delta(z, w) = [1 - \| \Phi(z^*, z^*) \Phi(w^*, w^*) \| \Phi(z^*, w^*) \|^{-2}]^{1/2} \]

for \( z, w \in B^n \) (see [3, p. 180]).

**Proposition 1.1.** Let \( z \) and \( w \) be points in \( B^n \).

(a) \( \delta(z, w) = \tanh d(z, w) \).

(b) \( \delta(g(z), g(w)) = \delta(z, w) \) for any element \( g \in U(1, n; C) \).

(c) \( d(z, w) = (1/2) \log (1 + \delta(z, w))(1 - \delta(z, w))^{-1} \).

(d) \( d(z, w) \geq \delta(z, w) \).

**Proof.** (a) It is seen that

\[ \tanh^2 d(z, w) = 1 - \sech^2 d(z, w) \]
\[ = 1 - \left[ \| \Phi(z^*, z^*) \Phi(w^*, w^*) \|^{-1/2} \| \Phi(z^*, w^*) \| \right]^{-2} \]
\[ = 1 - \left[ \| \Phi(z^*, z^*) \Phi(w^*, w^*) \| \Phi(z^*, w^*) \|^{-2} \right] \]
\[ = \delta^2(z, w). \]

Thus \( \delta(z, w) = \tanh d(z, w) \).

(b) This follows from (a) and the invariance of \( d \) under \( U(1, n; C) \).

(c) This is immediate.

(d) By (a), \( \delta(z, w) = \tanh d(z, w) \leq d(z, w) \).

**Proposition 1.2.** The function \( \delta \) is a distance function on \( B^n \).

**Proof.** By (a) in Proposition 1.1,

\[ \delta(z, w) \geq 0 \text{ and } \delta(z, w) = 0 \iff z = w; \]
\[ \delta(z, w) = \delta(w, z). \]

Therefore we have only to prove the triangle inequality. Let \( x, y \) and \( z \) be points in \( B^n \).

Using (a) in Proposition 1.1 and the addition theorem on \( \tanh \), i. e.

\[ \tanh[d(x, y) + d(y, z)] = \{ \tanh d(x, y) + \tanh d(y, z) \} \]
\[ \quad \times \left( 1 + \tanh d(x, y) \cdot \tanh d(y, z) \right)^{-1}, \]

we see that

\[ \delta(x, y) + \delta(y, z) = \tanh d(x, y) + \tanh d(y, z) \]
\[ \quad = \tanh[d(x, y) + d(y, z)] \times \{ 1 + \tanh d(x, y) \cdot \tanh d(y, z) \} \]
\[ \quad \geq \tanh d(x, z) = \delta(x, z). \]

Set
\[ \delta_\sigma(z, w) = [1 - \{\Phi(z^*, z^*) \Phi(w^*, w^*) \Phi(z^*, w^*)\}^{-2}]^{1/2} \]

and let
\[ d_\sigma = (1/2) \log (1 + \delta_\sigma)(1 - \delta_\sigma)^{-1}. \]

It is easy to see that \( \delta_1 = \delta \) and \( d_1 = d \).

**Proposition 1.3.**
(a) The functions \( \delta_\sigma \) and \( d_\sigma \) are increasing functions of \( \sigma > 0 \).
(b) \( \delta_\sigma = \tanh d_\sigma \).
(c) \( \sech d_\sigma = \sech^\sigma d \).
(d) If \( \sigma \in (0, 1) \), then \( d_\sigma \) is a distance function on \( B^n \).

**Proof.** (a) This is immediate.
(b) We see that
\[ \tanh d_\sigma = \tanh[(1/2) \log (1 + \delta_\sigma)(1 - \delta_\sigma)^{-1}] = \delta_\sigma. \]
(c) The equality (a) in Proposition 1.1 yields
\[ \sech^2 d_\sigma = 1 - \tanh^2 d_\sigma = (1 - \delta^2)\sigma = (1 - \tanh^2 d)^\sigma \]
\[ = \sech^{2\sigma} d. \]
(d) We have only to prove the triangle inequality. Since \( U(1, n; C) \) is transitive on \( B^n \), it is sufficient to show that
\[ d_\sigma(z, w) \leq d_\sigma(z, 0) + d_\sigma(w, 0) \quad \text{for } z, w \in B^n. \]

Set \( t_1 = \delta_\sigma(z, 0), t_2 = \delta_\sigma(w, 0) \) and \( t_3 = \delta_\sigma(z, w) \). Then (1) is equivalent to the following inequality:
\[ (1 + t_3)(1 - t_3)^{-1} \leq (1 + t_1)(1 - t_1)^{-1}(1 + t_2)(1 - t_2)^{-1}. \]

By (2), we obtain
\[ t_3^2 \leq [(t_1 + t_2)(1 + t_1 t_2)^{-1}]^2. \]

From this it follows that
\[ (1 - t_3^2) \geq 1 - [(t_1 + t_2)(1 + t_1 t_2)^{-1}]^2 \]
\[ = (1 - t_1^2)(1 - t_2^2)(1 + t_1 t_2)^{-2}. \]

We note that
\[ 1 - \delta_\sigma(z, w)^2 = (1 - \delta(z, w)^2)^\sigma. \]

By using (3) and (4), we have
\[ 1 - \delta(z, w)^2 \geq (1 - \delta(z, 0)^2)(1 - \delta(w, 0)^2)(1 + t_1 t_2)^{-2/\sigma}. \]

This implies that
\[ \Phi(z^*, w^*) \Phi(z^*, w^*)^{-1} \] for \( z^* = (1, z_1, z_2, ..., z_n) \in \pi^{-1}(z) \) and \( w^* = (1, w_1, w_2, ..., w_n) \in \pi^{-1}(w) \). Therefore we have only to prove that
\[ |\Phi(z^*, w^*)| \leq (1 + t_1 t_2)^{1/\alpha}. \quad (5) \]

We can show that if \( \alpha \in (0, 1) \), then the inequality (5) is true. In fact,
\[
(1 + t_1 t_2)^{1/\alpha} \geq 1 + \left( \frac{1}{\alpha} \right) t_1 t_2 \\
\geq 1 + \left( \frac{1}{\alpha} \right) \left[ 1 - \left( 1 - \| z \|^2 \right) \left( 1 - \| w \|^2 \right) \right]^{1/2} \\
\geq 1 + \left( \frac{1}{\alpha} \right) \left( 1 - \| z \|^2 \right)^{1/2} \left( 1 - \| w \|^2 \right)^{1/2} \\
= 1 + \| z \| \| w \| \geq |\Phi(z^*, w^*)|.
\]

Thus our proof is complete.

**Proposition 1.4.**
\[ \delta(z, w) \leq (\| z \| + \| w \|)(1 + \| z \| \| w \|)^{-1} \quad \text{for } z, w \in B^n. \]

To prove Proposition 1.4, we need a lemma.

**Lemma 1.5.** If \( 0 \leq r < 1 \), then a function \( f(x) = (r + x)(1 + rx)^{-1} \) is increasing in \( x \geq 0 \).

**Proof of Proposition 1.4.** By (b) in Proposition 1.1, we may assume that \( z = (r, 0, ..., 0) \) and \( w = (w_1, w_2, ..., w_n) \), where \( 0 \leq r < 1 \). Let \( z^* = (1, r, 0, ..., 0) \in \pi^{-1}(z) \) and \( w^* = (1, w_1, w_2, ..., w_n) \in \pi^{-1}(w) \). It follows from Lemma 1.5 that
\[
\delta(z, w)^2 = 1 - (1 - r^2)(1 - \| w \|^2)(1 - rw_1)^{-2} \\
\leq 1 - (1 - r^2)(1 - \| w_1 \|^2)(1 - rw_1)^{-2} \\
= |r - w_1|^2(1 - rw_1)^{-2} \\
\leq (r + \| w_1 \|)^2(1 + r \| w_1 \|)^{-2} \\
\leq (r + \| w \|)^2(1 + r \| w \|)^{-2}. \]

**Proposition 1.6.** Let \( g \) be an element of \( U(1, n; C) \). For \( z, w \in B^n \)
\[ \| g(z) \| \leq (\| z \| + \| g(0) \|)(1 + \| z \| \| g(0) \|)^{-1}. \]

**Proof.** From (a) in Proposition 1.1 it follows that
\[ \| g(0) \| = \delta(0, g(0)) = \delta(g^{-1}(0), 0) = \| g^{-1}(0) \|. \]

By using this equality and Proposition 1.4, we have
\[ \| g(z) \| = \delta(0, g(z)) \\
= \delta(g^{-1}(0), z) \\
\leq (\| z \| + \| g^{-1}(0) \|)(1 + \| g^{-1}(0) \|)^{-1} \\
\leq (\| z \| + \| g(0) \|)(1 + \| z \| \| g(0) \|)^{-1}. \]
3. Counting function $n(r, z)$. Let $n(r, z) = \# \{ g \in G \mid \|g(z)\| < r \}$. 

**Theorem 2.1.** If $\|z\| < r < 1$, then 

$$n\left(\frac{r - \|z\|}{1 - r\|z\|}, 0\right) \leq n(r, z) \leq n\left(\frac{r + \|z\|}{1 + r\|z\|}, 0\right).$$

**Proof.** Proposition 1.6 implies that 

$$\{ g \in G \mid \|g(z)\| < r \} \supset \{ g \in G \mid \|g(0)\| < (r - \|z\|)(1 - r\|z\|)^{-1}\}$$

for $\|z\| < r < 1$. If $\|g(0)\| < r$, then 

$$\|g(z)\| \leq (\|z\| + \|g(0)\|)(1 + \|z\|\|g(0)\|^{-1}) \leq (r + \|z\|)(1 + r\|z\|)^{-1}$$

Therefore we have our desired inequalities. 

Let $D_0$ be the Dirichlet polyhedron for $G$ centered at 0. We note that $D_0$ is expressed as 

$$D_0 = \{ z \in B^n \mid \|g(z)\| > \|z\| \text{ for all } g \in G - \{\text{identity}\} \}.$$ 

Let $dV$ be the volume element which is induced from the metric $g$. It is easy to see that $dV(z) = (i/2)^n(1 - \|z\|^2)^{-(n+1)}dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_n$ for $z = (z_1, \ldots, z_n) \in B^n$. We use $\text{vol}(A)$ for the volume of $A$ measured by $dV$. 

**Theorem 2.2.** Suppose that the volume of $D_0$ is finite. Let $a \in D_0$ and $\|a\| < \rho < 1$. There exists $r_0$ such that the following inequality is satisfied for $r_0 \leq r \leq 1$. 

$$A(1 - r)^{-n} \leq n(r, a) \leq B(1 - r)^{-n},$$

where $A$ is a constant, which depends only on $\rho$ and $B$ is a numerical constant. 

**Proof.** Let $g_0, g_1, \ldots$ be the complete list of elements in $G$. In virtue of [3, Proposition 4.1], we have only to prove that 

$$A(1 - r)^{-n} \leq n(r, a).$$

We choose $r_0$ such that $\rho < r_0 < 1$. Let $F_0$ be the part of $D_0$ which lies outside of $\|z\| = r_0$, where we take $1 - r_0$ so small that the volume of $F_0$ is less than $\varepsilon$. Let $F_\kappa$ be the image of $F_0$ under $g_\kappa \in G$ and put $F = \bigcup_{\kappa \neq 0} F_\kappa$. Denote $F \cap \{ z \mid \|z\| < r \}$ by $F(r)$, where $r_0 < r < 1$. Then 

$$\text{vol}(F(r)) = \int_{F_0} n(r, z)dV(z)$$

$$\leq \text{constant} \cdot (1 - r)^{-n}\text{vol}(F_0)$$

$$\leq \text{constant} \cdot \varepsilon(1 - r)^{-n}. \quad (6)$$

Put $H_0 = D_0 - F_0$ and let $(H_\kappa)$ be its image under $G$ and $H = \bigcup_{\kappa \neq 0} H_\kappa$. Let $H(r) = H \cap \{ z \mid \|z\| < r \}$. It is seen that $F(r) \cup H(r) = \{ z \mid \|z\| < r \}$ and $F(r) \cap H(r) = \phi$. 

Therefore
\[ \text{vol}(F(r)) + \text{vol}(H(r)) = \text{vol}(\{ z \| z \| < r \}) \geq \text{constant} \cdot (1 - r)^{-n}. \] (7)

It follows from (6) and (7) that
\[ \text{vol}(H(r)) \geq \text{constant} \cdot (1 - r)^{-n} - \text{constant} \cdot \epsilon(1 - r)^{-n}. \] (8)

Let \( r_0^* \) be the \( \delta \)-diameter of \( H_0 \) and set \( r_1 = (r + r_0^*)(1 + r r_0^*)^{-1} \). Using [3, Proposition 2.1], we see that, if \( H_k \cap \{ z \| z \| < r \} \neq \phi \), then \( H_k \) is included in \( \{ z \| z \| < r_1 \} \). Hence \( H(r) \) is contained in \( \bigcup_{k \geq 0} H_k \), where
\[ H_k \subset \{ z \| z \| < r_1 \}. \] (9)

Since \( a \in D_0 \cap \{ z \| z \| < \rho \} \), the number of \( \{ H_k \} \) satisfying (9) is less than \( n(a, r_1) \). Therefore we have
\[ \text{vol}(H(r)) \leq n(r_1, a) \text{vol}(H_0). \] (10)

By (8) and (10),
\[ n(r_1, a) \geq \text{constant} \cdot (1 - r)^{-n}(\text{vol}(H_0))^{-1}. \]

Noting that \((1 - r_1)(1 + r^* r_0^*)^{-1} = 1 - r \), we obtain
\[ n(r_1, a) \geq A(1 - r)^{-n}. \]

Writing \( r \) instead of \( r_1 \), we have
\[ n(r, a) \geq A(1 - r)^{-n} \quad \text{for} \quad r_0 \leq r < 1. \]

Thus our proof is complete.

References
3 ) S. Kamiya : On subgroups of convergence or divergence type of \( U(1, n; C) \), Math. J. Okayama Univ. vol. 26 (1984), 179–191.