Spinor Formalism to the Poincaré Gauge Theory

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On the base of the invariance of a system under the Poincaré gauge transformations, five identities are obtained according to Nöther's theorem. Using some of these identities and field equations, a different approach to the Poincaré gauge theory by means of a spinor formalism is presented.

§ 1. Introduction

Since the gauge theory of gravity was first formulated by Utiyama[1)] in 1956 on an invariant property of system under the local Lorentz transformations, the gauge theory on space-time symmetries has been investigated by many authors.[2~10)]

Poincaré gauge theory on the base of a symmetry of system under the local Poincaré transformations (hereafter called Poincaré gauge transformations) was first studied by Kibble.[2)] He introduced two kinds of gauge fields, i.e., the translation $c_k{}^{\mu}$ and the Lorentz $A_{km}{}^{\mu}$ gauge fields, to keep a system invariant under the Poincaré gauge transformations. However, he assumed only the Lagrangian linear in the Lorentz gauge field strength by analogy with Einstein gravity. This type Lagrangian does not contain any kinetic terms for the Lorentz gauge field. Accordingly, the Lorentz gauge field was not a propagating field in his theory.

Against it, Hayashi[3)] proposed the most general Lagrangian quadratic or less in the first derivatives of $c_k{}^{\mu}$ and $A_{km}{}^{\mu}$, and suggested the existence of propagating massive as well as massless gauge fields.

In this paper we shall adopt the Lagrangian proposed by Hayashi, and develop an argument in terms of spinors instead of usual tensors.

A spinor technique adopted here has been growing up with general relativity,[10)~16)] and especially used successfully to the study of gravitational radiation.[17)~20)]

Accordingly, it seems that a spinor formalism is more useful for an argument of propagating massless gauge fields. (Under this formalism we will investigate the possible existence of massless Lorentz gauge fields propagating with positive energy in linear field approximation in the forthcoming paper,)

In § 2 we recapitulate Kibble-Hayashi's method, and deduce five identities according to Noether's theorem.[21)] § 3 is devoted to preparations for spinor approach. In § 4 we present a spinor formalism to Poincaré gauge theory. The last section is devoted to concluding remark.

§ 2. Preliminaries

As preparations to later section, we review briefly the Poincaré gauge theory in this section.

2.1. Action

we consider a set of matter fields $q = \{ q^A / A = 1, 2, \cdot\cdot\cdot, N \}$ with the Lagrangian density

$$L_M = L_M(q, q_{,K}). \qquad (q_{,K} = \partial q / \partial x^K)$$

The action $\int d^4x L_M$ is assumed to be invariant under Poincaré group.

Let us now postulate the invariance of the action under Poincaré gauge transformations which are defined by replacing ten parameters in ordinary Poincaré transformations by arbitrary functions of the coordinates: *)

$$\delta x^\mu = \xi^\mu(x) \qquad (2.1.1)$$

$$\delta q(x) = (i/2)\,\omega_{km}(x)\; S^{km}\; q(x) \qquad (2.1.2)$$

Here $\xi^\mu(x)$ and $\omega_{km}(x)$ are ten arbitrary infinitesimal functions of the coordinates, and the S^{km} are six infinitesimal generators of the Lorentz group, satisfying the commutation relations

$$[S^{km}, S^{np}] = i\,(\eta^{kn} S^{mp} + \eta^{mp} S^{kn} - \eta^{kp} S^{mn} - \eta^{mn} S^{kp}). \qquad (2.1.3)$$

*) We use the Greek indices for the coordinate indices and the Latin indices, for the local Lorentz indices. The Latin indices are raised or lowered with the flat-space metric $\eta_{km} = diag.(1, -1, -1, -1)$.

In order to keep the action invariant, the derivative $q_{,K}$ must be replaced by the "covariant" one $D_K q$ in the original Lagrangian density L_M. Here $D_K q$ is defined by introducing the two gauge fields, that is, the translational gauge field $c_k{}^\mu$ and the Lorentz gauge field $A_{km\mu} (= -A_{mk\mu})$, namely

$$D_k q = b_k{}^\mu \{q_{,\mu} + (i/2) A_{km\mu} S^{km} q\} \qquad (2.1.4)$$

with

$$b_k{}^\mu = \delta_k{}^\mu + c_k{}^\mu .$$

The field $b_k{}^\mu$, which is defined in terms of $c_k{}^\mu$, is called the vier-bein or tetrad field.

The Poincaré gauge-invariant action is given by

$$I_M = \int d^4x \, L_M (q, D_K q), \qquad (2.1.5)$$

where $L_M = bL_M (q, D_k q)$ with $b = -\det(b_{k\mu})$.

The field $b_{k\mu}$ *is the inverse of* $b^{k\mu}$, satisfying

$$b_{k\mu} b^{k\nu} = \delta_\mu{}^\nu , \qquad b_{k\mu} b^{m\mu} = \delta_k{}^m . \qquad (2.1.6)$$

The two field strengths $\mathscr{C}_{kmn}$ and F_{kmnp} for the translation and Lorentz gauge fields are obtained by calculating the commutator $(D_m D_n - D_n D_m)$q. We easily find

$$(D_m D_n - D_n D_m) q = (i/2) F_{klmn} S^{kl} q + \mathscr{C}_{kmn} D^k q \qquad (2.1.7)$$

where*) $\mathscr{C}_{kmn} = C_{kmn} + 2A_{k(mn)}$, (2.1.8)

$$F_{kmnp} = 2(A_{km\nu,\mu} + A_{kr\mu} A^r{}_{m\nu}) \, b_{[n}{}^\mu b_{p]}{}^\nu \qquad (2.1.9)$$

with $c_{kmn} = 2b_{k\mu,\nu} \, b_{[m}{}^\mu b_{n]}{}^\nu$, (2.1.10)

$$A_{kmn} = A_{km\mu} b_n{}^\mu . \qquad (2.1.11)$$

*) We adopt the standard convension that round brackets around indicies denote that the symmetric part is being taken and square brackets, the antisymmetric part.

The Poincaré gauge-invariant action for the gauge fields is determined in terms of the two field strenghs $\mathscr{C}_{kmn}$, F_{kmnp} by

$$I_G = \int d^4x \, b(\alpha \, {}^T\mathscr{C}_{kmn} \, {}^T\mathscr{C}^{kmn} + \beta \, {}^V\mathscr{C}_k \, {}^V\mathscr{C}^k + \gamma \, {}^A\mathscr{C}_k \, {}^A\mathscr{C}^k$$

$$+ a_1 A_{kmnp} A^{kmnp} + a_2 B_{kmnp} B^{kmnp} + a_3 C_{kmnp} C^{kmnp} + a_4 E_{km} E^{km} + a_5 G_{km} G^{km} + a_6 F^2 + aF) \tag{2. 1. 12}$$

where a, α, β, γ and a_i $(i = 1, 2, \cdot\cdot\cdot, 6)$ are constant parameters (only five of six parameters a_i are independent),[9] and ${}^{T}\mathscr{C}_{kmn}, \cdot\cdot\cdot ; A_{kmnp}, \cdot\cdot\cdot, F$, which are the irreducible components of Lorentz (local) 4-tensors $\mathscr{C}_{kmn}$ and F_{kmnp} respectively, are defined by

$${}^{T}\mathscr{C}_{kmn} = \mathscr{C}_{(km)n} - (1/3)(\eta_{km}{}^{V}\mathscr{C}_n - \eta_n({}_k{}^{V}\mathscr{C}_{m)}), \tag{2. 1. 13}$$

$${}^{V}\mathscr{C}_k = \mathscr{C}^m{}_{mk}, \tag{2. 1. 14}$$

$${}^{A}\mathscr{C}_K = (1/3!)\,\varepsilon_{kmnp}\mathscr{C}^{mnp}, \tag{2. 1. 15}$$

and

$$A_{kmnp} = (1/6)(F_{kmnp} - F_{knmp} + F_{kpmn} - F_{mpkn} + F_{npkm} - F_{nmkp}), \tag{2. 1. 16}$$

$$B_{kmnp} = (1/4)(D_{kmnp} + D_{kpnm} + D_{npkm} + D_{nmkp}), \tag{2. 1. 17}$$

$$C_{kmnp} = (1/2)(D_{kmnp} - D_{npkm}), \tag{2. 1. 18}$$

$$E_{km} = F_{[km]}, \tag{2. 1. 19}$$

$$G_{km} = F_{(km)} - (1/4)\,\eta_{km} F \tag{2. 1. 20}$$

with

$$F_{km} = \eta^{np} F_{knmp},$$

$$F = \eta^{km} F_{km} \tag{2. 1. 22}$$

and

$$D_{kmnp} = F_{kmnp} - (1/2)(F_{kn}\eta_{mp} + F_{mp}\eta_{kn} - F_{mn}\eta_{kp} - F_{kp}\eta_{mn} + (1/6)(\eta_{kn}\eta_{mp} - \eta_{kp}\eta_{mn})F. \tag{2. 1. 23}$$

Here ε_{kmnp} is a completely antisymmetric Lorentz tensor and $\varepsilon^{0123} = 1$, $\varepsilon_{0123} = -1$.

When making use of the identity*)

$$-R = (2/3)\,{}^{T}C_{kmn}{}^{T}C^{kmn} - (2/3)\,{}^{V}C_k{}^{V}C^k + (3/2)\,{}^{A}C_k{}^{A}C^k + b^{-1}(2bb^{m\mu}\ {}^{V}C_m)_{,\mu}, \tag{2. 1. 24}$$

then the above action (2. 1. 12) will be rewritten in more useful form

$$I_G = \int d^4x\ \{abR + L_{\mathscr{C}} + L_F + (2abb^{m\mu}\ {}^{V}\mathscr{C}_m)_{,\mu}\} \tag{2. 1. 25}$$

where

$$L_{\mathscr{C}} = bL_{\mathscr{C}} = b(c_1{}^{T}\mathscr{C}_{kmn}{}^{T}\mathscr{C}^{kmn} + c_2{}^{V}\mathscr{C}_k{}^{V}\mathscr{C}^k + c_3{}^{A}\mathscr{C}_k{}^{A}\mathscr{C}^k) \tag{2. 1. 26}$$

with

$$c_1 = \alpha + (2/3)\,a, \quad c_2 = \beta - (2/3)\,a, \quad C_3 = \gamma + (3/2)\,a,$$

and

$$L_F = bL_F = b(a_1 A_{kmnp} A^{kmnp} + a_2 B_{kmnp} B^{kmnp} + a_3 C_{kmnp} C^{kmnp} + a_4 E_{km} E^{km} + a_5 G_{km} G^{km} + a_6 F^2). \qquad (2.1.27)$$

Combining I_M of (2.1.5) and I_G of (2.1.25), we get the Poincaré gauge-invariant action for the whole system:

$$I = I_M + I_G = \int d^4x \{L + (2abb^{m\mu} V\mathscr{C}_m)_{,\mu}\} \qquad (2.1.28)$$

with

$$L = bL = L_M + abR + L_{\mathscr{C}} + L_F. \qquad (2.1.29)$$

2.2. Field equations From the action I of (2.1.28), we get the three field equations by the variational principle.

One of them is that for a matter field q, which is determined with the concrete expression for L_M.

*) R is a Riemann scalar curvature defined by the metric $g_{\mu\nu} = b_{k\mu} b^k{}_\nu$, and ${}^T C_{kmn}$, etc., are the components of C_{kmn} just defined like as ${}^T\mathscr{C}_{kmn}$, etc.,.

Another one is that for the field $b_{k\mu}$:

$$2aG^{km} = -T^{km} \qquad (2.2.1)$$

where G^{km} is an Einstein tensor $G_{\alpha\beta}$ ($= R_{\alpha\beta} - (1/2) g_{\alpha\beta} R$) in local form, and T_{km} is a local form*) of the energy-momentum tensor $T_{\mu\nu}$ for a whole system except for the Einstein gravity, namely

$$T^{km} = b^K{}_\mu b^m{}_\nu T^{\mu\nu} = T_{(M)}{}^{km} + T_{(\mathscr{C})}{}^{km} + T_{(F)}{}^{km} \qquad (2.2.2)^{**)}$$

with

$$T_{(M)}{}^{km} = \frac{\partial L_M}{\partial D_m q} D^k q - \eta^{km} L_M, \qquad (2.2.3)$$

$$T_{(\mathscr{C})}{}^{km} = \nabla_n I^{kmn} + K_{nr}{}^k I^{[nr]m} - \eta^{km} L_{(\mathscr{C})} \qquad (2.2.4)$$

and

$$T_{(F)}{}^{km} = F_{pqn}{}^k H^{pqnm} - \eta^{km} L_F. \qquad (2.2.5)$$

Incidentally, the conservation law of energy-momentum can be then expressed in the form

$$(a \underset{(G)}{\tau}{}^{\mu\nu} + \tau^{\mu\nu})_{,\mu} = 0, \qquad (2.2.6)$$

where

$$\tau^{\mu\nu} = b\ T^{\mu\nu,}$$

$$\underset{(G)}{\tau^{\mu}}{}_{\nu} = \frac{\partial G}{\partial b_{k}{}^{\lambda},{}_{\mu}} b_{k}{}^{\lambda},{}_{\nu} - \delta^{\mu}{}_{\nu} G.$$

*) The transposition of any tensor with coordinate indices (called world tensor) into the local form (called Lorentz tensor) is generally performed by using the field $b_{k}{}^{\mu}$.

**) T^{km} is essentially symmetric, because of the Lorentz gauge invariance of system.

Here G is a part of bR, not containing the second derivatives of $g_{\mu\nu}$.

The remainder is that for the Lorentz gauge field $A_{km\mu}$:

$$\nabla_p H^{kmnp} + \Delta^{k}{}_{rp} H^{rmnp} + \Delta^{m}{}_{rp} H^{krnp} = - S^{kmn}, \qquad (2.\ 2.\ 7)$$

where $S^{km\mu} = (S^{kmn} b_{n}{}^{\mu})$ is a spin-angular momentum tensor for a whole system:

$$S^{km\mu} = S_{(M)}{}^{km\mu} + S_{(\mathscr{C})}{}^{km\mu} + S_{(F)}{}^{km\mu} \qquad (2.\ 2.\ 8)$$

with

$$S_{(M)}{}^{km\mu} = \frac{\partial L_M}{\partial A_{km\mu}} = (i/2) b_1{}^{\mu} \frac{\partial L_M}{\partial D_1 q} S^{km} q, \qquad (2.\ 2.\ 9)$$

$$S_{(\mathscr{C})}{}^{km\mu} = \frac{\partial L_{\mathscr{C}}}{\partial A_{km\mu}} = b_n{}^{\mu} I^{[km]n}, \qquad (2.\ 2.\ 10)$$

and

$$S_{(F)}{}^{km\mu} = \frac{\partial L_F}{\partial A_{km\mu}} = b_n{}^{\mu} b_p{}^{\nu} (H^{krnp} A^{m}{}_{r\nu} - H^{mrnp} A^{k}{}_{r\nu}), \qquad (2.\ 2.\ 11)$$

and in terms of it, the conservation law of spin-angular momentum can be described in the form

$$(b\ S^{km\mu}),{}_{\mu} = 0. \qquad (2.\ 2.\ 12)$$

Here I^{kmn}, H^{kmnp}, and their "covariant" derivatives $\nabla_n I^{kmn}$ and $\nabla_p H^{kmnp}$ are defined as follows:

$$I^{kmn} = 2\frac{\partial L_{\mathscr{C}}}{\partial \mathscr{C}_{kmn}} = 4c_1{}^{T}\mathscr{C}^{k[mn]} + 4c_2 \eta^{k[m} V\mathscr{C}^{n]}$$

$$-(2/3) c_3 \varepsilon^{kmnp} {}^{A}\mathscr{C}_p, \qquad (2.\ 2.\ 13)$$

$$H_{kmnp}=2\frac{\partial L_F}{\partial F_{kmnp}}=4a_1A^{kmnp}+4a_2B^{km[np]}+4a_3C^{kmnp}$$

$$+ a_4 (E^{kn}\eta^{mp} + E^{mp}\eta^{kn} - E^{kp}\eta^{mn} - E^{mn}\eta^{kp})$$
$$+ a_5(G^{kn}\eta^{mp} + G^{mp}\eta^{kn} - G^{kp}\eta^{mn} - G^{mn}\eta^{kp})$$
$$+2a_6 (\eta^{kn}\eta^{mp} - \eta^{kp}\eta^{mn}) F \quad (2.\ 2.\ 14)$$

and

$$\nabla_n I^{kmn} = I^{kmn}{}_{,n} - \Delta^k{}_{rn}I^{rmn} - \Delta^m{}_{rn}I^{krn} - \Delta^n{}_{rn}I^{kmr}, \quad (2.\ 2.\ 15)$$

$$\nabla_p H^{kmnp} = H^{kmnp}{}_{,p} - \Delta^k{}_{rp}H^{rmnp} - \Delta^m{}_{rp}H^{krnp}$$
$$-\Delta^n{}_{rp}H^{kmrp} - \Delta^p{}_{rp}H^{kmnr} \quad (2.\ 2.\ 16)$$

with

$$\Delta_{kmn} = \Delta_{km\mu}\, b_n{}^{\mu} = (1/2)(C_{kmn} + C_{mnk} + C_{nmk})$$
$$= -\Delta_{mkn}. \quad (2.\ 2.\ 17)$$

The "covariant" derivative ∇_k is generally defined by

$$\nabla_k\, q = b_k{}^{\mu}\nabla_{\mu}\, q,$$
$$\nabla_{\mu}\, q = q,_{\mu} - (i/2)\Delta_{mn\mu}S^{mn}q. \quad (2.\ 2.\ 18)$$

Then the "covariant" derivative D_k of (2. 1. 4) is related to the ∇_k, using the relation $A_{kmn} = K_{kmn} - \Delta_{kmn}$: (2. 2. 19)

$$D_k q = \nabla_k q + (i/2)K_{mnk}S^{mn}q \quad (2.\ 2.\ 20)$$

with

$$K_{kmn} = K_{km\mu}\, b_n{}^{\mu} = (1/2)(\mathscr{C}_{kmn} + \mathscr{C}_{mnk} + \mathscr{C}_{nmk})$$
$$= -K_{mkn}. \quad (2.\ 2.\ 21)$$

2.3. Identities The action I of (2. 1. 28) is invariant under the Poincaré gauge transformations, even if the divergence term is neglected. From this invariance of the action I, the following identities are obtained in accordance with Noether's theorem:[21)*)]

$$\nabla_p F^{+kmnp} + K^k{}_{rp}F^{+rmnp} + K^m{}_{rp}F^{+krnp} = 0, \quad (2.\ 3.\ 1)$$
$$\nabla_n \mathscr{C}^{+}{}_{km}{}^{n} - K_{krn}\mathscr{C}^{+rn}{}_{m} = -F^{+}{}_{knm}{}^{n}, \quad (2.\ 3.\ 2)$$
$$\nabla_p R^{+kmnp} = 0, \quad (2.\ 3.\ 3)$$
$$R^{+kmn}{}_{m} = 0 \quad (2.\ 3.\ 4)$$

and

$$F^k{}_{rnp}H^{mrnp} - F^m{}_{rnp}H^{krnp} = F_{rnp}{}^{m}H^{rnpk} - F_{rnp}{}^{k}H^{rnpm}, \quad (2.\ 3.\ 5)$$

where it should be noticed that the Poincaré gauge invariance is preserved, even if any of L_{M}, abR, $L_{(\mathscr{C})}$ and L_{F} is substituted for L in the action I, and that + symbol on the right-above of F, $\mathscr{C}$ and R stands for the right duals of them, e. g.,

$$F^{+\mathrm{kmnp}} = (1/2\,!)\, F^{\mathrm{kmrs}}\, \varepsilon_{\mathrm{rs}}{}^{\mathrm{np}}, \text{ etc.,.}$$

*) It should be remarked that the "covariant" derivatives of η_{km}, $\varepsilon_{\mathrm{kmnp}}$ and their associates, i.e., $\nabla_{\mathrm{n}}\eta_{\mathrm{km}}$, $\nabla_{\mathrm{r}}\varepsilon_{\mathrm{kmnp}}$, etc., are all zero.

Substituting (2.2.19) for A_{kmn} in F_{kmnp}, F_{kmnp} is resolved like

$$\mathrm{F}_{\mathrm{kmnp}} = \mathrm{R}_{\mathrm{kmnp}} + \mathrm{F}_{\mathrm{kmnp}}\,(\mathrm{K}) \tag{2. 3. 6}$$

with

$$R_{kmnp} = 2\,(\varDelta_{km\mu,\nu} + \varDelta_{kr\mu}\varDelta^{r}{}_{m\nu})\, b_{n}{}^{[\mu}\, b_{p}{}^{\nu]} \tag{2. 3. 7}$$

and

$$F_{\mathrm{kmnp}}(K) = \nabla_{\mathrm{n}}K_{\mathrm{kmp}} - \nabla_{\mathrm{p}}K_{\mathrm{kmn}} + K_{\mathrm{krn}}\, K^{\mathrm{r}}{}_{\mathrm{mp}} - K_{\mathrm{krp}}\, K^{\mathrm{r}}{}_{\mathrm{mn}}. \tag{2. 3. 8}$$

§ 3. Spinor Approach

In this section, we shall obtain the expressions (2. 2. 1), (2. 2. 7), (2. 3. 1) and (2. 3. 8) written out in terms of spinors. The next some subsections are devoted to preparations to do so.

3.1. Fundamental spinor We consider the so-called spinor as a vector (called spin-vector) in spinor space S_2 which is a two-dimensional linear vector space over the field of complex numbers with an antisymmetric inner product, and with a conjugate space S_2^* associated with it. An any order spin-tensor is then defined in the same way as the ordinary tensor.

An n-th order spin-tensor is also called a spinor of n-th rank, simply a spinor in the case that there is no danger to be confused.

We denote the contravariant component of a spin-vector **u** in any frame of reference in S_2 by $u^{A}\,(A = 0,\ 1)$, and the corresponding of the conjugate **u*** by $u^{\dot{A}}\ (\dot{A} = \dot{0}\,,\,\dot{1})$. Then their covariant components u_{A} and $u_{\dot{A}}$ are respectively got by

$$u_{A} = u^{B}\ \varepsilon_{BA} \quad \text{and} \quad u_{\dot{A}} = u^{\dot{B}}\ \varepsilon_{\dot{B}\dot{A}}, \tag{3. 1. 1}$$

making use of the fundamental spinors ε_{AB} and $\varepsilon_{\dot{A}\dot{B}}$, which can be selected to be

$$\varepsilon_{AB}=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}=\varepsilon_{\dot{A}\dot{B}}\,. \tag{3. 1. 2}$$

Conversely, when the covariant components u_A and $u_{\dot{A}}$ are known, their contravariant components u^A and $u^{\dot{A}}$ are introduced by

$$u^A=\varepsilon^{AB}u_B \qquad \text{and} \qquad u^{\dot{A}}=\varepsilon^{\dot{A}\dot{B}}u_{\dot{B}} \tag{3. 1. 3}$$

where ε^{AB} and $\varepsilon^{\dot{A}\dot{B}}$ are the inverse of ε_{AB} and $\varepsilon_{\dot{A}\dot{B}}$, respectively:

$$\varepsilon^{AB}\varepsilon_{BC}=\varepsilon^{A}{}_{C}=-\varepsilon_{C}{}^{A}=-\delta_{C}{}^{A} \tag{3. 1. 4}$$

and the same one with dotted indices.

We shall often use the following relations involving the fudamental spinor:

$$\varepsilon_{AB}\varepsilon_{CD}+\varepsilon_{CA}\varepsilon_{BD}+\varepsilon_{BC}\varepsilon_{AD}=0, \tag{3. 1. 5}$$

$$\xi_A\varepsilon_{BC}+\xi_B\varepsilon_{CA}+\xi_C\varepsilon_{AB}=0, \tag{3. 1. 6}$$

$$\eta_{AB}-\eta_{BA}=\eta_{E}{}^{E}\varepsilon_{AB} \tag{3. 1. 7}$$

where ξ_A and η_{AB} may be arbitrary spinors of any rank.

Lastly, it should be noted that the norm of any spinor of odd rank is zero, because of the antisymmetry of the fundamental spinor. For example,

$$\xi_A\xi^A=\varepsilon_{AB}\xi^A\xi^B=-\xi^A\xi_A=0.$$

3.2 spinor equivalents of tensors The correspondence between tensors and spinors is obtained by making use of a mixed quantity $\sigma^{\mu}{}_{\dot{A}B}=b_{k}{}^{\mu}\,\sigma^{k}{}_{\dot{A}B}$, where $\sigma^{k}{}_{\dot{A}B}$ is algebraically determined by the following equations:

$$\sigma^{k\dot{A}B}=\sigma^{kB\dot{A}} \tag{3. 2. 1}$$

$$\sigma^{k}{}_{\dot{A}B}\,\sigma^{m\dot{A}}{}_{C}+\sigma^{m}{}_{\dot{A}B}\,\sigma^{k\dot{A}}{}_{C}=\eta^{km}\,\varepsilon_{BC}. \tag{3. 2. 2}$$

The spinor equivalent of any tensor is a quantity which has a dotted and an undotted spinor index for each tensor index. For example, the spinor representing the tensor $F_{\mu\nu}\,(=-F_{\nu\mu})$ (which is assumed to be real, and hereafter we shall treat only with real tensors) is given by

$$F_{\dot{A}B\dot{C}D}=F_{\mu\nu}\,\sigma^{\mu}{}_{\dot{A}B}\,\sigma^{\nu}{}_{\dot{C}D} \tag{3. 2. 3}$$

Conversely, the tensor $F_{\mu\nu}$ is expressed in terms of its spinor equivalent $F_{\dot{A}B\dot{C}D}$

$$F_{\mu\nu}=\sigma_{\mu}{}^{\dot{A}B}\,\sigma_{\nu}{}^{\dot{C}D}\,F_{\dot{A}B\dot{C}D}. \tag{3. 2. 4}$$

The spinor $F_{\dot{A}B\dot{C}D}$ has the symmetry

$$F_{\dot{A}B\dot{C}D}=-F_{\dot{C}D\dot{A}B}, \tag{3. 2. 5}$$

because of the antisymmetric property of the tensor $F_{\mu\nu}$, and as a result, we obtain the identity

$$F_{\dot{A}B\dot{C}D} = (1/2)\ (F_{\dot{A}B\dot{C}D} - F_{\dot{C}B\dot{A}D}) + (1/2)\ (F_{\dot{C}B\dot{A}D} - F_{\dot{C}D\dot{A}B}).$$

Applying (3. 1. 7) to two terms in the right-hand side, this identity can be also rewritten

$$F_{\dot{A}B\dot{C}D} = \varepsilon_{\dot{A}\dot{C}}\, \phi_{BD} + \varepsilon_{BD}\, \phi_{\dot{A}\dot{C}}, \tag{3. 2. 6}$$

where

$$\phi_{AB} = (1/2)\ F_{\dot{E}A}{}^{\dot{E}}{}_{B} = -\ (1/2)\ F^{\dot{E}}{}_{B\dot{E}A} = (1/2)\ F_{\dot{E}B}{}^{\dot{E}}{}_{A} = \phi_{BA}. \tag{3. 2. 7}$$

The spinor ϕ_{AB} is symmetric. On the other hand, taking the complex conjugate of ϕ_{AB} gives

$$\phi_{\dot{A}\dot{B}} = [\phi_{AB}]^{*} = (1/2)\ [F_{\dot{E}A}{}^{\dot{E}}{}_{B}]^{*} = (1/2)\ F_{\dot{A}E\dot{B}}{}^{E},$$

because of the reality of $F_{\mu\nu}$ and the Hermitian property (3. 2. 1) of $\sigma^{k\dot{A}B}$.

We thus see that the antisymmetric tensor $F_{\mu\nu}$ is equivalent to a symmetric spinor ϕ_{AB}. Incidentally, it is well-known that a symmetric spinor with two indices describes a vector field.

We are now in a position to consider the spinor equivalents*) of field strengths F_{kmnp} and $\mathscr{C}_{kmn}$.

From their definitions, it is easily known that the fields F_{kmnp} and $\mathscr{C}_{kmn}$ have the following symmetry properties:

$$F_{kmnp} = -\ F_{mknp} = -\ F_{kmpn}, \tag{3. 2. 8}$$

and

$$\mathscr{C}_{kmn} = -\ \mathscr{C}_{knm}. \tag{3. 2. 9}$$

Because of these symmetry properties, following the procedure outlined above for the tensor $F_{km}\ (= b_{k}{}^{\mu}\ b_{m}{}^{\mu}\ F_{\mu\nu})$, we see that the field strength F_{kmnp} is equivalent to three kinds of spinors, Ψ_{ABCD}, $X_{\dot{A}BCD}$, Φ_{AB} and one complex scalar Λ, and the field strength $\mathscr{C}_{kmn}$, equivalent to two types of spinors, $\psi_{\dot{A}BCD}$ and $\varphi_{\dot{A}B}$. Here Ψ_{ABCD} and Φ_{AB} are completely symmetric in their all indices, and $\psi_{\dot{A}BCD}$, in its three indices B, C, D.

On the other hand, $X_{\dot{A}BCD}$ is an irreducible spinor having the symmetry:

$$X_{\dot{A}BCD} = X_{\dot{B}ACD} = X_{\dot{A}BDC}. \tag{3. 2. 10}$$

Finally, it should be remarked that an irreducible spinor $\varphi_{\dot{A}B}$ is equivalent to a complex vector, or real two vectors.

By actually performing calculations, we find out

$$ {}^{V}\mathscr{C}_{\dot{A}B} = -3\,(\varphi_{\dot{A}B} + \varphi_{\dot{B}A}) \qquad (3.\ 2.\ 11)$$

and

$$ {}^{A}\mathscr{C}_{\dot{A}B} = -\,\mathrm{i}\,(\varphi_{\dot{A}B} - \varphi_{\dot{B}A}), \qquad (3.\ 2.\ 12)$$

where ${}^{V}\mathscr{C}_{\dot{A}B}$ and ${}^{A}\mathscr{C}_{\dot{A}B}$ are the spinor equivalents of the irreducible components, ${}^{V}\mathscr{C}_{k}$ and ${}^{A}\mathscr{C}_{k}$ of $\mathscr{C}_{kmn}$, and $\varphi_{A\dot{B}}$ is the complex conjugate of $\varphi_{\dot{A}B}$.

*) The spinor equivalent of a local tensor is defined by using the quantity $\sigma^{k}{}_{\dot{A}B}$ for $\sigma^{\mu}{}_{\dot{A}B}$.

In terms of these spinors, the spinor equivalents, $F_{\dot{A}B\dot{C}D\dot{E}F\dot{G}H}$ and $\mathscr{C}_{\dot{A}B\dot{C}D\dot{E}F}$, of F_{kmnp} and $\mathscr{C}_{kmn}$ are represented by

$$\begin{aligned} F_{\dot{A}B\dot{C}D\dot{E}F\dot{G}H} = {} & \{\Psi_{BDFH} + \Phi_{BH}\,\varepsilon_{DF} + \Phi_{DF}\,\varepsilon_{BH} \\ & + \Lambda\,(\varepsilon_{BF}\,\varepsilon_{DH} + \varepsilon_{BH}\,\varepsilon_{DF})\}\ \varepsilon_{\dot{A}\dot{C}}\,\varepsilon_{\dot{E}\dot{G}} \\ & + \{\Psi_{\dot{A}\dot{C}\dot{E}\dot{G}} + \Phi_{\dot{A}\dot{G}}\,\varepsilon_{\dot{C}\dot{E}} + \Phi_{\dot{C}\dot{E}}\,\varepsilon_{\dot{A}\dot{G}} \\ & + \Lambda^{*}\,(\varepsilon_{\dot{A}\dot{E}}\,\varepsilon_{\dot{C}\dot{G}} + \varepsilon_{\dot{A}\dot{G}}\,\varepsilon_{\dot{C}\dot{E}})\}\ \varepsilon_{BD}\,\varepsilon_{FH} \\ & + X_{\dot{A}\dot{C}FH}\,\varepsilon_{BD}\,\varepsilon_{\dot{E}\dot{G}} + X_{BD\dot{E}\dot{G}}\,\varepsilon_{\dot{A}\dot{C}}\,\varepsilon_{FH} \end{aligned} \qquad (3.\ 2.\ 13)$$

and

$$\begin{aligned} \mathscr{C}_{\dot{A}B\dot{C}D\dot{E}F} = {} & \{\psi_{\dot{A}BDF} + \varphi_{\dot{A}F}\,\varepsilon_{BD} + \varphi_{\dot{A}D}\,\varepsilon_{BF}\}\ \varepsilon_{\dot{C}\dot{E}} \\ & + \{\psi_{B\dot{A}\dot{C}\dot{E}} + \varphi_{B\dot{E}}\,\varepsilon_{\dot{A}\dot{C}} + \varphi_{B\dot{C}}\,\varepsilon_{\dot{A}\dot{E}}\}\ \varepsilon_{DF}, \end{aligned} \qquad (3.\ 2.\ 14)$$

where $\Psi_{\dot{A}\dot{B}\dot{C}\dot{D}}$, $\Phi_{\dot{A}\dot{B}}$, $X_{AB\dot{C}\dot{D}}$ and $\psi_{A\dot{B}\dot{C}\dot{D}}$ are the complex conjugates of the corresponding quantities.

3.3 Spin-connection In this subsection, we consider the "covariant" derivative ∇_{μ} of a spinor.

In such a case that a matter field q is a spin-vector u_{A}, we see from eq. (2. 2. 18) that

$$\nabla_{\mu}\,u_{A} = u_{A},_{\mu} + (1/2))\ \Delta_{km\mu}\,\sigma^{k\dot{E}B}\,\sigma^{m}{}_{\dot{E}A}\,u_{B}, \qquad (3.\ 3.\ 1)$$

on account of $(S^{km})_{A}{}^{B} = i\sigma^{k\dot{E}B}\,\sigma^{m}{}_{\dot{E}A}$.

Here, let us put

$$\Delta_{\dot{A}B\dot{C}D\mu} = \Delta_{km\mu}\,\sigma^{k}{}_{\dot{A}B}\,\sigma^{m}{}_{\dot{C}D}, \qquad (3.\ 3.\ 2)$$

then, because of the antisymmetry $\Delta_{km\mu} = -\,\Delta_{mk\mu}$

$\Delta_{\dot{A}B\dot{C}D\mu}$ is reduced just like as $F_{\dot{A}B\dot{C}D}$ in (3. 2. 6):

$$\Delta_{\dot{A}B\dot{C}D\mu} = \varepsilon_{\dot{A}\dot{C}} \Gamma_{BD\mu} + \varepsilon_{BD} \Gamma_{\dot{A}\dot{C}\mu}, \tag{3. 3. 3}$$

with

$$\Gamma_{AB\mu} = (1/2) \Delta_{\dot{E}A}{}^{\dot{E}}{}_{B\mu} \ \ (= \Gamma_{BA\mu}) \tag{3. 3. 4}$$

and its complex cojugate $\Gamma_{\dot{A}\dot{B}\mu}$.

We can thus rewrite the expression (3.3.1) as

$$\nabla_\mu \ u_A = u_{A,\ \mu} - \Gamma^B{}_{A\mu} \ u_B. \tag{3. 3. 5}$$

On the other hand, using this expression (3. 3. 5), we obtain the similar expression for $\nabla_\mu \ v^A$, where v^A is a contravariant component of a spinor v:

$$\nabla_\mu \ v^A = v^A{}_{,\ \mu} + \Gamma^A{}_{B\mu} \ v^B, \tag{3. 3. 6}$$

noting the relation

$$\nabla_\mu \ (u_A \ v^A) = (u_A \ v^A)_{,\ \mu}.$$

Generally, assuming the Leibniz rule, a similar formula for any spinor is obtained. Using the formula, it is shown that the "covariant" derivatives, $\nabla_\mu \ \varepsilon_{AB}$, $\nabla_\mu \ \sigma_k{}^{\dot{A}B}$ and their associates, are all zero:

$$\begin{aligned} \nabla_\mu \ \varepsilon_{AB} &= \varepsilon_{AB,\ \mu} - \Gamma^C{}_{A\mu} \ \varepsilon_{CB} - \Gamma^C{}_{B\mu} \ \varepsilon_{AC} \\ &= \Gamma_{AB\mu} - \Gamma_{BA\mu} = 0 \end{aligned} \tag{3. 3. 7}$$

and since, noting the relation $\nabla_\mu \ v^k = v^k{}_{,\ \mu} - \Delta^k{}_{m\mu} \ v^m$, for a spinor equivalent $v^{\dot{A}B}$ of vector v^k,

$$\begin{aligned} \nabla_\mu \ v^{\dot{A}B} &= \nabla_\mu \ (v^k \ \sigma_k{}^{\dot{A}B}) = \nabla_\mu \ v^k \cdot \sigma_k{}^{\dot{A}B} + v^k \ \nabla_\mu \ \sigma_k{}^{\dot{A}B} \\ &= v^{\dot{A}B}{}_{,\ \mu} + \Gamma^{\dot{A}}{}_{\dot{C}\mu} \ v^{\dot{C}B} + \Gamma^B{}_{D\mu} \ v^{\dot{A}D} = \nabla_\mu \ v^k \cdot \sigma_k{}^{\dot{A}B}, \end{aligned}$$

therefore

$$\begin{aligned} \nabla_\mu \ \sigma_k{}^{\dot{A}B} = \sigma_k{}^{\dot{A}B}{}_{,\ \mu} + \Delta^m{}_{k\mu} \ \sigma_m{}^{\dot{A}B} + \Gamma^{\dot{A}}{}_{\dot{C}\mu} \ \sigma_k{}^{\dot{C}B} \\ + \Gamma^B{}_{D\mu} \ \sigma_k{}^{\dot{A}D} = 0 \end{aligned} \tag{3. 3. 8}$$

and so on.

And also, as a result, we find out again

$$\nabla_\mu \ \eta_{km} = \nabla_\mu \ (\sigma_k{}^{\dot{A}B} \ \sigma_{m\dot{A}B}) = 0 \tag{3. 3. 9}$$

4. Spinor formalism

In this section, we shall find again the various expressions mentioned in section 2 in spinor form.

4.1 Field equations in spinor form First of all, we consider eq. (2. 2. 1).

According to the procedure given in the previous section, we know that a spinor equivalent $T_{\dot{A}B\dot{C}D}$ of a symmetric tensor T_{km}*) is reduced as

$$T_{\dot{A}B\dot{C}D} = \chi_{\dot{A}\dot{C}BD} + \lambda\varepsilon_{\dot{A}\dot{C}}\varepsilon_{BD}, \qquad (4.1.1)$$

where

$$\chi_{\dot{A}\dot{C}BD} = (1/2)\,(T_{\dot{A}B\dot{C}D} + T_{\dot{C}B\dot{A}D}) \qquad (4.1.2)$$

having the symmetry properties

$$\chi_{\dot{A}\dot{C}BD} = \chi_{\dot{C}\dot{A}BD} = \chi_{\dot{A}\dot{C}DB} = \chi_{BD\dot{A}\dot{C}} \qquad (4.1.3)$$

,and

$$\lambda = (1/4)\,T_{E\dot{F}}{}^{E\dot{F}} = (1/4)\,T^{m}{}_{m} = (1/4)\,T^{\mu}{}_{\mu}. \qquad (4.1.4)$$

*) T_{km} may not be manifestly symmetric, but essecially must be so, because of the Lorentz gauge invariance. Accordingly, it should be understood that T_{km} means a symmetric part of it, whenever it has not a manifest symmetry.

Now, we obtain the following two spinor equations, instead of one tensor equation (2. 2. 1):

$$\underset{(G)}{X}{}_{\dot{A}\dot{C}BD} = (1/4a)\,\chi_{\dot{A}\dot{C}BD} \qquad (4.1.5)$$

and

$$\underset{(G)}{\Lambda} = (1/12a)\,\lambda. \qquad (4.1.6)$$

Here $\underset{(G)}{X}{}_{\dot{A}\dot{C}BD}$ and $\underset{(G)}{\Lambda}$, together with a completely symmetric spinor $\Psi_{(G)ABCD}$, represent a spinor equivalent $R_{\dot{A}B\dot{C}D\dot{E}F\dot{G}H}$ of Riemann curvature tensor $R_{\mu\nu\lambda\kappa}$ ($= b^{k}{}_{\mu}\, b^{m}{}_{\nu}\, b^{n}{}_{\lambda}\, b^{p}{}_{\kappa}\, R_{kmnp}$) in an exactly similar way to (3. 2. 13) without Φ_{AB}. However, it should be remarked that $\underset{(G)}{\Lambda}$ is real unlike Λ, because of the additional symmetries of R_{kmnp}, i.e.,

$$R_{kmnp} = R_{npkm}$$

and

$$R_{kmnp} + R_{knpm} + R_{kpmn} = 0.$$

On the other hand, after the tedious calculation we see that $\chi_{\dot{A}\dot{C}BD}$ and λ are written in terms of irreducible spinors as follows:

$$\chi_{\dot{A}\dot{C}BD} = \underset{(M)}{\chi}{}_{\dot{A}\dot{C}BD} = \underset{(\mathscr{E})}{\chi}{}_{\dot{A}\dot{C}BD} + \underset{(F)}{\chi}{}_{\dot{A}\dot{C}BD} \qquad (4.1.7)$$

and

$$\lambda = \underset{(M)}{\lambda} + \underset{(\mathscr{E})}{\lambda} + \underset{(F)}{\lambda} \qquad (4.1.8)$$

with

$$
\begin{aligned}
\underset{(\mathscr{C})}{\chi}{}_{\dot{A}\dot{C}BD} = {} & (3/2)\, c_1 \ \{ \nabla_{\dot{A}F}\, \psi_{\dot{C}BD}{}^{F} + \nabla_{\dot{C}F}\, \psi_{\dot{A}BD}{}^{F} + \nabla_{\dot{E}B}\, \psi_{D\dot{A}\dot{C}}{}^{\dot{E}} \\
& + \nabla_{\dot{E}D}\, \psi_{B\dot{A}\dot{C}}{}^{\dot{E}} + \psi_{\dot{A}BEF}\, \psi_{\dot{C}D}{}^{EF} \\
& + \psi_{\dot{A}DEF}\, \psi_{\dot{C}B}{}^{EF} + \psi_{B\dot{A}\dot{E}\dot{F}}\, \psi_{D\dot{C}}{}^{\dot{E}\dot{F}} \\
& + \psi_{D\dot{A}\dot{E}\dot{F}}\, \psi_{B\dot{C}}{}^{\dot{E}\dot{F}} \} \ + \\
& + (3/2)\ c_2 \ \{ \nabla_{\dot{A}B} (\varphi_{\dot{C}D} + \varphi_{D\dot{C}}) + \nabla_{\dot{C}D} (\varphi_{\dot{A}B} + \varphi_{B\dot{A}}) \\
& + \nabla_{\dot{C}B} (\varphi_{\dot{A}D} + \varphi_{D\dot{A}}) + \nabla_{\dot{A}D} (\varphi_{\dot{C}B} + \varphi_{B\dot{C}}) \} \\
& + (1/12)\ (9c_1 + 36c_2 + 8c_3) \ \{ \psi_{\dot{A}BDF}\, \varphi_{\dot{C}}{}^{F} + \psi_{\dot{C}DBF}\, \varphi_{\dot{A}}{}^{F} \\
& + \psi_{B\dot{A}\dot{C}\dot{E}}\, \varphi_{D}{}^{\dot{E}} + \psi_{D\dot{A}\dot{C}\dot{E}}\, \varphi_{B}{}^{\dot{E}} \} \\
& + (1/12)\ (27c_1 + 36c_2 - 8c_3) \ \{ \psi_{\dot{A}BDF}\, \varphi^{F}{}_{\dot{C}} + \psi_{\dot{C}DBF}\, \varphi^{F}{}_{\dot{A}} \\
& + \psi_{B\dot{A}\dot{C}\dot{E}}\, \varphi^{\dot{E}}{}_{D} + \psi_{D\dot{A}\dot{C}\dot{E}}\, \psi^{\dot{E}}{}_{B} \} \\
& - (1/3)\ (9c_2 - c_3)\ \{ \varphi_{\dot{A}D}\, \varphi_{\dot{C}B} + \varphi_{\dot{C}D}\, \varphi_{\dot{A}B} + \varphi_{D\dot{A}}\, \varphi_{B\dot{C}} \\
& + \varphi_{D\dot{C}}\, \varphi_{B\dot{A}} \} \\
& - (1/3)\ (9c_2 + c_3)\ \{ \varphi_{\dot{A}D}\, \varphi_{B\dot{C}} + \varphi_{\dot{C}D}\, \varphi_{B\dot{A}} + \varphi_{D\dot{A}}\, \varphi_{\dot{C}B} \\
& + \varphi_{D\dot{C}}\, \varphi_{\dot{A}B} \} \ , \qquad (4.\ 1.\ 9)
\end{aligned}
$$

$$
\begin{aligned}
\underset{(F)}{\chi}{}_{\dot{A}\dot{C}BD} = {} & \Psi_{BDEF}\, (g_1\, X_{\dot{A}\dot{C}}{}^{EF} + g_2\, X^{EF}{}_{\dot{A}\dot{C}}) + \Psi_{\dot{A}\dot{C}\dot{E}\dot{F}}\, (g_1\, X_{BD}{}^{\dot{E}\dot{F}} + g_2\, X^{\dot{E}\dot{F}}{}_{BD}) \\
& + \Phi_{EB}\, (g_1\, X_{\dot{A}\dot{C}}{}^{E}{}_{D} + g_3\, X^{E}{}_{D\dot{A}\dot{C}} + \Phi_{ED} (g_1\, X_{\dot{A}\dot{C}}{}^{E}{}_{B} + g_3\, X^{E}{}_{B\dot{A}\dot{C}}) \\
& + \Phi_{\dot{E}\dot{A}} (g_1\, X_{BD}{}^{\dot{E}}{}_{\dot{C}} + g_3\, X^{\dot{E}}{}_{\dot{C}BD}) + \Phi_{\dot{E}\dot{C}} (g_1\, X_{BD}{}^{\dot{E}}{}_{\dot{A}} + g_3\, X^{\dot{E}}{}_{\dot{A}BD}) \\
& + g_4 (\Lambda - \Lambda^*) (X_{\dot{A}\dot{C}BD} - X_{BD\dot{A}\dot{C}}) + g_5 (\Lambda + \Lambda^*) ({}_{\dot{A}\dot{C}BD} + X_{BD\dot{A}\dot{C}}), \\
& \qquad (4.\ 1.\ 10)
\end{aligned}
$$

$$
\begin{aligned}
\underset{(\mathscr{C})}{\lambda} = {} & - (9/2)\, c_2 (\nabla_{\dot{E}F}\, \varphi^{\dot{E}F} + \nabla_{\dot{E}F}\, \varphi^{F\dot{E}}) \\
& - (3/4)\, c_1 (\psi_{\dot{I}JEF}\, \psi^{\dot{I}JEF} + \psi_{IJ\dot{E}\dot{F}}\, \psi^{IJ\dot{E}\dot{F}}) \\
& - (1/2)\, (9c_2 - c_3) (\varphi_{\dot{E}F}\, \varphi^{\dot{E}F} + \varphi_{E\dot{F}}\, \varphi^{E\dot{F}}) \\
& - (9c_2 + c_3)\ \ \varphi_{\dot{E}F}\, \varphi^{F\dot{E}} \qquad (4.\ 1.\ 11)
\end{aligned}
$$

$$\underset{(F)}{\lambda} = 0,$$

and $\underset{(M)}{\chi}{}_{\dot{A}\dot{C}BD}$ and $\underset{(M)}{\lambda}$ are not determined until a matter field q is concretely fixed. Here we put

$$
\begin{aligned}
g_1 &= 2(2a_3 - a_5), \\
g_2 &= -2(3a_2 + 2a_3 + a_5), \\
g_3 &= -2(2a_3 + 2a_4 + a_5), \\
g_4 &= 8(a_1 + a_3),
\end{aligned}
$$

and

$$g_5 = -4(a_5 + 12a_6).$$

Next, let us consider eq. (2. 2. 7), which is straightforwardly written out

$$\nabla_{\dot{G}H} H^{\dot{A}B\dot{C}D\dot{E}F\dot{G}H} + K^{\dot{A}B}{}_{\dot{K}L\dot{G}H} H^{\dot{K}L\dot{C}D\dot{E}F\dot{G}H} - K^{\dot{C}D}{}_{\dot{K}L\dot{G}H} H^{\dot{K}L\dot{A}B\dot{E}F\dot{G}H} + (1/2)(I^{\dot{A}B\dot{C}D\dot{E}F} - I^{\dot{C}D\dot{A}B\dot{E}F}) = - \underset{(M)}{S}{}^{\dot{A}B\dot{C}D\dot{E}F}.$$

Since this equation has a symmetric property just like as $\mathscr{C}_{\dot{A}B\dot{C}D\dot{E}F}$, it can be decomposed to the following two equations:

$$\begin{aligned} &8a_4 \nabla^{\dot{A}}{}_H \Phi^{BH} + 12a_4 \psi^{\dot{A}B}{}_{FH} \Phi^{FH} \\ &+ 2\nabla_{\dot{G}H} \{(2a_3 - a_5) X^{\dot{A}\dot{G}BH} - (2a_3 + a_5) X^{BH\dot{A}\dot{G}}\} \\ &\quad + 2\psi_{\dot{G}HF}{}^B \{(2a_3 - a_5) X^{\dot{A}\dot{G}FH} - (2a_3 + a_5) X^{FH\dot{A}\dot{G}}\} \\ &\quad + 2(\varphi_{\dot{G}H} + 3\varphi_{H\dot{G}}) \{(2a_3 - a_5) X^{\dot{A}\dot{G}BH} - (2a_3 + a_5) X^{BH\dot{A}\dot{G}}\} \\ &+ 12\nabla^{\dot{A}B} \{(a_1 + 6a_6)\Lambda - (a_1 - 6a_6)\Lambda^*\} \\ &\quad - 12(\varphi^{\dot{A}B} + 3\varphi^{B\dot{A}}) \{(a_1 + 6a_6)\Lambda - (a_1 - 6a_6)\Lambda^*\} \\ &+ 6a_2 \Psi^{BGFH} \psi^{\dot{A}}{}_{GFH} + (9c_2 + 2c_3)\varphi^{\dot{A}B} + (9c_2 - 2c_3)\varphi^{B\dot{A}} = 6 \underset{(M)}{\Theta}{}^{\dot{A}B} \end{aligned} \tag{4. 1. 12}$$

and

$$\begin{aligned} &6a_2 \nabla^{\dot{A}}{}_H \psi^{BCDH} + 3a_2(\varphi^{\dot{A}}{}_H + 3\varphi_H{}^{\dot{A}}) \Psi^{BCDH} + 12a_2 \psi^{\dot{A}(B}{}_{FH} \Psi^{CD)FH} \\ &+ 4a_4 \nabla^{\dot{A}(B} \Phi^{CD)} - 6a_4 (\varphi^{\dot{A}(B} + 3\varphi^{(B\dot{A}}) \Phi^{CD)} \\ &- 2\nabla_{\dot{G}}{}^{(B} \{(2a_3 - a_5) X^{\dot{A}\dot{G}CD)} - (2a_3 + a_5) X^{CD)\dot{A}\dot{G}}\} \\ &\quad + (\varphi_{\dot{G}}{}^{(B} + 3\varphi^{(B}{}_{\dot{G}}) \{(2a_3 - a_5) X^{\dot{A}\dot{G}CD)} - (2a_3 + a_5) X^{CD)\dot{A}\dot{G}}\} \\ &\quad - 4\psi_{\dot{G}H}{}^{(BC} \{(2a_3 - a_5) X^{\dot{A}\dot{G}H/D)} - (2a_3 + a_5) X^{D)H\dot{A}\dot{G}}\} - \\ &\qquad -8 \{(a_1 + 6a_6)\Lambda - (a_1 - 6a_6)\Lambda^*\} \psi^{\dot{A}BCD} - 3c_1 \psi^{\dot{A}BCD} = -2 \underset{(M)}{\Pi}{}^{\dot{A}BCD}, \end{aligned} \tag{4. 1. 13}$$ *)

where $\underset{(M)}{\Theta}{}_{\dot{A}B}$ and $\underset{(M)}{\Pi}{}_{\dot{A}BCD}$ are irreducible spinors of $\underset{(M)}{S}{}_{\dot{A}B\dot{C}D\dot{E}F}$, just defined like as $\varphi_{\dot{A}B}$ and $\psi_{\dot{A}BCD}$ in (3. 2. 14) respectively.

*) $\psi_{\dot{G}H}{}^{(BC} X^{\dot{A}\dot{G}H/D)} = (1/3) \{\psi_{\dot{G}H}{}^{BC} X^{\dot{A}\dot{G}HD} + \psi_{\dot{G}H}{}^{CD} X^{\dot{A}\dot{G}HB} + \psi_{\dot{G}H}{}^{DB} X^{\dot{A}\dot{G}HC}\}$

4.2. Identities in spinor form In this subsection, we shall consider only the identities (2. 3. 1) and (2. 3. 8), which will be thought of the field equations in our later treatment. The identities (2. 3. 4) and (2. 3. 5) are trivial, when they are rewritten in spinor form, and also an identity (2. 3. 2) is automatically satisfied, making use of the definition (2. 3. 8). And the identity (2. 3. 4) is a famous Bianchi identity, whose spinor form will be found in a great deal of literatures.[10),17),20)]

First, let us consider (2. 3. 1), which becomes on account of (2. 3. 3) and (2. 3. 6)

$$\nabla_p F^{+kmnp}(K) + K^k{}_{rp} F^{+rmnp} + K^m{}_{rp} F^{+krnp} = 0.$$

Noting the relation

$$\varepsilon_{\dot{A}B\dot{C}D\dot{E}F\dot{G}H} = \sigma^k{}_{\dot{A}B}\sigma^m{}_{\dot{C}D}\sigma^n{}_{\dot{E}F}\sigma^p{}_{\dot{G}H}\varepsilon_{kmnp}$$
$$= i(\varepsilon_{BD}\varepsilon_{FH}\varepsilon_{\dot{A}\dot{E}}\varepsilon_{\dot{C}\dot{G}} - \varepsilon_{BF}\varepsilon_{DH}\varepsilon_{\dot{A}\dot{C}}\varepsilon_{\dot{E}\dot{G}}), \qquad (4.\ 2.\ 1)$$

after the lengthy calculations, we shall find out the following two identities, which are the irreducible components of a spinor equivalent of (2. 3. 1):

$$\nabla_{\dot{A}}{}^F \underset{(K)}{\Psi}_{BCDF} - 2\psi_{\dot{A}EF(B}\Psi_{CD)}{}^{EF} + (1/2)(\varphi_{\dot{A}}{}^E + 3\varphi^E{}_{\dot{A}})\Psi_{EBCD}$$
$$- \nabla^{\dot{G}}{}_{(B}\underset{(K)}{X}_{CD)\dot{A}\dot{G}} + 2\psi^{\dot{G}H}{}_{(BC}X_{D)H\dot{G}\dot{A}} + (1/2)(\varphi^{\dot{G}}{}_{(B} + 3\varphi_{(B}{}^{\dot{G}})X_{CD)\dot{A}\dot{G}}$$
$$- \nabla_{\dot{A}(B}\Phi_{CD)} + (3/2)(\varphi_{\dot{A}(B}\Phi_{CD)} + 3\varphi_{(B\dot{A}}\Phi_{CD)}) + 2\psi_{\dot{A}BCD}\Lambda = 0 \qquad (4.\ 2.\ 2)$$

and

$$\nabla_{\dot{G}H}\underset{(K)}{X}_{BH\dot{A}\dot{G}} + \psi_{\dot{G}HEB}X^{EH\dot{G}}{}_{\dot{A}} + (\varphi^{\dot{G}H} + 3\varphi^{H\dot{G}})X_{BH\dot{A}\dot{G}}$$
$$- 2\nabla_{\dot{A}}{}^H\Phi_{BH} + 3\psi_{\dot{A}BEH}\Phi^{EH}$$
$$+ 3\nabla_{\dot{A}B}\underset{(K)}{\Lambda} - 3(\varphi_{\dot{A}B} + 3\varphi_{B\dot{A}})\Lambda + \psi_{\dot{A}EFH}\Psi_B{}^{EFH} = 0. \qquad (4.\ 2.\ 3)$$

Finally, we get four identities from a spinor equivalent of (2. 3. 8), i.e.,

$$2\nabla^{\dot{E}}{}_{(A}\psi_{\dot{E}BCD)} - 2\psi_{\dot{E}F(AB}\psi^{\dot{E}F}{}_{CD)}$$
$$+ (\varphi_{\dot{E}(A} + 3\varphi_{(A\dot{E}})\psi^{\dot{E}}{}_{BCD)} = 2\underset{(K)}{\Psi}_{ABCD}, \qquad (4.\ 2.\ 4)$$

$$4\nabla^{\dot{E}}{}_{(B}\psi_{D)\dot{A}\dot{C}\dot{E}} - 4\psi_{(B\dot{A}\dot{E}\dot{G}}\psi_{D)\dot{C}}{}^{\dot{E}\dot{G}} + 2(\varphi_{(D}{}^{\dot{E}} + 3\varphi^{\dot{E}}{}_{(D})\psi_{B)\dot{A}\dot{C}\dot{E}}$$
$$+ \nabla_{\dot{A}(B}(\varphi_{D)\dot{C}} + 3\varphi_{\dot{C}D)}) + \nabla_{\dot{C}(B}(\varphi_{D)\dot{A}} + 3\varphi_{\dot{A}D)})$$
$$- (\varphi_{(B\dot{A}} + 3\varphi_{\dot{A}(B})(\varphi_{D)\dot{C}} + 3\varphi_{\dot{C}D)}) = 4\underset{(K)}{X}_{\dot{A}\dot{C}BD}, \qquad (4.\ 2.\ 5)$$

$$2\nabla^{\dot{E}F}\psi_{\dot{E}FAB} + 3(\varphi^{\dot{E}F} + 3\varphi^{F\dot{E}})\psi_{\dot{E}FAB}$$
$$-2\nabla^{\dot{E}}{}_{(A}(\varphi_{\dot{E}B)} + 3\varphi_{B)\dot{E}}) = 8\Phi_{AB} \qquad (4.\ 2.\ 6)$$

and

$$3\nabla^{\dot{E}F}(\varphi_{\dot{E}F} + 3\varphi_{F\dot{E}}) + (3/2)(\varphi_{\dot{E}F} + 3\varphi_{F\dot{E}})(\varphi^{\dot{E}F} + 3\varphi^{F\dot{E}})$$
$$- 2\psi_{\dot{E}FGH}\psi^{\dot{E}FGH} = -24\underset{(K)}{\Lambda}, \qquad (4.\ 2.\ 7)$$

where a subscript (K) was adopted by reason of representing the irreducible spinors of $F_{\dot{A}B\dot{C}D\dot{E}F\dot{G}H}(K)$, just defined like as the correspondings of $F_{\dot{A}B\dot{C}D\dot{E}F\dot{G}H}$.

5. Concluding remark

A spinor formalism introduces the notion of a null tetrad into a theory in a remarkably natural way. Accordingly, this approach seems to be useful for a research on massless gauge fields propagating with positive energy. In forthcoming paper we shall investigate this possibility.

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Appendix: Identities induced by the Poincaré gauge invariance

Under the Poincaré gauge transformations (2. 1. 1) and (2. 1. 2), two fields $b_k{}^\mu$ and $A_{km}{}^\mu$ transform as

$$\delta b_k{}^\mu = -\omega^m{}_k b_m{}^\mu + \xi^\mu{}_{,\nu} b_k{}^\nu,$$

$$\delta A_{km\mu} = -\omega^n{}_k A_{nm\mu} - \omega^n{}_m A_{kn\mu} - \omega_{km,\mu} - A_{km\nu}\xi^\nu{}_{,\mu}.$$

From the invariance of an action (2. 1. 28) (neglected the divergence term) under these transformations, we get the following identical relations, according to Noether's theorem:

$$(i/2)[L']_q S^{km} q + (1/2)([L']_\mu{}^k b^{m\mu} - [L']_\mu{}^m b^{k\mu}) + [L']^{kn\mu} A^m{}_{n\mu} - [L']^{mn\mu} A^k{}_{n\mu} + [L']^{km\mu}{}_{,\mu} = 0, \qquad (A.\ 1)$$

$$[L']_q q_{,\mu} + [L']_\nu{}^k b_k{}^\nu{}_{,\mu} + [L']_\mu{}^k b_k{}^\nu{}_{,\nu} + [L']_\mu{}^k{}_{,\nu} b_k{}^\nu + [L']^{km\nu} A_{km\nu,\mu} - [L']^{km\nu} A_{km\mu,\nu} - [L']^{km\nu}{}_{,\nu} A_{km\mu} = 0, \qquad (A.\ 2)$$

$$\frac{\partial}{\partial x^\mu}\left\{(i/2)\frac{\partial L'}{\partial q_{,\mu}} S^{km} q + (1/2)\left(\frac{\partial L'}{\partial b_k{}^\nu{}_{,\mu}} b^{m\nu} - \frac{\partial L'}{\partial b_m{}^\nu{}_{,\mu}} b^{k\nu}\right) + \frac{\partial L'}{\partial A_{kn\nu,\mu}} A^m{}_{n\nu} - \frac{\partial L'}{\partial A_{mn\nu,\mu}} A^k{}_{n\nu} - [L']^{km\mu}\right\} = 0, \qquad (A.\ 3)$$

$$(i/2)\frac{\partial L'}{\partial q_{,\mu}}S^{km}q+(1/2)\left(\frac{\partial L'}{\partial b_k{}^\nu{}_{,\mu}}b^{m\nu}-\frac{\partial L'}{\partial b_m{}^\nu{}_{,\mu}}b^{k\nu}\right)$$

$$+\frac{\partial L'}{\partial A_{kn\nu,\mu}}A^m{}_{n\nu}-\frac{\partial L'}{\partial A_{mn\nu,\mu}}A^k{}_{n\nu}-[L']^{km\mu}$$

$$-\frac{\partial}{\partial x^\nu}\left(\frac{\partial L'}{\partial A_{km\mu,\nu}}\right)=0, \qquad (A.4)$$

$$\frac{\partial L'}{\partial A_{km\mu,\nu}}+\frac{\partial L'}{\partial A_{km\nu,\mu}}=0, \qquad (A.5)$$

$$\frac{\partial}{\partial x^\mu}\{[L']_\nu{}^k b_k{}^\mu-[L']^{km\mu}A_{km\nu}+a(\delta^\mu{}_\nu \boldsymbol{K}^\lambda{}_{,\lambda}-\boldsymbol{K}^\mu{}_{,\nu})-\tilde{\tau}_\nu{}^\mu\}=0, \qquad (A.6)$$

$$\frac{\partial}{\partial x^\mu}\left\{\frac{\partial L'}{\partial b_\kappa{}^\nu{}_{,\mu}}b_\kappa{}^\lambda-\frac{\partial L'}{\partial A_{km\lambda,\mu}}A_{km\nu}+2a\frac{\partial \boldsymbol{K}^\mu}{\partial g^{\nu\kappa}}g^{\lambda\kappa}\right.$$

$$\left.+2a\frac{\partial \boldsymbol{K}^\mu}{\partial g^{\nu\kappa}{}_{,\gamma}}g^{\lambda\kappa}{}_{,\gamma}-a\frac{\partial \boldsymbol{K}^\mu}{\partial g^{\alpha\beta}{}_{,\lambda}}g^{\alpha\beta}{}_{,\nu}\right\}+[L']_\nu{}^k b_k{}^\lambda-[L']^{km\lambda}A_{km\nu}$$

$$-\tilde{\tau}_\nu{}^\lambda+a(\delta^\lambda{}_\nu \boldsymbol{K}^\kappa{}_{,\kappa}-\boldsymbol{K}^\lambda{}_{,\nu})=0, \qquad (A.7)$$

$$\frac{\partial}{\partial x^\mu}\left(a\frac{\partial \boldsymbol{K}^\mu}{\partial g^{\nu\alpha}{}_{,\lambda}}g^{\alpha\kappa}+a\frac{\partial \boldsymbol{K}^\mu}{\partial g^{\nu\alpha}{}_{,\kappa}}g^{\alpha\lambda}\right)$$

$$+\frac{1}{2}\left\{\frac{\partial L'}{\partial b_k{}^\nu{}_{,\kappa}}b_k{}^\lambda+\frac{\partial L'}{\partial b_k{}^\nu{}_{,\lambda}}b_k{}^\kappa-\left(\frac{\partial L'}{\partial A_{km\lambda,\kappa}}+\frac{\partial L'}{\partial A_{km\kappa,\lambda}}\right)A_{km\nu}\right.$$

$$+2a\frac{\partial \boldsymbol{K}^\kappa}{\partial g^{\nu\alpha}}g^{\lambda\alpha}+2a\frac{\partial \boldsymbol{K}^\lambda}{\partial g^{\nu\alpha}}g^{\kappa\alpha}$$

$$+2a\frac{\partial \boldsymbol{K}^\kappa}{\partial g^{\nu\alpha}{}_{,\gamma}}g^{\lambda\alpha}{}_{,\gamma}+2a\frac{\partial \boldsymbol{K}^\lambda}{\partial g^{\nu\alpha}{}_{,\gamma}}g^{\kappa\alpha}{}_{,\gamma}$$

$$\left.-a\frac{\partial \boldsymbol{K}^\kappa}{\partial g^{\alpha\beta}{}_{,\lambda}}g^{\alpha\beta}{}_{,\nu}-a\frac{\partial \boldsymbol{K}^\lambda}{\partial g^{\alpha\beta}{}_{,\kappa}}g^{\alpha\beta}{}_{,\nu}\right\}=0$$

and (A.8)

$$\frac{\partial \boldsymbol{K}^\mu}{\partial g^{\alpha\nu}{}_{,\lambda}}g^{\alpha\kappa}+\frac{\partial \boldsymbol{K}^\mu}{\partial g^{\alpha\nu}{}_{,\kappa}}g^{\alpha\lambda}+\frac{\partial \boldsymbol{K}^\lambda}{\partial g^{\alpha\nu}{}_{,\mu}}g^{\alpha\kappa}$$

$$+\frac{\partial \boldsymbol{K}^\lambda}{\partial g^{\alpha\nu}{}_{,\kappa}}g^{\alpha\mu}+\frac{\partial \boldsymbol{K}^\kappa}{\partial g^{\alpha\nu}{}_{,\lambda}}g^{\alpha\mu}+\frac{\partial \boldsymbol{K}^\kappa}{\partial g^{\alpha\nu}{}_{,\mu}}g^{\alpha\lambda}=0\,. \qquad (A.9)$$

where $L=L'+a\boldsymbol{K}^\mu{}_{,\mu}$

$$[L']_q=\frac{\partial L'}{\partial q}-\frac{\partial}{\partial x^\mu}\left(\frac{\partial L'}{\partial q_{,\mu}}\right),$$

$$[L']_{\mu}{}^{k}=\frac{\partial L'}{\partial b_{k}{}^{\mu}}-\frac{\partial}{\partial x^{\nu}}\left(\frac{\partial L'}{\partial b_{k}{}^{\mu}{}_{,\ \nu}}\right),$$

$$[L']^{km\mu}=\frac{\partial L'}{\partial A_{km\mu}}-\frac{\partial}{\partial x^{\nu}}\left(\frac{\partial L'}{\partial A_{km\mu,\ \nu}}\right),$$

and

$$\tilde{\tau}_{\nu}{}^{\mu}=\frac{\partial L'}{\partial q_{,\ \mu}}q_{,\ \nu}+\frac{\partial L'}{\partial b_{k}{}^{\lambda}{}_{,\ \mu}}b_{k}{}^{\lambda}{}_{,\ \nu}+\frac{\partial L'}{\partial A_{km\lambda,\ \mu}}A_{km\lambda,\ \nu}-\delta_{\nu}{}^{\mu}L'$$

$$=a\underset{(G)}{\tau_{\nu}{}^{\mu}}+\tau_{\nu}{}^{\mu}+\text{divergence term.} \qquad \text{(A. 10)}$$

From (A. 1) we see at once that the field equation for a field $b_{k}{}^{\mu}$ is essentially symmetric, when another equations are fulfilled:

Putting $[L']_{q}=0$ and $[L']^{km\mu}=0$ in (A. 1),

we find

$$[L']^{km}=[L']^{mk}$$

with $[L']^{km}=b^{k\mu}[L']_{\mu}{}^{m}$.

When putting

$$[L']_{\mu}{}^{m}=\mathrm{b}\ \mathrm{b}_{k\mu}\ \mathrm{J}^{km},$$

$$[L']^{km\mu}=\mathrm{b}\ \mathrm{b}_{n}{}^{\mu}\ \mathrm{N}^{kmn},$$

(A. 1) and (A. 2) are rewritten respectively as

$$D_{n}N^{kmn}+{}^{V}\mathscr{C}_{n}N^{kmn}+J^{[mk]}=0 \qquad \text{(A. 1)}'$$

or

$$\nabla_{n}N^{kmn}+K^{k}{}_{rn}N^{rmn}+K^{m}{}_{rn}N^{krn}+J^{[mk]}=0,$$

and

$$D_{k}J_{p}{}^{k}+{}^{V}\mathscr{C}_{k}J_{p}{}^{k}-\mathscr{C}_{kmp}J^{km}-F_{kmnp}N^{kmn}=0 \qquad \text{(A. 2)}'$$

or

$$\nabla_{k}J_{p}{}^{k}-K_{kmp}J^{[km]}-F_{kmnp}N^{kmn}=0.$$

From these identities we can get the identities (2. 3. 1) ~ (2. 3. 5), noting that the Poincaré gauge invariance is preserved, even if any of *abR*, $L_{\mathscr{C}}$ and L_{F} is substituted for L in the action I: Substituting L_{F} for L' in (A. 1)', we get an identity (2. 3. 5).

And, substituting L_F and L_G for L' in (A. 2)', we get the identities (2. 3. 1) and (2. 3. 2) resepectively.

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