A Certain Theorem on the Unitary Group $U(1, n; F)$

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Let $F$ denote the field $R$ of real numbers, the field $C$ of complex numbers, or the division ring of real quaternions $K$. Let $V=V$, $(F^n)$ denote the (right) vector space $F^{n+1}$, together with the unitary structure defined by the $F$-Hermitian form

$$\phi(z, w) = -\bar{z}_0w_0 + \bar{z}_1w_1 + \bar{z}_2w_2 + \cdots + \bar{z}_nw_n$$

for $z = (z_0, z_1, \ldots, z_n)^T$, $w = (w_0, w_1, \ldots, w_n)^T$ (where T denotes the transpose).

An automorphism $g$ of $V$ will be called unitary transformation. ($g$ must be $F$-linear and $\phi(g(z), g(w)) = \phi(z, w)$, for all $z, w \in V$.) We denote the group of all unitary transformations by $U(1, n; F)$.

Our purpose of this paper is to prove the following theorem.

**Theorem.** Let $G$ be a discrete subgroup of $U(1, n; F)$. Let $a=(\beta_1, \beta_2, \cdots, \beta_n)^T \in H^n(F)$, either (i) $\sum_{f_k \in G} (1-|f_k(a)||) < \infty$, or

(ii) $\sum_{f_k \in G} (1-|f_k(a)||) = \infty$

is independent of $a \in H^n(F)$, where $||a|| = (\sum_{j=1}^n |\beta_j|^2)^{1/2}$.

$G$ is called of convergence, or of divergence type, according as the case (i), or (ii).

1. Let us begin with recalling some notation and definitions.

Let $V_\perp = \{ z \in V : \phi(z, z) < 0 \}$. Obviously $V_\perp$ is invariant under $U(1, n; F)$. Let $P(V)$ be the projective space obtained from $V$. This is defined as usual, by the equivalence relation in $V - \{ 0 \} : u \sim v$ if there exists $\lambda \in F^*$ (the multiplicative group in $F$) such that $u = v\lambda$. $P(V)$ is the set of equivalence classes, with the quotient topology. Let $P : V - \{ 0 \} \rightarrow P(V)$ denote the projection map. We define $H^n(F) = P(V_\perp)$. If $g \in U(1, n; F)$, then $g(V_\perp) = V_\perp$ and $g(v\lambda) = g(v)\lambda$. Therefore $U(1, n; F)$ acts in $P(V)$, leaving $H^n(F)$ invariant.

2. Now we are ready to prove our theorem.

**Proof of Theorem.** We denote $(1, 0, \cdots, 0)^T$ and $(\alpha_1, \alpha_2, \cdots, \alpha_{n+1})^T$ by $O$ and
$A$, respectively. Let $P(O) = 0$ and $P(A) = a$. Set
\[ [A, B] = [[\Phi(A, B) \cdot \Phi(B, A)]^{-\frac{1}{2}}]^{-1} \]
for $A, B \in V$. It is clear that $[A, B]$ is invariant under $U(1, n; F')$. Let
\[
 f_k = \begin{pmatrix}
 a_{1,1}^{(k)} & a_{1,2}^{(k)} & \cdots & a_{1,n+1}^{(k)} \\
 & \ddots & & \\
 a_{n+1,1}^{(k)} & a_{n+1,2}^{(k)} & \cdots & a_{n+1,n+1}^{(k)}
\end{pmatrix}.
\]
We note that
\[
 f_k(A) = (\sum_{j=1}^{n+1} a_{1,j}^{(k)} \alpha_j, \sum_{j=1}^{n+1} a_{2,j}^{(k)} \alpha_j, \ldots, \sum_{j=1}^{n+1} a_{n+1,j}^{(k)} \alpha_j),
\]
\[
f_k(O) = (a_{1,1}^{(k)}, a_{2,1}^{(k)}, \ldots, a_{n+1,1}^{(k)}),
\]
\[
 \Phi(f_k(A), f_k(O)) = -\sum_{j=1}^{n+1} a_{1,j}^{(k)} \alpha_j a_{1,1}^{(k)} + \sum_{m=2}^{n+1} \left( \sum_{j=1}^{n+1} a_{m,j}^{(k)} \alpha_j \right) a_{m,1}^{(k)},
\]
\[
 \Phi(f_k(O), f_k(A)) = -|a_{1,1}^{(k)}|^2 + \sum_{m=2}^{n+1} |a_{m,1}^{(k)}|^2,
\]
\[
 ||P(f_k(A))||^2 = \sum_{j=1}^{n+1} (|\sum_{m=2}^{n+1} a_{m,j}^{(k)} \alpha_j|^2 - |a_{1,j}^{(k)} \alpha_j|^2),
\]
\[
 ||P(f_k(O))||^2 = \sum_{m=2}^{n+1} |a_{m,1}^{(k)}|^2 - |a_{1,1}^{(k)}|^2,
\]
\[
 1 - ||P(f_k(A))||^2 = \left( \left| a_{1,1}^{(k)} \right|^2 - \sum_{m=2}^{n+1} |a_{m,1}^{(k)}|^2 \right) - |a_{1,1}^{(k)}|^2 - 2 \left( \sum_{m=2}^{n+1} |a_{m,1}^{(k)}|^2 \right) - |a_{1,1}^{(k)}|^2,
\]
\[
 1 - ||P(f_k(O))||^2 = \left( \left| a_{1,1}^{(k)} \right|^2 - \sum_{m=2}^{n+1} |a_{m,1}^{(k)}|^2 \right) - |a_{1,1}^{(k)}|^2 - 2 \left( \sum_{m=2}^{n+1} |a_{m,1}^{(k)}|^2 \right) - |a_{1,1}^{(k)}|^2,
\]
\[
 \|P(f_k(A)) \cdot ||P(f_k(O))\| = \left[ \sum_{m=2}^{n+1} \left( |\sum_{j=1}^{n+1} a_{m,j}^{(k)} \alpha_j|^2 - |a_{1,j}^{(k)} \alpha_j|^2 \right) \right. \\
\left. \sum_{j=1}^{n+1} (|\sum_{m=2}^{n+1} a_{m,j}^{(k)} \alpha_j|^2 - |a_{1,j}^{(k)} \alpha_j|^2) \right]^{\frac{1}{2}}.
\]

Using the above, we have
\[
 1 - ||a||^2 = [A, O]^2 = [f_k(A), f_k(O)]^2
\]
\[
 = \langle \Phi(f_k(A), f_k(A)) \cdot \Phi(f_k(O), f_k(O)) \rangle \Phi(f_k(A), f_k(O)) \rangle^{\frac{1}{2}}
\]
\[
 = \left( -\sum_{j=1}^{n+1} |a_{1,j}^{(k)} \alpha_j|^2 + \sum_{m=2}^{n+1} |a_{m,1}^{(k)} \alpha_j|^2 \right)
\]
\[
 - |a_{1,1}^{(k)}|^2 + \sum_{m=2}^{n+1} |a_{m,1}^{(k)}|^2
\]
\[
 - \sum_{j=1}^{n+1} a_{1,j}^{(k)} \alpha_j a_{1,1}^{(k)} + \sum_{m=2}^{n+1} \left( \sum_{j=1}^{n+1} a_{m,j}^{(k)} \alpha_j \right) a_{m,1}^{(k)}
\]
\[
 = (1 - ||P(f_k(A))||^2) (1 - ||P(f_k(O))||^2)
\]
\[
 |1 - \sum_{m=2}^{n+1} \left( \sum_{j=1}^{n+1} a_{m,j}^{(k)} \alpha_j a_{m,1}^{(k)} \right) / \sum_{j=1}^{n+1} a_{1,j}^{(k)} \alpha_j a_{1,1}^{(k)}|^{-2}.
\]
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\[
\leq \left( 1 + \|P(f_k(A))\| \right) \left( 1 - \|P(f_k(O))\| \right) \left( 1 + \|P(f_k(O))\| \right) \left( 1 - \|P(f_k(O))\| \right)
\]

\[
\left( 1 - \sum_{m=1}^{n-1} \sum_{j=1}^{n+1} a_{m,j}^{(k)} a_{m,1}^{(k)} / \sum_{i=1}^{n+1} a_{i,1}^{(k)} a_{i,1}^{(k)} \right)^{-2}
\]

\[
\leq 4 \left( 1 - \|P(f_k(A))\| \right) \left( 1 - \|P(f_k(O))\| \right) \left( 1 - \|P(f_k(A))\| \right) \left( 1 - \|P(f_k(O))\| \right)^{-2}
\]

\[
\leq \frac{4(1 - \|P(f_k(A))\|)(1 - \|P(f_k(O))\|)^{-1} \cdot \|P(f_k(O))\|^{-2}}{4(1 - \|P(f_k(O))\|)(1 - \|P(f_k(A))\|)^{-1}}.
\]

Therefore we see that

\[
\frac{1}{4} \left( 1 - \|a\|^2 \right) \left( 1 - \|P(f_k(O))\| \right) \leq 1 - \|P(f_k(A))\| \leq 4 \left( 1 - \|P(f_k(O))\| \right) \left( 1 - \|a\|^2 \right)^{-1}.
\]

Thus our theorem is completely proved.

References
