Poison transforms for some principal series representations of $Sp(n, \mathbb{R})$

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1 Introduction
Let $G = Sp(n, \mathbb{R})$ ($n \geq 2$), $K$ a maximal compact subgroup of $G$, and let $P_\theta$ be a parabolic subgroup of $G$ with a Langlands decomposition $P_\theta = M_\theta A_\theta N_\theta$, where $M_\theta \simeq \{\pm 1\} \times Sp(n-1, \mathbb{R})$. We consider an induced representation of $G$ from $P_\theta$, which is induced from a holomorphic representation of $M_\theta$, a character of $A_\theta$, and the trivial representation of $N_\theta$. We consider the problem of characterizing the image of the Poisson transform from the principal series representation to a homogeneous line bundle over $G/K$. The main result (Theorem 3.1) asserts that the Poisson transform is injective under certain conditions on parameter and the image is characterized by second-order differential equations, which are given by a $K$-covariant differential operator between homogeneous vector bundles over $G/K$. As a corollary we obtain a characterization of the images of degenerate series representations on $G/P_\theta$ under the Poisson transform (Corollary 3.2).

For the Furstenberg boundary of a Riemannian symmetric space and the Shilov boundary of Hermitian symmetric space of tube type, there are several studies on the Poisson transform\(^{5,9,11}\). We believe that it is of importance to construct differential equations that characterize the image of the Poisson transform explicitly for other boundary components of a symmetric space and this article gives a new example on this problem.

2 Notation and preliminary results
2.1 Notation
Let

\[ G = Sp(n, \mathbb{R}) = \{ g \in SL(2n, \mathbb{R}); 'gJg = J \}, \]

where

\[ J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \]
and $I_n$ is $n \times n$ identity matrix. The group $K = O(2n) \cap Sp(n, \mathbb{R})$ is a maximal compact subgroup of $G$, which is isomorphic to $U(n)$ by

$$
\begin{pmatrix}
A & B \\
-B & A
\end{pmatrix} \in K \mapsto A + \sqrt{-1}B \in U(n).
$$

Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of $G$ and $K$ respectively. Let $\theta$ denote the corresponding Cartan involution of $G$ and $\mathfrak{g}$. We have a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{p}$ is the $-1$-eigenspace of $\theta$ in $\mathfrak{g}$.

For $l \in \mathbb{Z}$ let $\tau_l$ denote the one-dimensional representation of $U(n)$ given by $\tau_l(x) = (\det x)^l (x \in U(n))$ and we denote corresponding representation of $K$ and $\mathfrak{k}$ by the same notation.

Let $E_{ij}$ denote the $n \times n$ matrix with $(i, j)$-entry 1 and all other entries being 0. We choose a Cartan subalgebra $\mathfrak{t}$ of $u(n)$ to be the set of diagonal matrices. We define $e_i \in \sqrt{-1}\mathfrak{t}^*$ by $e_i(E_{ij}) = \delta_{ij}$ ($1 \leq i, j \leq n$). Let $\Delta$ denote the root system of $(\mathfrak{g}, \mathfrak{t})$ and $\Delta^+$ be the positive system of $\Delta$ given by

$$
\Delta^+ = \{2e_i, e_i \pm e_k; 1 \leq i \leq n, 1 \leq j < k \leq n\}.
$$

For $\gamma \in \Delta$ let $\mathfrak{g}_{\gamma} \subset \mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}$ denote the root space for $\gamma$. Let $\mathfrak{p}^+ = \sum_{\gamma \in \Delta^+} \mathfrak{g}_{\gamma}$, where $\Delta^+$ is the set of non-compact positive roots.

We put

$$
X_i = \begin{pmatrix}
E_{ii} & 0 \\
0 & -E_{ii}
\end{pmatrix} \in \mathfrak{p} \quad (1 \leq i \leq n)
$$

and $a = \sum_{i=1}^n \mathbb{R} X_i$. Then $a$ is a maximal abelian subspace of $\mathfrak{p}$. We put $X_0 = X_1 + \cdots + X_n$. Let $e_i$ ($1 \leq i \leq n$) be the linear form on $a$ given by $e_i(X_j) = \delta_{ij}$. Let $\Sigma$ denote the restricted root system of the pair $(\mathfrak{g}, a)$ and $\Sigma^+$ be the positive system of $\Sigma$ given by

$$
\Sigma^+ = \{2e_i, e_i \pm e_k; 1 \leq i \leq n, 1 \leq j < k \leq n\}.
$$

For $a \in \Sigma$ let $\mathfrak{g}^a \subset \mathfrak{g}$ be the root space for $a$. For $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{R})$ we have dim $\mathfrak{g}^a = 1$ for all $a \in \Sigma$. We put $\rho = \frac{1}{2} \sum_{\alpha \in \Delta} \alpha$. For any $\lambda \in \mathfrak{a}^*$ let $A_\lambda$ be the element of $\mathfrak{a}$ determined by $B(H, A_\lambda) = \lambda(H)$ for all $H \in \mathfrak{a}$, where $B$ denotes the Killing form of $\mathfrak{g}$. For $\lambda, \mu \in \mathfrak{a}^*$ we put $\langle \lambda, \mu \rangle = B(A_\lambda, A_\mu)$. Since $\{e_1, \ldots, e_n\}$ forms a basis of $\mathfrak{a}^*$, any $\lambda \in \mathfrak{a}^*$ can be written as $\lambda = \sum_{i=1}^n \lambda_i e_i$ ($\lambda_i \in \mathbb{C}$). We identify $\mathfrak{a}^*$ with $\mathbb{C}^n$ by $\lambda \mapsto (\lambda_1, \ldots, \lambda_n)$. In this identification we have $\rho = (n, n-1, \ldots, 1)$.

Let $A$ be the analytic subgroups of $G$ corresponding to $a$. Let $n^+ = \sum_{\alpha \in \Sigma^+} \mathfrak{g}^\alpha$ and $n^- = \theta(n^+)$. Let $N^+$ and $N^-$ be the corresponding analytic subgroups of $G$. Let $M$ be the centralizer of $a$ in $K$. The subgroup $P = MAN^+$ is a minimal parabolic subgroup of $G$.

Put $a_i = e_i - e_{i+1}$ ($1 \leq i \leq n-1$) and $a_n = 2e_n$. Then the set of simple roots is $\Psi = \{a_1, a_2, \ldots, a_n\}$. Let $\{H_1, \ldots, H_n\}$ denote the basis of $a$ which is dual to $\Psi$. We consider subset

$$
\Theta = \{a_1, \ldots, a_n\}.$$

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of $\Psi$ and corresponding standard parabolic subgroup $P_\theta$ of $G$ with the Langlands decomposition $P_\theta = M_\theta A_\theta N_\theta$ such that $A_\theta \subset A$. Then the Lie algebra $a_\theta$ of $A_\theta$ and its orthogonal complement $a(\theta)$ in $a$ are given by

$$a_\theta = \mathbb{R} X_i, \quad a(\theta) = \bigoplus_{i=1}^{n} \mathbb{R} X_i,$$

and $M_\theta \cong \{ \pm 1 \} \times \text{Sp}(n-1, \mathbb{R})$. Put $K_\theta = M_\theta \cap K$ and define a closed subgroup $B_\theta = K_\theta A_\theta N_\theta$ of $G$. Notice that the pairs $(K, K_\theta) \cong (U(n), \{ \pm 1 \} \times U(n-1))$ is a Gelfand pair.

2.2 Eigenspaces of invariant differential operators

We review the main result of Shimeno\textsuperscript{9)}, which gives a characterization of the image of the Poisson transform.

For a real analytic manifold $X$ we denote by $B(X)$ the space of all hyperfunctions on $X$. Let $\lambda \in \mathfrak{a}_\mathbb{R}$ and $l \in \mathbb{Z}$. We define

$$B(G/P, L_{\lambda, l}) = \{ f \in B(G) ; f(gm) = e^{(l-\rho)\log a} \tau(m)^{-1} f(g) \}
\quad g \in G, m \in M, a \in A, n \in N^+$$

and

$$B(G/K ; \tau_l) = \{ u \in B(G) ; u(gk) = \tau(k)^{-1} u(g) \text{ for any } g \in G, k \in K \}.$$  

For $f \in B(G/P ; L_{\lambda, l})$, we define the Poisson integral $\mathcal{P}_{\lambda, l} f$ by

$$\mathcal{P}_{\lambda, l} f(g) = \int_{K} f(gk) \tau_l(k) dk.$$  

Here $dk$ denotes the invariant measure on $K$ with total measure 1.

Let $\mathcal{D}_l(G/K)$ denote the algebra of invariant differential operators on $B(G/K ; \tau_l)$ and $L_l \in \mathcal{D}_l(G/K)$ denote the Laplace-Beltrami operator acting on $B(G/K ; \tau_l)$. We have the Harish-Chandra isomorphism

$$\gamma_l : \mathcal{D}_l(G/K) \cong S(a_\mathbb{C})^w,$$

where $S(a_\mathbb{C})^w$ denotes the set of $W$-invariant elements in the symmetric algebra $S(a_\mathbb{C})$. Let $\mathcal{A}(G/K, \mathcal{M}_{\lambda, l})$ denote the space of all real analytic functions in $B(G/K, \tau_l)$ satisfying the system of differential equations, 

$$\mathcal{M}_{\lambda, l} : D u = \gamma_l(D)(\lambda) u, \quad D \in \mathcal{D}_l(G/K).$$

We define

$$e_{\lambda, l} = \prod_{1 \leq j < k \leq n} \Gamma\left(\frac{1}{2}(1+\lambda_j+\lambda_k)\right) \Gamma\left(\frac{1}{2}(1+\lambda_j-\lambda_k)\right)$$
$$\times \prod_{1 \leq j < k \leq n} \Gamma\left(\frac{1}{2}(\lambda_j+1+l)\right) \Gamma\left(\frac{1}{2}(\lambda_j+1-l)\right).$$

Theorem 2.1 (Shimeno\textsuperscript{9}) If $\lambda \in \mathfrak{a}_\mathbb{R}$ satisfies the condition
\[ -2 \left< \lambda, a \right> < \left< a, a \right> < 1, 2, 3, \ldots \text{ for all } a \in \Sigma^+ \quad (2.2) \]

\[ e_{\lambda, l} \neq 0, \quad (2.3) \]

then the Poisson transform \( \mathcal{P}_{\lambda, l} \) is a \( G \)-isomorphism of \( B(G/P; L_{\lambda, l}) \) onto \( \mathcal{A}(G/K; M_{\lambda, l}) \).

Under the condition of the above theorem, the inverse of the Poisson transform is given by the boundary value map up to a non-zero constant multiple.

### 3 The Poisson transforms and the Hua equations

#### 3.1 Poisson transform for principal series representations

Let \( s \in \mathbb{C}, \ l \in \mathbb{Z}, \) and \( \varepsilon \in \{+1, -1\} \). We put

\[ \lambda_{\varepsilon, l, e} = (s, -\varepsilon l+n-1, -\varepsilon l+n-2, \ldots, -\varepsilon l+1) \in \mathfrak{a}^\varepsilon, \]

and \( \lambda^\theta = \lambda_{\varepsilon, l, e} \). Throughout this section we denote \( \lambda_{\varepsilon, l, e} \) by \( \lambda \) for simplicity.

We define

\[ B(G/B_\theta; L_{\lambda, l}) = \{ f \in B(G); f(gmnan) = \tau(m)^{-1}e^{(s-\varepsilon l)\rho+\lambda}\phi(g) \}, \]

\[ g \in G, \ m \in K_\theta, a \in A_\theta, n \in N_\theta \} \].

The algebra \( \mathbb{D}(M_\theta/K_\theta) \) of invariant differential operators on \( B(M_\theta/K_\theta, \tau) \) acts from the right on \( B(G/B_\theta; L_{\lambda, l}) \). Let \( B(\Theta; s, l, \varepsilon) \) denote the subspace of \( B(G/B_\theta; L_{\lambda, l}) \) consisting of solutions of the system

\[ v : Df = \chi_{\lambda}(D)f, \quad D \in \mathbb{D}(M_\theta/K_\theta). \]

Let \( m_\theta = \theta + p_\theta \) be the Cartan decomposition corresponding to \( \Theta \mid m_\theta \) and put \( v_\theta^\pm = v^\pm \cap v_{\theta,c}. \) Each element of \( v_\theta^\pm \) acts on \( B(\Theta; s, l, \varepsilon) \) from the right as a differential operator, where \( v_\theta \) denotes \( v_\theta^\varepsilon \) for \( \varepsilon = \pm 1 \). We define

\[ B(G/P_\theta; s, l, \varepsilon) = \{ f \in B(\Theta; s, l, \varepsilon); v_\theta f = 0 \}. \]

We define

\[ B(G/P_\theta; s) = \{ f \in B(G); f(gmnan) = e^{(s-\varepsilon l)\rho+\lambda}\phi(g) \}, \]

\[ g \in G, \ m \in M_\theta, a \in A_\theta, n \in N_\theta \} \],

which equals \( B(\Theta; s, 0, +1) \cap B(\Theta; s, 0, -1) \).

We recall the definition of partial Poisson transforms after Shimeno\(^{10}\). For \( f \in B(G/P; L_{\lambda, l}) \) we define

\[ \mathcal{P}_{\theta, s, l, \varepsilon, f}(x) = \int_{K_\theta} f(xk)\tau_l(k)dk. \quad (3.1) \]

The image of \( \mathcal{P}_{\theta, s, l, e}^\theta(x) \) is contained in \( B(\Theta; s, l, \varepsilon) \). Let \( 1_{\lambda, l} \) be the element of \( B(G/P; L_{\lambda, l}) \) such that \( 1_{\lambda, l}(x) = \tau_l \) and define \( \phi_l^\theta = \mathcal{P}_{\lambda, l, s, l, \varepsilon}^\theta \) and \( = \phi_{l,-l}(x^{-1}), x \in G \). For \( f \in B(\Theta; s, l, \varepsilon) \) we define

\[ \mathcal{P}_{\theta, s, l, \varepsilon, f}(x) = \int_{K} f(xk)\tau_l(k)dk. \quad (3.2) \]

Then we have \( \mathcal{P}_{\lambda, l} = \mathcal{P}_{\theta, s, l, \varepsilon}^\theta \mathcal{P}_{\varepsilon, l, \varepsilon}^\theta \). A straightforward calculation shows that
(3.3)

\[
(P_{\theta,s,\ell,e})(x) = \int_k P^\theta_{\ell,e}(k^{-1}x)f(k)dk.
\]

We write \(P_{\theta,s} = P_{\theta,s,0,e}\) for simplicity.

If \(\ell > n-1\), then it follows from Theorem 5.1 and Theorem 5.10 in Shimeno\(^{10}\) that \(\phi_{\ell,i}^\theta|M_\theta\) is a vector in the holomorphic discrete series representation of \(M_\theta\) with lowest \(K\)-type \(\tau_i\). Thus the function \(\phi_{\ell,i}^\theta\) is contained in \(\mathcal{B}(G/P_\theta; l, e, s)\).

For dominant integral weight \(\mu \in \sqrt{-1}\mathbb{I}^*\), let \(V_\mu\) denote the irreducible representation of \(K\) with highest weight \(\mu\). Let \(\{E_i\}\) be a basis of \(\nu^\varphi\) and \(\{E_i^\varphi\}\) be the dual basis of \(\mathfrak{p}^-\) with respect to \(B\). Let \(\rho_\varphi\) denote the projection of \(\nu^\varphi\otimes\nu^\varphi\) onto the \(K\)-irreducible component with highest weight \(\gamma_1 + \gamma_2\). We define an element of \(\mathfrak{U}(\mathfrak{g}_\varphi)\otimes V_{n+n}\) by

\[
\mathcal{H}_\varphi = \sum_{i,j} E_i^\varphi E_j^\varphi \rho_\varphi(E_i \otimes E_j),
\]

which does not depend on the choice of basis. Notice that \(\mathcal{H}_\varphi^\theta\) defines a homogeneous differential operator from \(\mathcal{C}^\infty(G/K; \tau_i)\) to \(\mathcal{C}^\infty(G/K; \tau_i \otimes \text{Ad}_K|V_{n+n})\). We define \(\mathcal{H}_\varphi^e\) similarly. We denote \(\mathcal{H}_\varphi^\pm\) for \(e = \pm 1\) by \(\mathcal{H}_\varphi^\pm\).

We state the main result of this article:

**Theorem 3.1** If \(s \in \mathbb{C}, l \in \mathbb{Z}, \) and \(e \in \{+1, -1\}\) satisfies condition,

\[
\{s+e-l-n+1, s, s-eL, \frac{1}{2}(s+1-|l|)\} \cap \{0, -1, -2, \ldots\} = \emptyset,
\]

then the partial Poisson transform \(P_{\theta,s,\ell,e}\) is a \(G\)-isomorphism of \(\mathcal{B}(G/P_\theta; s, l, e)\) onto the space of analytic functions \(u\) in \(\mathcal{B}(G/K, \tau_i)\) that satisfy

\[
\mathcal{H}_\varphi^s u = 0,
\]

\[
L_i u = \langle \lambda^\theta_{a,i,e}, \lambda^\theta_{a,i,e} > - < \rho, \rho > \rangle u.
\]

**Corollary 3.2** If \(s \in \mathbb{C}\) satisfies the condition,

\[
s-n+1 \notin \{0, -1, -2, \ldots\}
\]

then the partial Poisson transform \(P_{\theta,s}\) is a \(G\)-isomorphism of \(\mathcal{B}(G/P_\theta; s)\) onto the space of analytic functions \(u\) on \(G/K\) that satisfy

\[
\mathcal{H}_s^\varphi u = 0,
\]

\[
\mathcal{H}_s^\varphi u = 0,
\]

\[
L_i u = \langle \lambda^s_\varphi, \lambda^s_\varphi > - < \rho, \rho > \rangle u.
\]

The proof of the theorem is divided into four steps;

1. The image of \(\mathcal{B}(G/P_\theta; s, l, e)\) under \(P_{\theta,s,\ell,e}\) satisfies (3.6) (Proposition 3.3),
2. Solutions of (3.6) and (3.7) satisfy \(\mathcal{M}_{\lambda_{\ell\varphi}}\) (Proposition 3.5),
3. Under condition (3.5), \(P_{\theta,s,\ell,e}\) gives an isomorphism of \(\mathcal{B}(\Theta; s, l, e)\) onto the joint eigenspace of \(\mathcal{M}_{\lambda_{\ell\varphi}}\) (Proposition 3.9),
4. Under condition (3.17) boundary values of solutions of (3.6) and (3.7) are contained in \(\mathcal{B}(G/P_\theta; s, l, e)\) (Proposition 3.11).

**Proposition 3.3** For any \(f\) in \(\mathcal{B}(G/P_\theta; s, l, e)\), \(u = P_{\theta,s,\ell,e}(f)\) satisfies (3.6).

**Proof.** We consider the universal covering group of \(G\) and may assume that \(l \in \mathbb{C}\). It
is sufficient to show that \( u = P_{\lambda, t}^\theta \) satisfies (3.6). We put \( F = \mathcal{H}^\theta P_{\lambda, t}^\theta \). If \( l < -n \), then the restriction of \( F \) to \( \text{Sp}(n-1, \mathbb{R}) \subset M_{\theta} \) is a vector in the holomorphic discrete series representation of lowest \( K \)-type \( \tau_i \) that is \( U(n-1) \)-finite of type \( (\tau_i \otimes \text{Ad}_X| V_{n+r_2})| U(n-1) \). Since the holomorphic discrete series of \( \text{Sp}(n-1, \mathbb{R}) \) with lowest \( U(n-1) \)-type \( \tau_i \) equals \( S(\psi_2) \otimes \tau_i \) as \( U(n-1) \)-modules, \( F \) must be identically zero. We have \( F = 0 \) for all \( l \) by analytic continuation. We can show in the same way that \( \mathcal{H}^\theta P_{\lambda, t+1}^\theta = 0 \). □

Remark 3.4 The use of operator \( \mathcal{H}^\theta \) is inspired by Miyazaki, Oda\(^6\) and Iida\(^4\), where they construct differential equations for Whittaker functions or matrix coefficients of principal series representations of \( \text{Sp}(2, \mathbb{R}) \) by using \( K \)-covariant differential operators between homogeneous vector bundles over \( G/K \) (shift operators in their terminology).

3.2 Radial parts of the Hua equations

Proposition 3.5 Any solution of (3.6) and (3.7) satisfies \( \mathcal{M}_{\lambda, t} \).

Let \( \varphi_{\lambda, t} \) denote the Poisson integral of the function \( 1_{\lambda, t} \in B(G/P; L_{\lambda, t}) \) with \( 1_{\lambda, t}|_K = \tau_{-t} \), i.e.,

\[
\varphi_{\lambda, t}(g) = \int_K \tau_i(k^{-1} g^{-1} k) \exp \left< -\lambda - \rho, H(g^{-1} k) \right> \; dK.
\]

We shall prove that \( \varphi_{\lambda, t} \) is a unique solution of (3.6) and (3.7) such that \( u(kx) = \tau_i(k)^{-1} u(x) \) for all \( k \in K \) and \( x \in G \) (Corollary 3.8). Then we can prove Proposition 3.5 in the same way as the proof of Theorem 3.3 in Shimeno\(^1\)). In the proof of Theorem 3.3 in Shimeno\(^1\)) we use a characterization of joint eigenfunctions of \( \mathcal{D}(G/K) \) by means of an integral formula (Helgason\(^3\), Ch IV, Proposition 2.4), which can easily be generalized to the case of a homogeneous line bundle.

We define elements \( T_i \) \((i = 1, \ldots, n)\), \( X_{\pm 2te_i} \in \mathfrak{g}_{\pm 2te_i} \) \((i = 1, \ldots, n)\) and \( X_{\pm e_i \pm e_k} \in \mathfrak{g}_{\pm e_i \pm e_k} \) \((1 \leq j \neq k \leq n)\) by

\[
T_i = \begin{pmatrix} 0 & E_{ii} \\ -E_{ii} & 0 \end{pmatrix}, \quad (1 \leq i \leq n),
\]

\[
X_{\pm 2te_i} = \begin{pmatrix} E_{ii} & \sqrt{-1}E_{ii} \\ \sqrt{-1}E_{ii} & -E_{ii} \end{pmatrix}, \quad (1 \leq i \leq n),
\]

\[
X_{\pm e_i \pm e_k} = \begin{pmatrix} E_{jk} + E_{kj} & \sqrt{-1}(E_{jk} + E_{kj}) \\ \sqrt{-1}(E_{jk} + E_{kj}) & -E_{jk} - E_{kj} \end{pmatrix}, \quad (1 \leq j < k \leq n)
\]

\[
X_{\pm e_i \pm e_k} = \begin{pmatrix} E_{jk} - E_{kj} & -\sqrt{-1}(E_{jk} + E_{kj}) \\ -\sqrt{-1}(E_{jk} + E_{kj}) & E_{jk} - E_{kj} \end{pmatrix}, \quad (1 \leq j < k \leq n)
\]

and \( \mathcal{X}_\beta = \sum_{\mu = 2}^{10} \mathcal{X}(\beta = 2\varepsilon_1, 2\varepsilon_2, \varepsilon_1 + \varepsilon_2) \).

For \( m, l \in \mathbb{Z} \) let \( h(t; m, l) \) be the function on \( \mathbb{R}^n \) given by

\[
h(t; m, l) = \left( \prod_{i=1}^n \cosh t \right)^{\frac{1}{2}(m+l)} \left( \prod_{i=1}^n \sinh t \right)^{\frac{1}{2}(l-m)}
\]

A function \( u \) on \( G \) is called \((\tau_-, \tau_-)\)-spherical when it satisfies
where \( u(k_1 g k_2) = \tau_m(k_1)^{-1} u(g) \tau_2(k_2)^{-1} \) for all \( g \in G, k_1, k_2 \in K \).

We will calculate \((r_m, r_-)\)-radial parts of (3.6). For \(Sp(2, \mathbb{R})\) this was done by Iida\(^4\).

**Proposition 3.6** If \( \phi \in C^\infty(G) \) is a \((r_m, r_-)\)-spherical solution of (3.6), then the function

\[
\phi(t) = h(t; \epsilon \ell, \epsilon \ell) u \left( \exp \left( \sum_{j=1}^{n} t_j X_j \right) \right)
\]

satisfies

\[
(2 \partial_t \partial_{\ell} + (\coth(t_j + t_k) - \coth(t_j - t_k)) \partial_t + (\coth(t_j + t_k) + \coth(t_j - t_k)) \partial_{\ell}) \phi = 0
\]

for all \( 1 \leq j < k \leq n \).

**Lemma 3.7** The highest weight vector of the irreducible \(K\)-module \( \mathcal{V}_{\gamma_1 \gamma_2} \subset p^+ \otimes p^+ \) is up to constant given by

\[
X_{\gamma_1} \otimes X_{\gamma_2} + X_{\gamma_2} \otimes X_{\gamma_1} - \frac{1}{2} X_{\gamma_1 + \gamma_2} \otimes X_{\gamma_1 + \gamma_2}.
\]

**Proof.** Since each weight space of \( p^+ \otimes p^+ \) is multiplicity one, it is enough to show that (3.11) is a highest weight vector with weight \( \gamma_1 + \gamma_2 \). Since \( a + \beta \notin \Delta \) for all \( a \in \{ \pm \epsilon_1 \pm \epsilon_2, \pm 2 \epsilon_1, \pm 2 \epsilon_2 \} \) and

\[
\beta \in \{ \epsilon_j - \epsilon_k ; 1 \leq j < k \leq n \} \setminus \{ \epsilon_1 - \epsilon_2 \}.
\]

Thus it suffices to show that (3.11) is annihilated by \( X_{\epsilon_1 - \epsilon_2} \). It follows easily from direct computation. So we omit it. □

**Proof of Proposition 3.6.** We prove the proposition for \( \epsilon = -1 \). The case of \( \epsilon = +1 \) can be proved in a similar way.

The coefficient of vector (3.11) in \( \mathcal{H}_\alpha \) is up to constant given by

\[
\|2 \epsilon_1\|^2 \|2 \epsilon_2\|^2 (X_{-2 \epsilon_1} X_{-2 \epsilon_2} + X_{-2 \epsilon_2} X_{-2 \epsilon_1}) - 2 \| \epsilon_1 + \epsilon_2 \|^4 X_{\gamma_1 - \gamma_2}.
\]

which is a non-zero constant multiple of

\[
2X_{-2 \epsilon_1} X_{-2 \epsilon_2} - \frac{1}{2} X_{\gamma_1 - \gamma_2}^2.
\]

It follows from Proposition 6.6 (ii) and Proposition 7.1 in Iida\(^4\) that \((r_m, r_-)\)-radial part of element (3.13) gives differential equation (3.10) for \( j = 1, k = 2 \). For each \( \sigma \in S_n \) we get equation (3.10) for \( j = \sigma(1), k = \sigma(2) \) from the coefficient

\[
2X_{-2 \epsilon_{\sigma(1)}} X_{-2 \epsilon_{\sigma(2)}} - \frac{1}{2} X_{\epsilon_{\sigma(1)} - \epsilon_{\sigma(2)}}^2
\]

in \( \mathcal{H}_\alpha u = 0 \) of the highest weight vector with respect to the ordering \( \epsilon_{\sigma(1)} > \epsilon_{\sigma(2)} > \ldots > \epsilon_{\sigma(n)} \). □

Let \( u \) be a \((r_m, r_-)\)-spherical solution of (3.7). By Proposition 2.6 in Shimeno\(^10\), the function \( \phi \) given by (3.9) is a solution of the differential equation...
\[
\begin{align*}
&\left(\sum_{i=1}^{n} \frac{\partial^2}{\partial t_i^2} + \sum_{1 \leq j < k \leq n} ((\coth(t_j + t_k) + \coth(t_j - t_k)) \partial t_j \\
&\quad + (\coth(t_j + t_k) - \coth(t_j - t_k)) \partial t_k) \\
&\quad + \sum_{i=1}^{n} (\epsilon 2m \coth t_i + 2(1 - \epsilon l - \epsilon m) \coth 2t_i) \partial t_i\right) \phi \\
&= (s^2 - (n - \epsilon l)^2) \phi.
\end{align*}
\]

**Corollary 3.8** Assume that \( \phi \) is a \( W \)-invariant solution of (3.10) for \( 1 \leq j < k \leq n \) and (3.15) that is analytic at \( t = 0 \). Then \( \phi \) is a constant multiple of the hypergeometric function \( F(\exp(\sum_{i=1}^{n} t_i X_i); \lambda^{\alpha}, k) \) of Heckman and Opdam, where \( k \) is given by \( k_{\lambda_{\epsilon}, \epsilon} = 1/2 (1 - \epsilon l) \) and \( k_{\alpha, \epsilon} = 1/2 (1 - \epsilon l - \epsilon m) \) (1 \( \leq i \leq n \)). In particular, if \( l = m \), then \( \phi(t) \) is a constant multiple of \( h(t; \epsilon l, \epsilon l, \epsilon) \).

**Proof.** By the change of variables \( y_i = -\sinh^2 t_i (1 \leq i \leq n) \), the system of differential equations (3.10) for \( 1 \leq j < k \leq n \) and (3.15) become a system of differential equation that was investigated by Debiard and Gaveau. By Theorem 41, there is a unique solution up to constant subject to condition that it is \( W \)-invariant and analytic at \( t = 0 \). Moreover, by Corollary 41 it is a joint eigenfunction of commutative family of \( W \)-invariant differential operators, which turns out to be the hypergeometric function of Heckman and Opdam for the root system of type \( BC_n \). The latter statement follows from Remark 3.8 in Shimeno.

### 3.3 Boundary value map

If \( s \in \mathbb{C} \) satisfies condition

\[
\frac{1}{2} < \im\lambda - \lambda, H_1 > \in \{0, 1, 2, \ldots\} \quad \text{for all} \quad \bar{w} \in W_\phi \setminus W \quad \text{with} \quad \bar{w} \neq \bar{1},
\]

then we can define the boundary value map (cf. Section 4 of Shimeno),

\[
\beta_{\theta, s, l, \epsilon} : A(G/K; M_{\alpha, i}) \to B(\Theta; s, l, \epsilon).
\]

Condition (3.16) is equivalent to

\[
\{s + \epsilon l - n + 1, s, s - \epsilon l\} \cap \{0, -1, -2, \ldots\} = \emptyset.
\]

**Proposition 3.9** Assume that \( s, l, \) and \( \epsilon \in \{+1, -1\} \) satisfy condition (3.5). Then the partial Poisson transform \( \mathcal{P}_{\theta, s, l, \epsilon} \) is a \( G \)-isomorphism of \( B(\Theta; s, l, \epsilon) \) onto \( A(G/K; M_{\alpha, i}) \).

**Proof.** We consider the universal covering group of \( G \) and may assume that \( l \in \mathbb{C} \). First assume conditions (2.2) and (2.3) so that the Poisson transform \( \mathcal{P}_{\lambda, i} \) is bijective. It follows from Proposition 4.13 in Shimeno that

\[
\beta_{\lambda, s, l, \epsilon} = e^\theta(\lambda, l) \mathcal{P}_{\lambda, s, l, \epsilon} \mathcal{P}_{\lambda, l}^{-1}.
\]

Here
\[ c^\omega(\lambda, l) = c 2^{-s} \frac{\Gamma(s) \Gamma\left( \frac{1}{2} (s - \varepsilon l) \right) \Gamma\left( \frac{1}{2} (s + \varepsilon l - n + 2) \right) \Gamma\left( \frac{1}{2} (s + \varepsilon l + 1) \right)}{\Gamma\left( \frac{1}{2} (s + 1 + l) \right) \Gamma\left( \frac{1}{2} (s + 1 - l) \right) \Gamma\left( \frac{1}{2} (s - \varepsilon l - n - 1) \right) \Gamma\left( \frac{1}{2} (s - \varepsilon l + 1) \right)}, \]

where \( c \) is a non-zero constant (cf. Section 4 in Shimeno\(^{10}\)).

Since \( V_{x_j} = V_{x_0, i, e}, \) we have

\[ i(5 + e_l + 1))' \]

\[ V_{x_j} = V_{x_0, i, e} = C_9(A, l)ld. \quad (3.18) \]

Equation (3.18) holds under condition (3.17) by analytic continuation. Therefore the inverse of \( V_{x_j} \) is given by \( c^\omega(\lambda, l)^{-1} \beta_{b, l, i, e} \) under conditions (3.17) and \( c^\omega(\lambda, l) \neq 0, \) which are equivalent to (3.5). \( \square \)

From the Iwasawa decomposition \( g = \xi \Theta a \Theta n^{-1} = \xi \Theta a \Theta n\Theta(\Theta)^{-} \) and the Poincaré-Birkoff-Witt theorem it follows that

\[ U(g_c) = U(g_c)\xi \oplus U(\alpha_c + n_c)\Theta \oplus U(n(\Theta)\tilde{c} + a_c). \quad (3.19) \]

Let \( \pi \) be the projection of \( U(g_c) \) to \( U(n(\Theta)\tilde{c} + a_c) \) with respect to this decomposition. Let \( \iota_{b, l, e} \) be the algebra homomorphism of \( U(n(\Theta)\tilde{c} + a_c) \) to \( U(n(\Theta)\tilde{c} + a(\Theta)^{-}) \) such that \( \iota_{b, l, e}(Y) = Y \) if \( Y \in a(\Theta) \) and \( \iota_{b, l, e}(Y) = <\lambda - \rho, Y> \) if \( Y \in a_\theta \). We state the following proposition, which is a special case of Theorem 4.4 in Shimeno\(^{10}\).

**Proposition 3.10** Assume that \( \lambda = \lambda_{b, l, e} \) satisfies condition (3.16). Let \( u \in A(G/K, M_{b, l}) \) and \( U \in U(g_c) \). If \( Uu = 0 \) then \( \iota_{b, l, e} \pi(U)\beta_{b, l, i, e}(u) = 0. \)

**Proposition 3.11** Assume condition (3.17). Then boundary values \( \beta_{b, l, i, e}(u) \) of solutions of (3.6) and (3.7) satisfy

\[ \nu_{b, l, i, e}(u) = 0. \quad (3.20) \]

**Proof.** We prove the proposition for \( \varepsilon = -1. \) The case of \( \varepsilon = +1 \) can be proved in a similar way. We apply Proposition 3.10 to operators

\[ U_i = 2X_{-2\varepsilon_i} - \frac{1}{2} X_{-e_i - e_i} \quad (2 \leq i \leq n) \quad (3.21) \]

and

\[ U_{jk} = X_{-2\varepsilon_i} - X_{-e_j - e_k} \quad (2 \leq j < k \leq n). \quad (3.22) \]

Operator (3.21) is a coefficient in \( H^\Theta \) of weight vector of weight \( 2\varepsilon_i + 2\varepsilon_j \) as we see in the proof of Proposition 3.6. We see by direct computations that operator (3.22) is a coefficient in \( H^\Theta \) of weight vector of weight \( 2\varepsilon_i + \varepsilon_j + \varepsilon_k \).

We define elements \( E_{2\varepsilon_i} \in \mathfrak{g}_{\varepsilon_i} \) \((i = 1, \ldots, n)\) and \( E_{2\varepsilon_i + \varepsilon_k} \in \mathfrak{g}_{2\varepsilon_i + \varepsilon_k} \) \((1 \leq j \neq k \leq n)\) by

\[ E_{2\varepsilon_i} = \begin{pmatrix} 0 & E_{i} \\ 0 & 0 \end{pmatrix}, \quad (1 \leq i \leq n), \]

\[ E_{2\varepsilon_i + \varepsilon_k} = \begin{pmatrix} 0 & E_{i} + E_{k} \\ 0 & 0 \end{pmatrix}, \quad (1 \leq j < k \leq n) \]
\[ E_{\varepsilon_j - \varepsilon_k} = \begin{pmatrix} E_{jk} & 0 \\ 0 & -E_{kj} \end{pmatrix}, \quad (1 \leq j < k \leq n) \]

and \( E_{-\alpha} = \varepsilon E_{\alpha} (\alpha \in \Sigma^+) \).

We can show by direct computations that
\[
\tau_{\varepsilon, 1, \varepsilon \circ} \pi(U_i) = 2(s-n+1-l)(X_i - 2\sqrt{-1}E_{-\varepsilon_i} + l),
\]
which is identical to \( 2(s-n+1-l)X_{-\varepsilon_i} \) modulo \( \varepsilon \), and
\[
\tau_{\varepsilon, 1, \varepsilon \circ} \pi(U_{jk}) = 2(s-n+2-l)(E_{\varepsilon_j - \varepsilon_k} - \sqrt{-1}E_{-\varepsilon_j - \varepsilon_k}),
\]
which is identical to \( 2(s-n+2-l)X_{-\varepsilon_j - \varepsilon_k} \) modulo \( \varepsilon \). Since
\[
\{X_{-\varepsilon_i}, X_{-\varepsilon_j - \varepsilon_k} ; 2 \leq i \leq n, 2 \leq j < k \leq n\}
\]
forms a basis of \( \mathfrak{p}_\varepsilon \), we have (3.20) \( \square \)

**Remark 3.12** We can consider generalizations of Theorem 3.1 for
1. any Hermitian symmetric space,
2. parabolic subgroups that correspond to \( \Theta_k = \{a_k, a_{k+1}, \ldots, a_n\} \) \( (2 \leq k \leq n) \).

We will discuss these problems in a forthcoming paper.

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