On Pseudo-Affine Domains

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In what follows, all rings considered are commutative with identity.

We say that a ring $A$ is a Hilbert ring if each prime ideal of $A$ is an intersection of maximal ideals of $R$. It is known that a $k$-affine domain over a field $k$ is a Hilbert ring ([G, (31.11)]).

We say that a ring $A$ is a catenary ring if the following condition is satisfied: for any prime ideals $p$ and $q$ of $A$ with $p \subseteq q$, then exists a saturated chain of prime ideals starting from $p$ and ending at $q$, and all such chains have the same (finite) length. We say that a ring $A$ is a universally catenary ring if $A$ is Noetherian and every finitely generated $A$-algebra is catenary.

Let $k$ be a field and $R$ a $K$-affine domain. Then $R$ is Noetherian, Hilbert and catenary. Moreover $\dim R_m = Tr.\text{deg}_k R < +\infty$ for each maximal ideal $m$ of $R$.

Our objective in this paper is to investigate integral domains having these properties.

Throughout this paper, $k$ denotes a field and $R$ an integral domain containing $k$ and $K(R)$ denotes the quotient field of $R$ unless otherwise specified. Any unexplained terminology is standard, as in [M], [N].

**Definition 1.** An integral domain $R$ is called a pseudo-affine domain over $k$ (PAD($k$) for short) if the following conditions are satisfied:

(i) $R$ is Noetherian;
(ii) $R$ is Hilbert and catenary;
(iii) $\dim R_m = Tr.\text{deg}_k R < +\infty$ for each maximal ideal $m$ of $R$.

**Remark 2.** It is known that a $k$-affine domain is a PAD($k$) ([M, (5.6)]). A field $K$ containing $k$ is a PAD($k$) if and only if $K$ is algebraic over $k$.

The following Lemma 3 is shown in [O].

**Lemma 3.** Let $R$ be an integral domain containing a field $k$. Let

$$(0) = P_0 \subset P_1 \subset \cdots \subset P_r$$

be a strict ascending chain of prime ideals of $R$ and let $a_i \in P_i \setminus P_{i-1}$ $(1 \leq i \leq r)$. Then
$a_1, \ldots, a_r$ are algebraically independent over $k$.

Proof. Suppose that there exists a non-trivial polynomial $F(X_1, \ldots, X_r)$ in a polynomial ring $k[X_1, \ldots, X_r]$ such that $F(a_1, \ldots, a_r) = 0$. We can assume that $\deg F(X_1, \ldots, X_r)$ is minimal among such polynomials. Write:

$$0 = F(a_1, \ldots, a_r) = F_0(a_2, \ldots, a_r)a_1 + \cdots + F_n(a_2, \ldots, a_r)a_1^n.$$ 

Let $a'_2, \ldots, a'_r$ denote the images in $R/P$. By induction on $r$, we may assume that $a'_2, \ldots, a'_r$ are algebraically independent over $k$. Thus since $F(a'_1, \ldots, a'_r) = F_0(a'_2, \ldots, a'_r) = 0$, we have $F_0(X_2, \ldots, X_r) = 0$ in $k[X_2, \ldots, X_r]$. We have:

$$F(a_1, \ldots, a_r) = a_1(F(a_2, \ldots, a_r) + \cdots + F_n(a_2, \ldots, a_r)a_1^{n-1}) = 0,$$

and hence

$$F_1(a_2, \ldots, a_r) + \cdots + F_n(a_2, \ldots, a_r)a_1^{n-1} = 0.$$ 

By the minimality of $\deg F(X_1, \ldots, X_r)$, we conclude that:

$$F_1(X_2, \ldots, X_r) + \cdots + F_n(X_2, \ldots, X_r)X_1^{n-1} = 0$$

in $k[X_1, \ldots, X_r] = 0$ in $k[X_1, \ldots, X_r]$, a contradiction.


Proposition 4. $\dim R \leq \text{Tr.deg}_k R$.

Proof. This follows Lemma 3 immediately.


Corollary 4.1. Let $R$ be a PAD($k$) and let $p \in Ht_k(R)$. Then

(i) $\dim R/p = \dim R - 1$;
(ii) $\text{Tr.deg}_k R/p = \text{Tr.deg}_k R - 1$.

Proof. Since $R$ is catenary, $\dim R - 1 = \dim R/p$. By definiton, $\text{Tr.deg}_k R/p \leq \text{Tr.deg}_k R/p$ by Proposition 4. Thus $\text{Tr.deg}_k R/p = \text{Tr.deg}_k R - 1 = \dim R - 1 = \dim R/p$.


Proposition 5. Let $R$ be a PAD($k$) and let $p$ is a prime ideal of $R$. Then $R/p$ is also a PAD($k$).

Proof. Since $R$ is Hilbert (resp. Noetherian), so is $R/p$. Corollary 4.1 repeatedly, $\dim R/p = \dim R - \text{ht}(p) = \text{Tr.deg}_k R/p$.


Corollary 5.1. An integral domain which is a homomorphic image of a PAD($k$) is also a PAD($k$).

Proof. Let $p$ be a prime ideal of $\text{ht}(p) = 1$. Then it is clear that $R/p$ is a Hilbert ring. Hence $R/p$ is a PAD($k$) by Corollary 4.1. So we get our conclusion by induction on $\dim R$.


Proposition 6. Assume that $R$ is a PAD($k$). Then $\text{ht}(p) = \dim R - \dim R/p = \text{Tr.deg}_k R - \text{Tr.deg}_k R/p$ for each $p \in \text{Spec}(R)$.
Proof. This follows from the proof of Corollary 4.1 and \( \dim R = \text{Tr.deg}_a R \) and \( \dim R/\mathfrak{p} = \text{Tr.deg}_a R/\mathfrak{p} \) by definition.

**Lemma 7** ([G, (31.18)]). The following conditions are equivalent:
1. \( R \) is a Hilbert ring;
2. For each maximal ideal \( M \) of a polynomial ring \( R[X_1, \ldots, X_n] \), \( M \cap R \) is a maximal ideal of \( R \);
3. A polynomial ring \( R[X_1, \ldots, X_n] \) is a Hilbert ring;
4. \( R/I \) is a Hilbert ring for each proper ideal \( I \) of \( R \).

**Example.** Let \( k \) be a field and \( k[t] \) a polynomial ring. Put \( R = k[t]_{(t)} \). Then \( R[X]/(tX - 1) \cong k(t) \) and \( (tX - 1) \) is a maximal ideal of \( R[X] \) with \( R \cap (tX - 1) = \{0\} \). So \( R[X] \) is a Hilbert ring but is not a PAD(k).

**Lemma 8** ([G, (31.9)]). If \( R \) is a Hilbert ring and if \( M \) is a maximal ideal of a polynomial ring \( R[X_1, \ldots, X_n] \), then \( R[X_1, \ldots, X_n]/M \) is algebraic over \( R/M \cap R \).

L. J. Ratliff shows the following result:

**Lemma 9** (cf. [M, p. 31]) Let \( (A, m) \) be a Noetherian local domain. Then \( A \) is catenary if and only if \( ht(p) + \dim A/\mathfrak{p} = \dim A \) for each \( p \in \text{Spec}(A) \).

**Lemma 10.** Let \( R \) be a PAD(k), let \( R[X] \) be a polynomial ring and let \( P \) be a prime ideal of \( R[X] \) such that \((P \cap R)R[X] = P\). Then \( \dim R[X]/P = \text{Tr.deg}_a R[X]/P \).

**Proof.** Since \( R \) is a PAD(k), we have \( \dim R/P \cap R = \text{Tr.deg}_a R/P \cap R \) by Proposition 5. Since \((P \cap R)R[X] = P\), it follows that \( \dim R/P \cap R = \dim R[X]/P \). By the same reason, \( R[X]/P \) is algebraic over \( R/P \cap R \). Thus we have \( \text{Tr.deg}_a R/P \cap R = \text{Tr.deg}_a R[X]/P \). Hence \( \dim R[X]/P = \dim R/P \cap R = \text{Tr.deg}_a R/P \cap R = \text{Tr.deg}_a R[X]/P \).

**Proposition 11.** A PAD(k) is universally catenary.

**Proof.** We have only to prove a polynomial ring \( R[X] \) is catenary. Take \( P \in \text{Spec}(R[X]) \). First assume that \( P = pR[X] \) for some \( p \in \text{Spec}(R) \) i.e., \((P \cap R)R[X] = P\). Then \( ht(P) = ht(p) = (\dim R + 1) - (\dim R/p + 1) = \dim R[X] - \dim R[X]/pR[X] \) by Lemma 9. Second, assume that \((P \cap R) \neq P\). Then \( ht(p) - ht(P \cap R) = 1 \). Hence \( ht(P) = ht(P \cap R) + 1 = \dim R - \dim R/P \cap R + 1 = \dim R[X] - \text{Tr.deg}_a R/P \cap R = \dim R[X] - \text{Tr.deg}_a R/P \cap R = \dim R[X] - \text{Tr.deg}_a R[X]/P \cap R \). Therefore by Lemma 9, we conclude that \( R[X] \) is catenary.

Let \( A \) be a Noetherian domain and \( B \) a finitely generated extension domain. We say that the *dimension formula* holds between \( A \) and \( B \) if

\[
ht P = ht p + \text{Tr.deg}_a B - \text{Tr.deg}_a \mathfrak{p}, k(P)
\]
for every $P \in \text{Spec}(R)$, where $p = P \cap A$.

**Corollary 11.1.** Assume that $R$ is a PAD($k$). Then dimension formula holds between $R/p$ and $B$ for every prime ideal $p$ of $R$ and every finitely generated domain $B$ of $R/p$.

**Proof.** Since $R$ is universally catenary by Proposition 11, the conclusion follows from [M, (15.6)].

**Theorem 12.** The following conditions are equivalent:
(i) $R$ is a PAD($k$);
(ii) A polynomial ring $R[X_1, \ldots, X_n]$ is a PAD($k$);
(iii) Every integral domain containing $R$ which is finitely generated over $R$ is a PAD($k$).

**Proof.** (i) $\Rightarrow$ (ii) follows from Lemmas 7 and 8 because $\dim R[X_1, \ldots, X_n] = \dim R + n = \text{Tr.deg}_k R + n = \text{Tr.deg}_k R[X_1, \ldots, X_n]$. By Proposition 11, $R[X_1, \ldots, X_n]$ is catenary. (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) are immediately verified by Proposition 5.

**Proposition 13.** Let $R$ be a normal PAD($k$) and $A$ a Noetherian domain which is integral over $R$. Then $A$ is a PAD($k$).

**Proof.** Let $p$ be a prime ideal of $A$ with $\dim A/p = 1$. Since $A$ is integral and since $B$ is a Hilbert ring, $p$ is contained in infinitely many maximal ideals by Lying-Over Theorem. So by [G, (31, Ex. 22)], $A$ is a Hilbert ring. Let $M$ be a maximal ideal of $A$ and put $m = M \cap R$. Then $m$ is a maximal ideal of $R$. Note that $A/M$ is algebraic over $R/m$. Moreover $\dim A_m = \dim R_m$ by Going-Down Theorem. Since $\text{Tr.deg}_k A = \text{Tr.deg}_k R$ and $\dim R_m = \text{Tr.deg}_k R$, we have $\dim A_m = \text{Tr.deg}_k A$. It is easy to see that $A$ is catenary. Thus $A$ is PAD($k$).

**Proposition 14.** Let $A$ be a Noetherian domain containing $R$ with $K(A)$ algebraic over $K(R)$. If $A$ is faithfully flat over $R$ and $R$ is a PAD($k$), then $A$ is a PAD($k$).

**Proof.** Since a canonical morphism $\text{Spec}(A) \rightarrow \text{Spec}(R)$ is surjective. Let $M$ be a maximal ideal of $A$ and put $m = M \cap R$. Then $\text{Tr.deg}_k A = \text{Tr.deg}_k R = \dim R_m$ and $\dim R_m = \dim A_m$ by Going-Down Theorem. By the same way as the proof of Proposition 13, we can show that $A$ is Hilbert. Since $A$ is faithfully flat over the catenary ring $R$, $A$ is also catenary. Thus $A$ is a PAD($k$).

**Proposition 15.** Assume that $R$ be a PAD($k$) and let $m$ be a maximal ideal of $R$. Then $R/m$ is algebraic over $k$.

**Proof.** The field $R/m$ is a PAD($k$) by Proposition 5. So by the fact stated in Remark 2, $R/m$ is algebraic over $k$.

**Proposition 16.** Assume that $R$ is a normal PAD($k$). Let $L$ be a finite separable field extension of the quotient field $K(R)$ of $R$. Let $B$ be intermediate ring between $R$ and $L$ which is integral over $R$. Then $B$ is a PAD($k$).
Proof. By [N, (10.16)], the integral closure $R_L$ of $R$ in $L$ is a finite $R$-module. Hence $B$ is a finite $R$-module. So $B$ is $\text{PAD}(k)$ by Theorem 12. □

**Corollary 16.1.** Assume that $R$ is a $\text{PAD}(k)$ whose derived normal ring $\bar{R}$ is Noetherian. Let $L$ be finite separable extension field of the quotient field $K(R)$. Then the integral closure $R_L$ of $R$ in $L$ is a $\text{PAD}(k)$. 

Proof. By Proposition 13, $\bar{R}$ is a $\text{PAD}(k)$. Note that $R_L$ is integral closure of $\bar{R}$ in $L$. Since is Noetherian, $R_L$ is a $\text{PAD}(k)$ by Proposition 16. □

**Proposition 17.** Let $\bar{R}$ denote the derived normal domain of a Noetherian domain $R$. If $\bar{R}$ is a $\text{PAD}(k)$, then so is $R$.

Proof. The domain $R$ is Hilbert by Lying-Over Theorem, which is seen in the same maners of the proof of Proposition 13 because $\bar{R}$ is Hilbert and catenary (Proposition 11). Moreover $\text{Tr.deg}_k \bar{R}_m = \text{Tr.deg}_k \bar{R}_m = \dim \bar{R}_m \leq \dim R_m \leq \text{Tr.deg}_k \bar{R}_m$, where $M$ is maximal ideal of $\bar{R}$ lying over a maximal ideal $m$ of $R$. Hence $R$ is a $\text{PAD}(k)$. □

Let $A$ be a ring and $I$ an ideal of $A$. We recall that $J$ is called a reduction of $I$ if $J \subseteq I$ and $\sqrt{J^r} = I^{r+1}$ for at least one positive integer $r$ ([L], [O]). It is easy to see that $\sqrt{J} = \sqrt{I}$ and $ht(J) = ht(I)$.

**Proposition 18.** Assume that a Noetherian domain $R$ satisfies the condition : $\dim R = \text{Tr.deg}_k R := n$ and let $I$ be an ideal of $R$. Then $I$ has a reduction generated by $(n + 1)$-elements.

Proof. This follows from [L] or [O, (3.4)]. □

**Corollary 18.1.** Assume that $R$ is a $\text{PAD}(k)$ with $\dim R = n$. Then each ideal $I$ of $R$ has a reduction $J$ generated by $(n+1)$-elements.

**References**


