Further Development of Chebyshev Acceleration in connection with SAOR Method

Hiroshi NIKI, Ichi-ann OHSAKI and Masatoshi IKEUCHI

The authors are with the Graduate School of Science,
Okayama University of Science,
Ridai-1-1, Okayama-shi 700,
Japan

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Abstract :

The paper is concerned with an improvement over the symmetric accelerated overrelaxation (SAOR) method for an iterative solution of large linear systems. At first, the Chebyshev acceleration (or semi-iteration) procedure is introduced to the SAOR method, and the Non-Adaptive SAOR-SI algorithm is developed. Next, an adaptive procedure which estimates automatically the maximum eigenvalue of the SAOR iteration matrix is constructed. Moreover the Partial-Adaptive SAOR-SI algorithm including the adaptive procedure is proposed, and its characteristics are cleared with numerical results. A comparison with the optimum SOR algorithm is also given. It is finally proved that the proposed algorithms, Non-Adaptive SAOR-SI and Partial-Adaptive SAOR-SI algorithms, are efficient for the iterative solution.

1. Introduction

We are often faced with linear systems arising from numerical solution of partial differential equations by the finite difference method, the finite element method or the other methods. For example, the difference solution of elliptic partial differential equations is associated with steady states of potential, diffusion, fluid flow, and many other physical problems. In such the linear system, its coefficient matrix is frequently very large, i.e., number of unknowns is oftentimes a few hundred-sometimes several thousand! Taking account of storage and arithmetic requirements for current computers, it may be advantageous to solve the linear system with the iterative method rather than the direct method.

In this paper, we shall treat the iterative solution of large and sparse linear system. We study on further development of the symmetric accelerated overrelaxation (SAOR) method which has been introduced in [2, 5, 6] for solving the linear system

$$Au=b \tag{1}$$

where A is the NXN real and nonsingular matrix, b is the $NX1$ given vector and u is the $NX1$ vector to be determined. The SAOR method is understood to be a two-sweeps scheme consisting of the forward and the backward AOR methods. Under certain assumptions the SAOR method employing the optimum parameters (γ, ω) converges nearly as fast as the AOR method, but it is mostly slower than the AOR method in spite of extraneous works. The fact seems to preclude the SAOR method. However, by a combination with the acceleration procedure such as the conjugate gradient (CG) acceleration and the Chebyshev acceleration (or semi-iteration), it is expected that the SAOR method converges faster by an order-of-magnitude than the AOR method. Recently as an example of accelerated SAOR methods, based on the CG acceleration procedure, the Non-Adaptive SAOR-CG algorithm has been proposed in [6, 9] and then by applying the adaptive procedure which determines the SAOR parameters (γ, ω) automatically during the iteration process, the Adaptive SAOR-CG algorithm has been developed in [6, 9].

The objective in this paper is to present one more acceleration procedure with the Chebyshev acceleration on the basic SAOR method. In the Chebyshev acceleration, it is necessary to assume the SAOR parameters (γ, ω) and the spectral radius of SAOR iteration matrix $H(\gamma, \omega)$. Thus we can consider three versions of the Chebyshev acceleration: one is the non adaptive version (Non-Adaptive SAOR-SI algorithm) which estimates neither the SAOR parameters (γ, ω) nor the spectral radius $S(H(\gamma, \omega))$; the other is the partially adaptive version (Partial-Adaptive SAOR-SI algorithm) which improves adaptively the value of $S(H(\gamma, \omega))$ where the SAOR parameters (γ, ω) are fixed; another is the fully adaptive version (Full-Adaptive SAOR-SI algorithm) which estimates both of (γ, ω) and $S(H(\gamma,\omega))$. Among them, we will propose the Non-Adaptive SAOR-SI and Partial-Adaptive SAOR-SI algorithms. We also show some numerical results on the proposed algorithms and give a comparison with the SOR algorithm employing the optimum parameter (ω).

2. SAOR Method.

Assume that the coefficient matrix A of (1) is symmetric and positive definite. Without loss of generality, A may be then split into

$$A=I-L-U \tag{2}$$,

where I is the identity, and L and U are respectively the lower and upper triangular parts of A. For the n th iterated vector $u^{(n)}$, the SAOR method is defined [5, 6] as

$$u^{(n+1/2)}=\tilde{L}(\gamma, \omega)u^{(n)}+k_F \tag{3}$$

and

$$u^{(n+1)}=\tilde{U}(\gamma, \omega)u^{(n+1/2)}+k_B \tag{4}$$,

where γ and ω are respectively called the acceleration and overrelaxation parameters. Also $\tilde{L}(\gamma, \omega)$ and $\tilde{U}(\gamma, \omega)$ are respectively the corresponding iteration matrices to the forward AOR and backward AOR methods [5, 6] expressed as

$$\tilde{L}(\gamma, \omega)=(I-\gamma L)^{-1}[(1-\omega)I+(\omega-\gamma)L+\omega U]=I-\omega(I-\gamma L)^{-1}A \tag{5}$$

and

$$\tilde{U}(\gamma, \omega)=(I-\gamma U)^{-1}[(1-\omega)I+(\omega-\gamma)U+\omega L]=I-\omega(I-\gamma U)^{-1}A \tag{6}$$.

Eliminating $u^{(n+1/2)}$ from (3) and (4), we obtain

$$u^{(n+1)}=H(\gamma, \omega)u^{(n)}+k(\gamma, \omega)\ ; \tag{7}$$

$$H(\gamma, \omega)=\tilde{U}(\gamma, \omega)\tilde{L}(\gamma, \omega)=I-\omega^2(I-\gamma U)^{-1}M(I-\gamma L)^{-1}A \tag{8}$$

and

$$k(\gamma, \omega)=\tilde{U}(\gamma, \omega)k_F+k_B=\tilde{U}(\gamma, \omega)(I-\gamma L)^{-1}b+(I-\gamma U)^{-1}b \tag{9}$$,

where $H(\gamma, \omega)$ is the SAOR iteration matrix, and M is defined by

$$M=\frac{1}{\omega}[(2-\omega)I+(\omega-\gamma)B] \tag{10}$$,

in which $B(=L+U)$ is the Jacobi iteration matrix. Notice that for $\gamma=\omega$ $H(\gamma, \omega)$ is equivalent to the iteration matrix of the SSOR method [7].

3. Non-Adaptive SAOR-SI Algorithm

Now, let $A^{1/2}$ be the square root satisfying $(A^{1/2})^2=A$. Then we can define the matrices $H'(\gamma, \omega)$, $\tilde{L}'(\gamma, \omega)$ and $\tilde{U}'(\gamma, \omega)$ being similar to $H(\gamma, \omega)$, $\tilde{L}(\gamma, \omega)$ and $\tilde{U}(\gamma, \omega)$, respectively, as follows:

$$H'(\gamma, \omega)=A^{1/2}H(\gamma, \omega)A^{-1/2} = \tilde{U}'(\gamma, \omega)\tilde{L}'(\gamma, \omega) \tag{11},$$

where

$$\tilde{L}'(\gamma, \omega)=A^{1/2}\tilde{L}(\gamma, \omega)A^{-1/2} = I-\omega A^{1/2}(I-\gamma L)^{-1}A^{1/2} \tag{12}$$

and

$$\tilde{U}'(\gamma, \omega)=A^{1/2}\tilde{U}(\gamma, \omega)A^{-1/2} = I-\omega A^{1/2}(I-\gamma U)^{-1}A^{1/2} \tag{13}.$$

Since A is symmetric, we can readily see

$$\tilde{U}'(\gamma, \omega)=(\tilde{L}'(\gamma, \omega))^T \tag{14},$$

which in view of (8) gives rise to

$$H'(\gamma, \omega)=(\tilde{L}(\gamma, \omega))^T(\tilde{L}(\gamma, \omega)) \tag{15}.$$

If we choose γ and ω such that

$$0<\omega<2 \text{ and } \omega+\frac{2-\omega}{m(B)}<\gamma<\omega+\frac{2-\omega}{M(B)} \tag{16},$$

in which $m(B)$ and $M(B)$ are respectively the minimum and maximum eigenvalues of B, then the real symmetric matrix M defined by (10) is proved to be positive definite (see [2, 6]). From the relation in (8), we obtain

$$\begin{aligned} I-H'(\gamma, \omega)&=A^{1/2}(I-H(\gamma,\omega))A^{-1/2} \\ &=[\omega M^{1/2}(I-\gamma L)^{-1}A^{1/2}]^T[\omega M^{1/2}(I-\gamma L)^{-1}A^{1/2}] \end{aligned} \tag{17},$$

which is symmetric and positive definite. Hence we can use the $A^{1/2}$ as a symmetrization matrix [3] required in the application of the Chebyshev acceleration to the SAOR method.

Let us define the nth iterated vector $u^{(n)}$ during the Non-Adaptive SAOR-SI algorithm as

$$u^{(n+1)}=\rho_{n+1}(\nu_{n+1}\delta^{(n)}+u^{(n)})+(1-\rho_{n+1})u^{(n-1)} \tag{18},$$

where $\delta^{(n)}$ is the pseudo-residual vector represented by

$$\delta^{(n)}=H(\gamma, \omega)u^{(n)}+k(\gamma, \omega)-u^{(n)} \tag{19},$$

also ν_n and ρ_n are the Chebyshev parameters defined by

$$\nu_{n+1}=\frac{2}{2-S(H(\gamma, \omega))} \tag{20}$$

and

$$\begin{cases} \rho_1=1 \\ \rho_2=(1-\frac{1}{2}\sigma^2)^{-1} \\ \rho_{n+1}=(1-\frac{1}{4}\sigma^2\rho_n)^{-1}, \; n>2 \end{cases} \tag{21},$$

in which σ is given by

$$\sigma = \frac{S(H(\gamma, \omega))}{2-S(H(\gamma, \omega))} \tag{22}$$

In the non-adaptive algorithm the formulas (18)—(22) are simply iterated until a suitable criterion for convergence is achieved. The algorithm is shown in the flowchart of Figure 1.

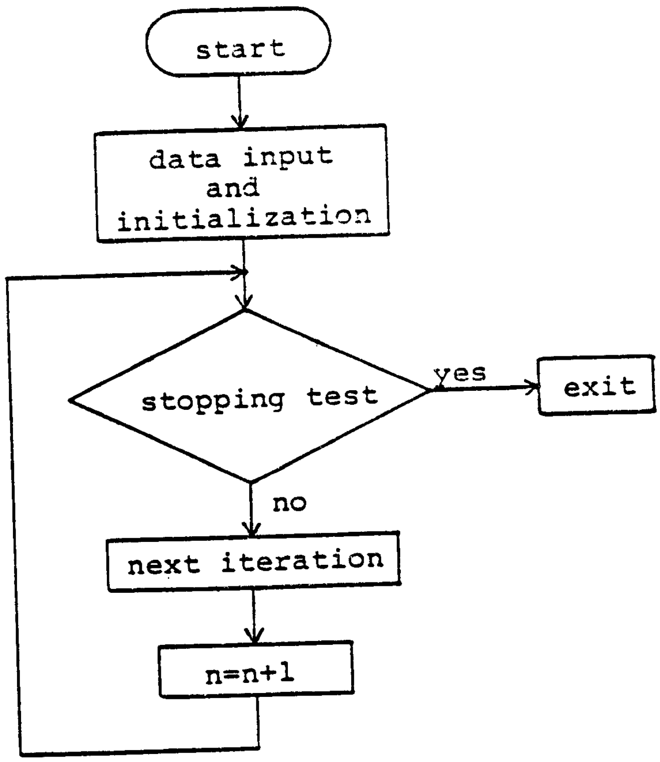

Figure 1: Flowchart of the Non-Adaptive SAOR-SI algorithm.

The eigenvalues of the SAOR iteration matrix $H(\gamma, \omega)$ are real, positive and less than unity. In the application of the Chebyshev acceleration to the SAOR method it is an idea to use the exactly maximum and minimum eigenvalues. However, the rapidity of the convergence is affected little by the minimum eigenvalue because the minimum eigenvalue of the SAOR iteration matrix is very close to zero. Thus it is sufficient to employ only the maximum eigenvalue. Also since the maximum eigenvalue plays an very important role for convergence in the Non-Adaptive SAOR-SI algorithm (see Table 1), we have to choose the maximum eigenvalue carefully as the initial data.

4. Partial-Adaptive SAOR-SI Algorithm

Let us introduce the partially adaptive procedure to the Non-Adaptive SAOR-SI algorithm for estimating the spectral radius (maximum eigenvalue) $S(H(\gamma, \omega))$

Table 1 Comparison of the algorithms.

1/h=	20	40	60	80	100
Optimum SOR algorithm	58	115	173	231	289
Non-Adaptive SAOR-CG algorithm					
$(\gamma, \omega)=(1.40, 1.54)$	13	23	34	44	53
$(\gamma, \omega)=(\text{opt}\gamma, \text{opt}\omega)$	14	20	24	28	32
Adaptive SAOR-CG algorithm					
(F=0.85)	16	27	41	45	50
Non-Adaptive SAOR-SI algorithm					
$(\gamma, \omega)=(1.40, 1.54)$ and ME=0.99	69	70	72	127	230
$(\gamma, \omega)=(\text{opt}\gamma, \text{opt}\omega)$ and optimum ME	15	21	27	30	34
Partial-Adaptive SAOR-SI algorithm (F=0.75)					
$(\gamma, \omega)=(1.40, 1.54)$	28	58	79	106	133
$(\gamma, \omega)=(\text{opt}\gamma, \text{opt}\omega)$	21	31	39	46	52

of the SAOR iteration matrix $H(\gamma, \omega)$. The Partial-Adaptive SAOR-SI algorithm is expressed as follows: for the nth iterated vector $u^{(n)}$

$$u^{(n+1)}=\rho_{n+1}(\nu_{n+1}\delta^{(n)}+u^{(n)})+(1-\rho_{n+1})u^{(n-1)} \tag{23},$$

where $\delta^{(n)}$ is the pseudo-residual vector which is the same form with the one given by (19), and ν_n and ρ_n are the Chebyshev parameters defined by

$$\nu_{n+1}=\frac{2}{2-S_E(H(\gamma, \omega))} \tag{24}$$

and

$$\rho_{n+1}=\begin{cases} 1; & n=s \\ (1-\frac{1}{2}\sigma_E{}^2)^{-1}; & n=s+1 \\ (1-\frac{1}{4}\sigma_E{}^2\rho_n)^{-1}; & n\geqq s+2 \end{cases} \tag{25},$$

in which

$$\sigma_E=\frac{S_E(H(\gamma, \omega))}{2-S_E(H(\gamma, \omega))} \tag{26}.$$

The partially adaptive version involves the parameter change test and the parameter estimation procedures.

(1) Parameter change test procedure.

We change $S(H(\gamma, \omega))$ whenever

$$\frac{\|\delta^{(n)}\|_A{}^{1/2}}{\|\delta^{(s)}\|_A{}^{1/2}}\geqq\left(\frac{2r^{p/2}}{1+r^p}\right)^F \tag{27},$$

where

$$p=n-s \tag{28}$$

and

$$r=\frac{1-\sqrt{1-\sigma_E^2}}{1+\sqrt{1-\sigma_E^2}} \tag{29}.$$

Here F is the damping factor to be selected in the interval [0, 1].

(2) Parameter estimation procedure.

Once we have decided to change $S(H(\gamma, \omega))$, we take new value of

$$[S'_E(H(\gamma, \omega))]_{\mathrm{NEW}}=\max(S_E(H(\gamma, \omega)),\ S'_E(H(\gamma, \omega))) \tag{30},$$

where $S'_E(H(\gamma, \omega))$ are determined by the Rayleigh quotient

$$S'_E(H(\gamma, \omega))=\frac{(W\delta^{(n)},\ WH(\gamma, \omega)\delta^{(n)})}{(W\delta^{(n)},\ W\delta^{(n)})} \tag{31}.$$

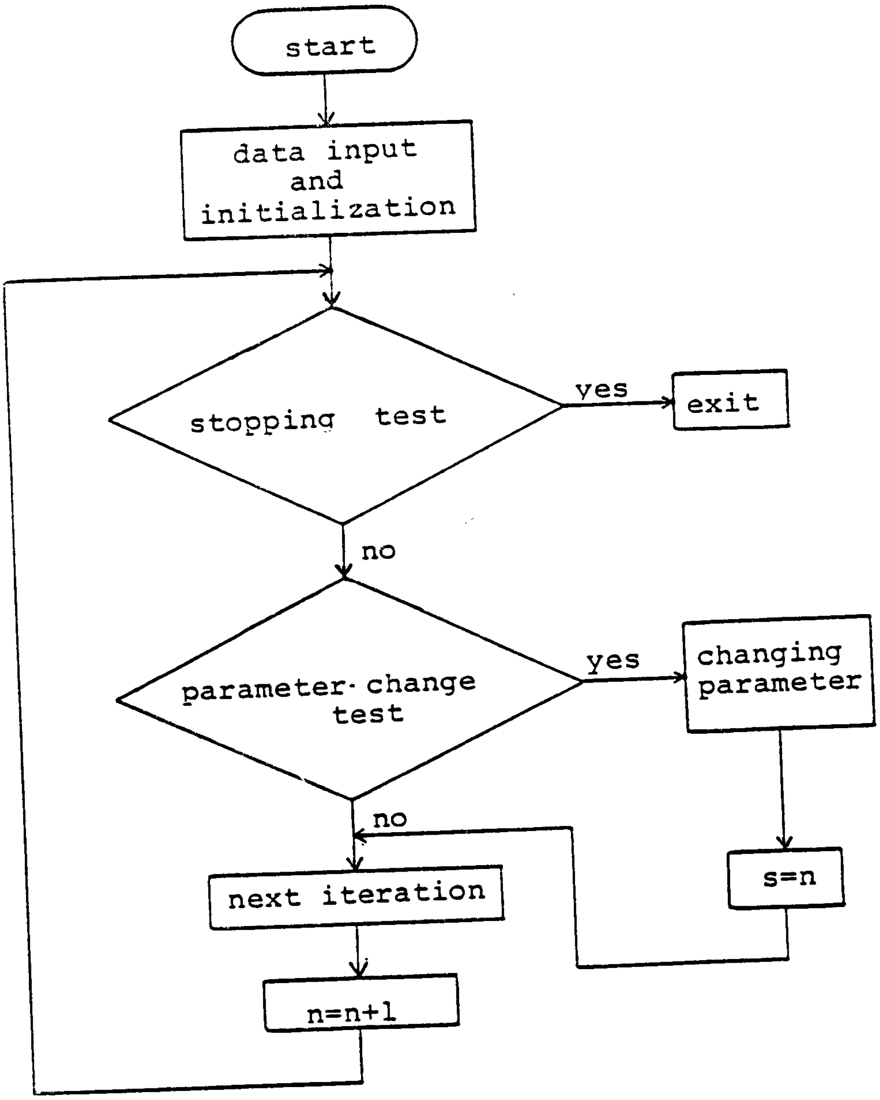

Figure 2: Flowchart of the Partial-Adaptive SAOR-SI algorithm.

If the new value is determined, we set $s=n$.

This partial adaptive procedure is shown in the flowchart of Figure 2.

In the partially adaptive procedure, there may be key question how we treat the symmetrization matrix $A^{1/2}$ in the computational program. It is almost impossible to make the matrix $A^{1/2}$ in practice, however for any vector v, by use of the transformation

$$(A^{1/2}v,\ A^{1/2}v)=(v,\ Av) \tag{32}$$

we can compute the $A^{1/2}$-norm with simple procedure. In the parameter estimation procedure, we employ the Rayleigh quotient with the symmetrization matrix $A^{1/2}$. If the spectral radius estimated is beyond the unity, we set $S(H(\gamma, \omega))=0.999$ and then we do not change $S(H(\gamma, \omega))$ any more.

5. Numerical Experiments.

In order to examine our algorithms we work out two types of model problems which involve the generalized Dirichlet problem with respect to the elliptic partial differential equation

$$\frac{\partial}{\partial x}\left(A\frac{\partial U}{\partial x}\right)+\frac{\partial}{\partial y}\left(C\frac{\partial U}{\partial y}\right)=0 \tag{33}$$

in the unit square ($0\leqq x\leqq 1$, $0\leqq y\leqq 1$), where $U=0$ is imposed on the whole boundary. Various choices of the coefficients $A(x, y)$ and $C(x, y)$ [8, 9] are considered. We now deal with the first type (model 1) that $A(x, y)=1$ and $C(x, y)=1$, i. e., the Laplace's equation

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=0 \tag{34}$$

Here the five-points difference formula is adopted for the discretization of the model problems. All the algorithms to be treated in the numerical experiments are terminated when the iterated vector $u^{(n)}$ is satisfied by the following criterion:

$$\|\varepsilon^{(n)}\|_{A^{1/2}}=\|u^{(n)}-\bar{u}\|_{A^{1/2}}<10^{-6} \tag{35}$$

where $\varepsilon^{(n)}$ is the nth error vector for the exact solution of $\bar{u}$. Also the initial vector $u^{(0)}$ is chosen such as all its elements are equal to be $1/(1/h-1)$, where h is the square mesh size.

(1) Characteristics of Chebyshev accelerations.

At first, we shall expose the characteristics of the Chebyshev accelerations on the SAOR method. Figure 3 shows the iteration numbers required for convergence in connection with the damping factor F in the Partial-Adaptive SAOR-SI algorithm.

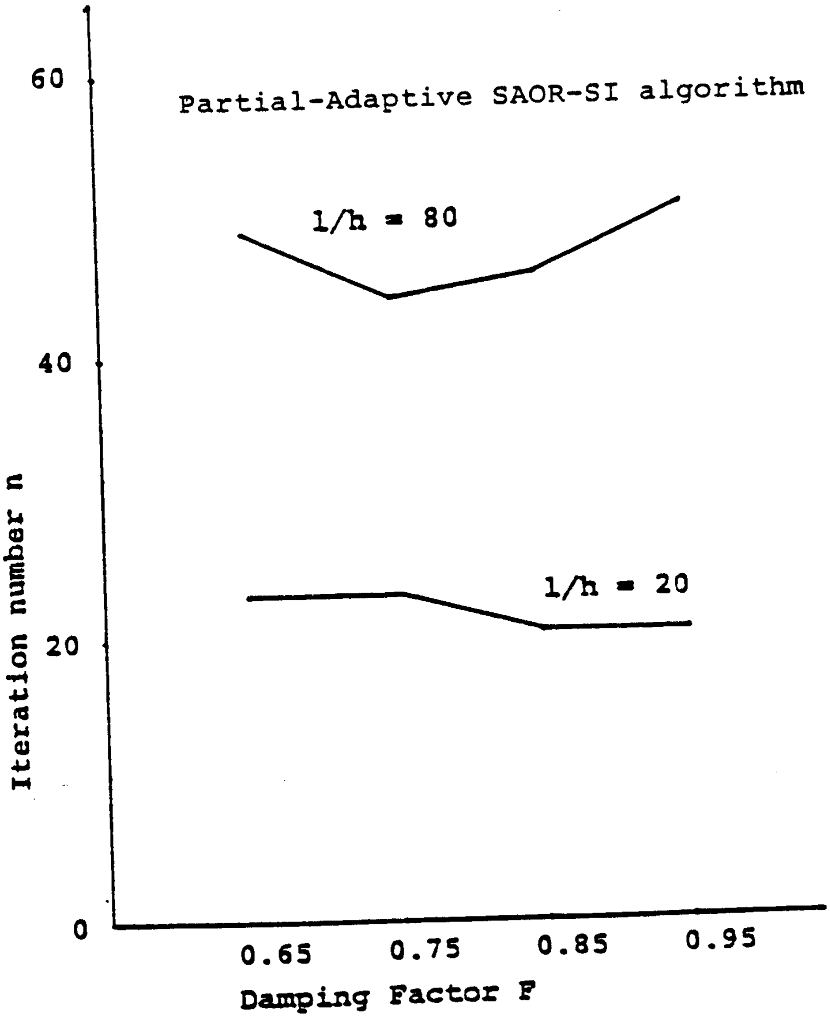

Figure 3: Iteration number versus damping factor for the Partial-Adaptive SAOR-SI algorithm.

If we work with F being very close to unity, we can see that the spectral radius $S(H(\gamma, \omega))$ of the SAOR iteration matrix $H(\gamma, \omega)$ is changing much frequently. With too small values of F, they are not changing often enough. However, as seen from the result in the Figure 3, the effectiveness of the adaptive procedure is relatively insensitive to F only in view of the iteration numbers required for convergence. Table 2, Table 3 and Table 4 show how the $S(H(\gamma, \omega))$ have changed during the adaptive processes with the damping factor F=0.65, 0.75 and 0.85, respectively. It can be seen from the result in tables that the number of parameter changes is too frequent when F is larger than some typical value, where the typical value in our Partial-Adaptive SAOR-SI algorithm is found out to be 0.75. Therefore such the selection of F causes the loss of computational time and the waste of arithmetic works for the adaptive procedure. Thus it can be suggested that the selection of F is so much important. Figure 4 shows the convergence domain

Table 2 Partial-Adaptive SAOR-SI algorithm (F=0.65).

	Iteration Number	S(H(γ, ω))
h=1/20	4	0.74958
(γ, ω) = (1.7795, 1.7617)	9	0.81042
	21	Convergence
h=1/40	3	0.76788
(γ, ω) = (1.8745, 1.8557)	7	0.88029
	19	0.90180
	31	Convergence
h=1/60	3	0.82702
(γ, ω) = (1.9205, 1.9012)	9	0.91158
	17	0.93307
	37	Convergence
h=1/80	3	0.86159
(γ, ω) = (1.9444, 1.9249)	9	0.91313
	15	0.94614
	47	Convergence
h=1/100	3	0.86779
(γ, ω) = (1.9491, 1.9296)	9	0.91740
	14	0.95437
	55	Convergence

Table 3 Partial-Adaptive SAOR-SI algorithm (F=0.75).

	Iteration Number	S(H(γ, ω))
h=1/20	3	0.70302
(γ, ω) = (1.7795, 1.7617)	5	0.78642
	12	0.81178
	21	Convergence
h=1/40	3	0.76788
(γ, ω) = (1.8745, 1.8557)	6	0.86041
	10	0.89626
	31	Convergence
h=1/60	3	0.82702
(γ, ω) = (1.9205, 1.9012)	7	0.88438
	11	0.92686
	30	0.93395
	39	Convergence
h=1/80	3	0.86159
(γ, ω) = (1.9444, 1.9249)	7	0.89285
	11	0.93243
	17	0.94779
	46	Convergence
h=1/100	3	0.86779
(γ, ω) = (1.9491, 1.9296)	7	0.86754
	11	0.93865
	16	0.95657
	52	Convergence

Table 4 Partial-Adaptive SAOR-SI algorithm (F=0.85).

	Iteration Number	$S(H(\gamma, \omega))$
$h=1/20$	2	0.53549
$(\gamma, \omega)=(1.7795, 1.7617)$	3	0.66000
	4	0.71843
	5	0.75114
	7	0.78580
	10	0.80772
	22	Convergence
$h=1/40$	2	0.62930
$(\gamma, \omega)=(1.8745, 1.8557)$	3	0.74619
	4	0.77775
	6	0.80794
	7	0.84581
	9	0.87051
	10	0.89021
	15	0.89937
	32	Convergence
$h=1/60$	2	0.67081
$(\gamma, \omega)=(1.9205, 1.9012)$	3	0.80207
	4	0.83417
	7	0.85074
	8	0.87761
	10	0.89210
	11	0.90950
	14	0.92131
	17	0.93071
	34	0.93381
	41	Convergence
$h=1/80$	2	0.67279
$(\gamma, \omega)=(1.9444, 1.9249)$	3	0.82563
	4	0.86837
	7	0.88104
	10	0.89841
	11	0.91549
	14	0.92971
	15	0.94025
	20	0.94627
	29	0.95031
	46	Convergence
$h=1/100$	2	0.65452
$(\gamma, \omega)=(1.9491, 1.9296)$	3	0.82459
	4	0.87519
	7	0.88801
	10	0.90357
	11	0.92038
	14	0.93654
	15	0.94857
	19	0.95435
	25	0.95891
	52	Convergence

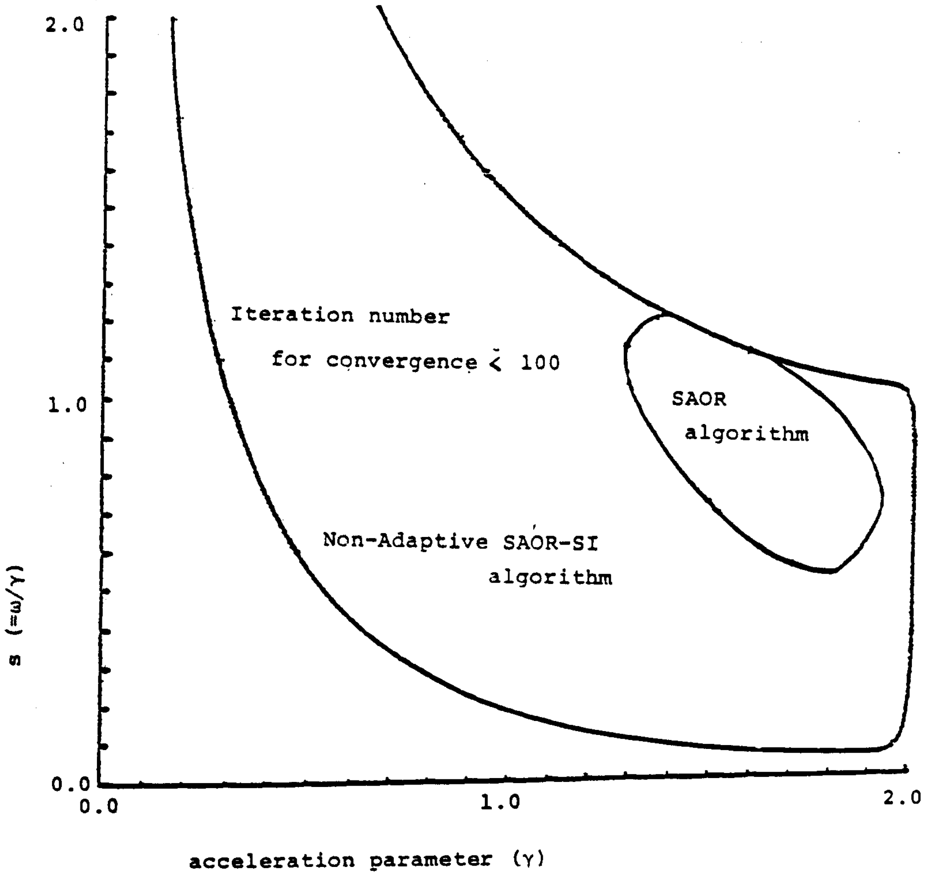

Figure 4: Convergence domain.
h = 1/20

to the SAOR parameters γ and $s(=\omega/\gamma)$ in the SAOR algorithm and Non-Adaptive SAOR-SI algorithm, where the input data $S(H(\gamma, \omega))=0.99$ is employed. By combining with the Chebyshev acceleration procedure, the convergence domain in the SAOR method is extended, and thus we can expect a fast convergence for a rough selection of (γ, ω) in both of the Non-Adaptive and Partial-Adaptive SAOR-SI algorithms.

(2) Comparison with other algorithms.

Here we present two examples for comparison. One is the comparison with the Partial-Adaptive SAOR-SI algorithm, the Non-Adaptive SAOR-SI algorithm and the optimum SOR algorithm. The other is the comparison with the Chebyshev acceleration and conjugate gradient (CG) accceleration on the SAOR method, i.e., the Non-Adaptive and Adaptive SAOR-CG algorithms. Table 1 gives the iteration numbers required for convergence in all the algorithms to be compared, i. e., the optimum SOR algorithm, the Non-Adaptive SAOR-SI algorithm, the Partial-Adaptive SAOR-SI algorithm, the Non-Adaptive SAOR-CG algorithm and the Adaptive SAOR-

CG algorithm. In three of the Non-Adaptive SAOR-SI algorithm, the Partial-Adaptive SAOR-SI algorthm and the Non-Adaptive SAOR-CG algorithm, the SAOR parameters are taken as $(\gamma, \omega)=(1.40, 1.54)$ and (γ_b, ω_b), where (γ_b, ω_b) are the optimum parameters determined experimentally. In the Adaptive SAOR-CG algorithm, the SAOR parameters (γ, ω) are determined automatically during the iteration process. Also the input data for the estimates of $S(H(\gamma, \omega))$ in the Non-Adaptive SAOR-SI algorithm are 0.99 and the experimentally convenient value. The SOR parameter ω is taken as $\omega=2(1+\sqrt{1-M(B)^2})^{-1}$, where $M(B)$ is the maximum eigenvalue of the Jacobi iteration matrix B. As expected, both of the Partial-Adaptive and the Non-Adaptive SAOR-SI algorithms have achieved the considerably faster convergence than the SOR algorithm. Next, let us consider a comparison with the Chebyshev acceleration and CG acceleration on the SAOR method. In the case of the non adaptive version, the effectiveness of both the algorithms with the optimum parameters are almost comparable. However, taking account that the Non-Adaptive SAOR-SI algorithm requires not only the SAOR parameters (γ, ω) but also $S(H(\gamma, \omega))$, it is advantageous for the CG acceleration to require only the SAOR parameters (γ, ω) and to be at least comparable in the effectiveness to the Non-Adaptive SAOR-SI algorithm. In the case of the adaptive version, we can suggest that the Partial-Adaptive SAOR-SI algorithm is inferior to the Adaptive SAOR-CG algorithm because of the same facts with the above.

(3) Further applications.

We try to test the feasibility and efficiency of the Chebyshev acceleration on the SAOR method for more general problems, i.e., we choose the coefficients $A(x, y)$ and $C(x, y)$ in (33) as in Table 5. Table 6 and Table 7 give how $S(H(\gamma, \omega))$

Table 5 Further application (model 2). $A(x, y)=C(x, y)=e^{10(x+y)}$

1/h	20	40	60	80	100
SOR algorithm	72	161	241	321	401
Non-Adaptive SAOR-SI algorithm $(\gamma, \omega)=(1.40, 1.54)$ ME $=0.99$	88	90	90	89	89
Partial-Adaptive SAOR-SI algorithm $(\gamma, \omega)=(\text{opt}\gamma, \text{opt}\omega)$ F $=0.65$	22	32	42	50	53
Adaptive SAOR-CG algorithm F $=0.85$	26	61	87	102	95

Table 6 Partial-Adaptive SAOR-SI algorithm (F=0.65).
(model 2)

	Iteration Number	S(H(γ, ω))
h=1/20	7	0.50367
(γ, ω) = (1.7795, 1.7617)	9	0.58417
	11	0.61273
	14	0.62291
	16	0.64710
	19	0.65234
	22	Convergence
h=1/40	4	0.57434
(γ, ω) = (1.8745, 1.8557)	6	0.61034
	7	0.68946
	8	0.71758
	9	0.73656
	14	0.75212
	16	0.76736
	22	0.78088
	25	0.79222
	29	0.80075
	32	Convergence
h=1/60	4	0.75123
(γ, ω) = (1.9205, 1.9012)	9	0.81664
	16	0.83132
	25	0.84939
	29	0.86199
	34	0.86709
	40	0.87400
	42	Convergence
h=1/80	4	0.79820
(γ, ω) = (1.9444, 1.9249)	5	0.84770
	11	0.86411
	18	0.87804
	21	0.88799
	30	0.89542
	34	0.90219
	40	0.90454
	47	0.91079
	50	Convergence
h=1/100	4	0.79256
(γ, ω) = (1.9491, 1.9296)	5	0.80113
	6	0.85462
	11	0.87241
	13	0.88708
	20	0.88816
	23	0.90050
	32	0.90342
	36	0.91076
	42	0.91410
	50	0.91869
	53	Convergence

Table 7 Partial-Adaptive SAOR-SI algorithm (F=0.75).
(model 2)

	Iteration Number	$S(H(\gamma, \omega))$
$h=1/20$	6	0.56285
$(\gamma, \omega) = (1.7795, 1.7617)$	9	0.58572
	10	0.61476
	14	0.62139
	16	0.64625
	18	0.65025
	20	0.65807
	22	Convergence
$h=1/40$	4	0.57437
$(\gamma, \omega) = (1.8745, 1.8557)$	6	0.61034
	7	0.68946
	8	0.71758
	9	0.73656
	16	0.76236
	22	0.77127
	24	0.77806
	26	0.78611
	28	0.79278
	30	0.79755
	32	0.80070
	35	Convergence
$h=1/60$	4	0.75123
$(\gamma, \omega) = (1.9205, 1.9012)$	8	0.76606
	9	0.81071
	11	0.82310
	12	0.82439
	19	0.83721
	20	0.84306
	24	0.84765
	32	0.85487
	36	0.86399
	41	0.86838
	44	0.87018
	47	Convergence
$h=1/80$	4	0.79820
$(\gamma, \omega) = (1.9444, 1.9249)$	5	0.84770
	11	0.86411
	18	0.86988
	19	0.87019
	20	0.87838
	23	0.88555
	26	0.88982
	34	0.89071
	38	0.89804
	43	0.90380
	46	0.90483
	50	0.90619
	54	0.90925
	57	Convergence

continued

h=1/100	4	0.82995
(γ, ω) = (1.6491, 1.9296)	6	0.83876
	11	0.87033
	13	0.88680
	26	0.89385
	28	0.89840
	30	0.90131
	32	0.90283
	34	0.90309
	44	0.90495
	46	0.90763
	48	0.91007
	51	0.91343
	54	0'91486
	58	0.91587
	62	0.91769
	64	Convergence

have changed during the adaptive process with the damping factors $F=0.65$ and $F=0.75$ and the SAOR parameters $(\gamma, \omega)=(1.40, 1.54)$. It shows that the choice of the damping factor $F=0.65$ is better than that of $F=0.75$. Thus it will be said that the choice of the damping factor depends on the problems, but it may be enough to choose the damping factor in our adaptive procedure $F=0.65-0.75$, or at least $F=0.55$ in order to work out effectively. For comparison purposes, we show the iteration numbers required for convergence in the Non-Adaptive and Partial-Adaptive SAOR-SI algorithms and the SOR algorithm with $\omega=2(1+\sqrt{1-M(B)^2})^{-1}$. It is clear from the result in Table 5 that the Non-Adaptive and Partial-Adaptive SAOR-SI algorithms guarantee their feasibility and efficiency for more general problems.

6. Concluding Remarks.

In this paper, we propose two versions of the Chebyshev acceleration procedure, i.e., Non-Adaptive SAOR-SI algorithm and Partial-Adaptive SAOR-SI algorithm. The present algorithms are based on (i) the formulation of the SAOR method, (ii) the introduction of the Chebyshev acceleration procedure and (iii) the development of the partially adaptive procedure.

In the numerical experiments we have shown that the Partial-Adaptive SAOR-SI algorithm estimates $S(H(\gamma, \omega))$ effectively and converges in a few iterations. It can be seen from the result in Table 1 and Table 5 for comparison purposes that the Non-Adaptive and the Partial-Adaptive SAOR-SI algorithms are far superior to the optimum SOR algorithm. However, in comparison with the Adaptive SAOR-CG algorithm we have obtained negative (or non positive) result in the following items:

(1) the iteration numbers required for convergence are comparable,

(2) the amount of the computational works in the Partial-Adaptive SAOR-SI algorithm are slightly smaller than that of the Adaptive SAOR-CG algorithm, and

(3) the Partial-Adaptive SAOR-SI algorithm requires the SAOR parameters (γ, ω) as the initial (input) data, but no parameter requires in the Adaptive SAOR-CG algorithm.

The disadvantage of (3) is a very important factor in the application to more general problems. This disadvantage may be removed by a development of the Full-Adaptive SAOR-SI algorithm, also by further application to more general problems.

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