Symmetric Accelerated Overrelaxation Method

Hiroshi NIKI*, Seiji YAMADA*, Masatoshi IKEUCHI*
and Hideo SAWAMI**

*Graduate School of Applied Mathematics, Okayama University of Science
**Department of Applied Mathematics, Okayama University of Science
Ridai-cho 1-1, Okayama 700, JAPAN

(Received September 24, 1982)

Contents :

Abstract

In this paper, a symmetric accelerated overrelaxation (SAOR) method is proposed, and the convergence theorem is derived. An SAOR-CG algorithm is also presented, which is formulated by introducing the conjugate gradient (CG) procedure to the SAOR method. Moreover, some results obtained in numerical experiments are shown.

1. Introduction

Iterative solutions of linear systems generated by discretizations of partial differential equations are studied. The linear systems are generally characterized by the sparseness of the matrices and a pattern of their nonzero elements. These facts are very advantageous for computers and has promoted the use of iterative methods. As the basic iterative methods[1], the Jacobi method, the Gauss-Seidel (GS) method and the successive overrelaxation (SOR) method are known. The accelerated overrelaxation (AOR) method[2], which has been recently proposed, covers[3] the basic iterative methods and includes two of the acceleration and relaxation parameters. Our purpose is to improve the AOR method. In this paper, we will first suggest that the AOR method is an extrapolated SOR (ESOR) method. Next, we will propose the symmetric AOR (SAOR) mothod[4] and give the convergence theorem. Moreover, in order to accelerate the SAOR method, the SAOR-CG algorithm will be presented by introducing the conjugate gradient (CG) procedure.[5,6] Some numerical results are also shown.

2. Basic Equations

We consider the linear system

$$Au = b \qquad (1),$$

where A is the coefficient matrix of order N and b is a given vector. Then, the unique solution $\bar{u}$

$$\bar{u} = A^{-1}b \qquad (2)$$

exists if A is nonsingular. So, we assume that A is nonsingular, and that all the diagonal elements of A are nonzeros. We treat the iterative method (or the stationary linear iterative method) expressed as

$$u^{(n+1)} = Gu^{(n)} + b \qquad (3),$$

for the iterated vector $u^{(n)}$ at the n-th stage ($n=0, 1, \dots$), where G is the iteration matrix and $u^{(0)}$ is arbitrary initial vector. If $I-G$ is nonsingular, there exists the unique solution u satisfying

$$(I-G)u = b \qquad (4).$$

The iterative method converges if and only if

$$\rho(G) < 1 \qquad (5),$$

where $\rho(G)$ is the spectral radius of G. We define the error vector $\varepsilon^{(n)}$ as

$$\varepsilon^{(n)} = u^{(n)} - \bar{u} \qquad (6).$$

Then, we have

$$\varepsilon^{(n)} = G\varepsilon^{(n-1)} = \dots = G^n\varepsilon^{(0)} \qquad (7)$$

and, for suitable vector and matrix norms $\|\cdot\|$,

$$\|\varepsilon^{(n)}\| \leqq \|G^n\| \cdot \|\varepsilon^{(0)}\| \qquad (8).$$

For arbitrary vector $\varepsilon^{(0)}$, the norm $\|G^n\|$ gives a sharp upper bound for the ratio $\|\varepsilon^{(n)}\| / \|\varepsilon^{(0)}\|$, that is,

$$\|G^n\| \geqq \|\varepsilon^{(n)}\| / \|\varepsilon^{(0)}\| \qquad (9).$$

Therefore, (3) converges if and only if (5) holds.[1]

3. AOR Method and Extrapolation Method

Now, we consider the coefficient matrix A in (1) splitted as

$$A = D - C_L - C_U \qquad (10),$$

where D is the diagonal part of A and C_L, C_U are the strictly lower and upper triangular part of A, respectively. Based on the splitting (10), a stationary linear iterative method is generally expressed as

$$(\alpha_1 D + \alpha_2 C_L) u^{(n+1)} = (\alpha_3 D + \alpha_4 C_L + \alpha_5 C_U) u^{(n)} + \alpha_6 b \qquad (11),$$

where α_i $(i=1, 2, \dots, 6)$ is the coefficients to be determined and $u^{(0)}$ is arbitrary initial vector. As a matter of course, since the coefficient α_1 is nonzero, we obtain

$$(D + \alpha_2' C_L) u^{(n+1)} = (\alpha_3' D + \alpha_4' C_L + \alpha_5' C_U) u^{(n)} + \alpha_6' b \qquad (12),$$

where $\alpha_i' = \alpha_i / \alpha_1$ $(i=2, 3, \dots, 6)$. Here, in order that (12) is consistent with the linear system (1), the following equation must be satisfied:

$$(1-\alpha_3')D + (\alpha_2' - \alpha_4')C_L - \alpha_5' C_U = \alpha_6' A \; ; \; \alpha_6' \neq 0 \qquad (13).$$

From (10) and (13), we have

$$1 - \alpha_3' = \alpha_6', \; \alpha_2' - \alpha_4' = -\alpha_6' \text{ and } -\alpha_5' = -\alpha_6' \qquad (14).$$

From (14), we can choose as

$$\alpha_2' = -\gamma, \; \alpha_3' = 1-\omega, \; \alpha_4' = \omega - \gamma, \text{ and } \alpha_5' = \alpha_6' = \omega \qquad (15),$$

where γ and $\omega \neq 0$ are any parameters. From (12) and (15), we have finaly the equation

$$(I - \gamma L) u^{(n+1)} = [(1-\omega) I + (\omega - \gamma) L + \omega U] u^{(n)} + \omega c \qquad (16),$$

where I is the unit matrix of order N, and L, U and c are

$$L = D^{-1} C_L, \; U = D^{-1} C_U \text{ and } c = D^{-1} b \qquad (17),$$

respectively. We thus define a general iterative method as follows.

Definition 1 (AOR method).[2]

The iterative method expressed as

$$u^{(n+1)} = L(\gamma, \omega) u^{(n)} + \omega (I - \gamma L)^{-1} c \qquad (18)$$

for the iterated vector $u^{(n)}$ at the n-th stage $(n=0, 1, 2, \dots)$ is called the accelerated overrelaxation (AOR) method, where $u^{(0)}$ is an arbitrary inital vector and $L(\gamma, \omega)$ *is the AOR iteration matrix defined by*

$$L(\gamma, \omega) = (I - \gamma L)^{-1} [(1-\omega) I + (\omega - \gamma) L + \omega U] \qquad (19).$$

Here, γ and $\omega \neq 0$ are the acceleration and relaxation parameters, respectively.

Table 1 Basic iterative methods.

(γ, ω)	Method	i-th Eigenvalue
(0, 1)	Jacobi Method	μ_i (real)
(1, 1)	Gauss-Seidel Method	μ_i^2 (real)
(γ, γ) or (ω, ω)	SOR Method	$\lambda_i(\gamma, \gamma)$ (complex)

The AOR method has two of the characteristics, as follows. At first, by specifying the pair (γ, ω) in (18), the basic iterative methods are obtained, as given in Table 1. Next, using the extrapolation parameter defined by $s=\omega/\gamma$ $(\gamma\neq 0)$, $L(\gamma, \omega)$ can be rewritten as

$$\begin{aligned} L(\gamma, \omega) &= L(\gamma, \gamma s) \\ &= sL(\gamma, \gamma)+(1-s)I \end{aligned} \qquad (20),$$

where $L(\gamma, \gamma)=(I-\gamma L)^{-1}[(1-\gamma)I+\gamma U]$ becomes the SOR iteration matrix with the acceleration parameter γ. Therefore, the AOR method is equivalent to the extrapolation method applied to the SOR method with $L(\gamma, \gamma)$, *i.e.*, to the extrapolated SOR (ESOR) method. The extrapolation method applied to the basic iterative methods are similarily obtained, as given in Table 2.

Table 2 Extrapolation methods.

(γ, ω)	Method	i-th Eigenvalue	Extrapolation Parameter
$(0, \omega)$	Extrapolated Jacobi Method	$(1-\omega)+\omega\mu_i$	ω
$(1, \omega)$	Extrapolated GS Method	$(1-\omega)+\omega\mu_i^2$	ω
$(\gamma, s\gamma)$	ESOR Method	$(1-s)+\lambda_i(\gamma, \gamma)$	s

4. Symmetric AOR method

Let us define an improvement over the AOR method.

Definition 2 (Unsymmetric AOR method).

The unsymmeric AOR (USAOR)method is defined as

$$u^{(n+1/2)} = L(\gamma_F, \omega_F)u^{(n)} + \omega_F(I-\gamma_F L)^{-1}c \qquad (21)$$

and

$$u^{(n+1)} = U(\gamma_B, \omega_B)u^{(n+1/2)} + \omega_B(I-\gamma_B U)^{-1}c \qquad (22),$$

for the iterated vector $u^{(n)}$ *at the n-th stage* $(n=0, 1, \dots)$, *where*

$$L(\gamma_F, \omega_F) = (I-\gamma_F L)^{-1}[(1-\omega_F)I+(\omega_F-\gamma_F)L+\omega_F U] \qquad (23)$$

and

$$U(\gamma_B, \omega_B) = (I-\gamma_B U)^{-1}[(1-\omega_B)I+(\omega_B-\gamma_B)U+\omega_B L] \qquad (24).$$

Also, γ_F, ω_F, γ_B and ω_B are the acceleration and relaxation parameters of which the subscripts F and B denote the forward and backward iterations, respectively. Note that (21) and (22) are the forward and backward AOR methods, respectively.

Lemma 1.

The product of the iteration matrices of the USAOR method defind by (21) and (22)

$$U(\gamma_B, \omega_B)L(\gamma_F, \omega_F) \tag{25}$$

has nonnegative eigenvalues if and only if

$$\gamma_B = \gamma_F \text{ and } \omega_B = \omega_F \qquad (26).$$

(Proof). Let $A^{1/2}$ be the positive definite matrix satisfying $(A^{1/2})^2 = A$. Then, we define the matrices $L'(\gamma_F, \omega_F)$ and $U'(\gamma_B, \omega_B)$ as

$$L'(\gamma_F, \omega_F) = A^{1/2}L(\gamma_F, \omega_F)A^{-1/2} \qquad (27),$$

$$U'(\gamma_B, \omega_B) = A^{1/2}U(\gamma_B, \omega_B)A^{-1/2} \qquad (28),$$

which are similar to $L(\gamma_P, \omega_F)$ and $U(\gamma_B, \omega_B)$, respectively.

On the other hand, from (23) and (24) we have

$$\begin{aligned} L(\gamma_F, \omega_F) &= I - \omega_F(I - \gamma_F L)^{-1}D^{-1}A \\ &= I - \omega_F(D - \gamma_F C_L)^{-1}A \end{aligned} \qquad (29),$$

and

$$\begin{aligned} U(\gamma_B, \omega_B) &= I - \omega_B(I - \gamma_B U)^{-1}D^{-1}A \\ &= I - \omega_B(D - \gamma_B C_U)^{-1}A \end{aligned} \qquad (30),$$

respectively. Also, we know

$$C_L = DL \text{ and } C_U = DU = C_L{}^T \qquad (31),$$

since A is symmetric. Thus, for (27) and (28) we obtain

$$L'(\gamma_F, \omega_F) = I - \omega_F A^{1/2}(D -- \gamma_F C_L)^{-1}A^{1/2} \tag{32}$$

and

$$[L'(\gamma_F, \omega_F)]^T = I - \omega_F A^{1/2}(D - \gamma_F C_U)^{-1}A^{1/2} \qquad (33).$$

Here, if $\gamma_B = \gamma_F$ and $\omega_B = \omega_F$, we have

$$[L'(\gamma_F,\omega_F)]^T = U'(\gamma_B, \omega_B) \qquad (34).$$

Therefore, we have

$$U'(\gamma_B, \omega_B)L'(\gamma_F, \omega_F) = [L'(\gamma_F, \omega_F)]^T L'(\gamma_F, \omega_F) \qquad (35).$$

Namely, (35) is symmetric matrix. Hence, it is obvious that (25) has nonnegative and real eigenvalues. (QED).

From Lemma 1, the symmetric AOR method is defined, as follows.

Deffinition 3 (Symmetric AOR method).

The symmetric AOR (SAOR) method is defined as

$$u^{(n+1/2)} = L(\gamma, \omega)u^{(n)} + \omega(I - \gamma L)^{-1}c \tag{36}$$

and

$$u^{(n+1)} = U(\gamma, \omega)u^{(n+1/2)} + \omega(I-\gamma U)^{-1}c \tag{37}.$$

for the iterated vector $u^{(n)}$ *at the n-th stage* ($n=0, 1, \ldots$).

For the convenience sake, (36) and (37) can be written into

$$u^{(n+1)} = H(\gamma, \omega)u^{(n)} + k \tag{38},$$

where the SAOR iteration matrix $H(\gamma, \omega)$ *is*

$$H(\gamma, \omega) = U(\gamma, \omega)L(\gamma, \omega) \tag{39}$$

and

$$k = \omega U(\gamma, \omega)(I-\gamma L)^{-1}c + \omega(I-\gamma U)^{-1}c \tag{40}.$$

From Lemma 1 and Definition 3, it is known that the SAOR iteration matrix $H(\gamma, \omega)$ has nonnegative and real eigenvalues.

Corollary 1.

If the parameters γ *and* ω *satisfy the equality of*

$$\omega = \gamma \tag{41},$$

the SAOR method is equivalent to the symmetric successive overrelaxation (SSOR) method.

5. Convergence Theorem of SAOR Method

In this chapter, we present the convergence the theorem of the SAOR method. We consider

$$H'(\gamma, \omega) = A^{1/2}H(\gamma, \omega)A^{1/2} \tag{42},$$

which is similar to the SAOR iteration matrix $H(\gamma, \omega)$. Then, (42) can be rewritten into

$$H'(\gamma, \omega) = I - \omega[(2-\gamma)P' + (\gamma-\omega)Q'] \tag{43},$$

where

$$P' = [A^{1/2}(D-\gamma C_U)^{-1}D^{1/2}]\,[A^{1/2}(D-\gamma C_U)^{-1}D^{1/2}]^T \tag{44}$$

and

$$Q' = [A^{1/2}(D-\gamma C_U)^{-1}A^{1/2}]\,[A^{1/2}(D-\gamma C_U)^{-1}A^{1/2}]^T \tag{45}.$$

Since $(D-\gamma C_U)$ is nonsingular, the matrices P' and Q' is symmetric and positive definite, which have the positive eigenvalues p and q. Then, the eigenvalue $h(\gamma, \omega)$ of $H'(\gamma, \omega)$ is given by

$$h(\gamma, \omega) = 1 - \omega[(2-\gamma)p + (\gamma-\omega)q] \tag{46}.$$

Therefore, the convergence theorem of the SAOR method can be derived, as follows.

Theorem 1.

If the parameters γ *and* ω *satisfy the relation*

$$2 > \gamma \geqq \omega > 0 \qquad (47),$$

then the SAOR method converges.

(Proof). We can rewrite (43) as

$$I - H'(\gamma, \omega) = \omega[(2-\gamma)P' + (\gamma-\omega)Q'] \qquad (48).$$

Since P′ and Q′ are symmetric and positive definite, and since $H'(\gamma, \omega)$ is nonnegative definite, $I - H'(\gamma, \omega)$ becomes positive definite for $2>\gamma\geqq\omega>0$. Therefore, all the eigenvalues $h(\gamma, \omega)$ are less than the unities. Hence, we readily find that the spectral radius $\rho(H(\gamma, \omega))<1$ for $2>\gamma\geqq\omega>0$.

Corollary 2.

For $\gamma=\omega$ the SAOR method (, i.e., the SSOR method) converges if

$$2 > \gamma > 0 \qquad (49).$$

6. Symmetric AOR-CG Algorithm

Let us show an example of the acceleration of the SAOR method. We introduce the conjugate gradient (CG) procedure. As stated in the previous chapter 5, we note that all the eigenvalues of $H(\gamma, \omega)$ are real and nonnegative, and that they are also less than the unities. Our CG procedure, which is called the SAOR-CG algorithm, can be formulated, as follows:

$$u^{(n+1)} = \rho_{n+1}[\nu_{n+1}\delta^{(n)} + u^{(n)}] + (1-\rho_{n+1})u^{(n-1)} \qquad (50),$$

where $\delta^{(n)}$ is the pseudo-residual vector defined by

$$\delta^{(n)} = H(\gamma, \omega)u^{(n)} + k - u^{(n)} \qquad (51).$$

Also, ν_n and ρ_n are given by

$$\nu_{n+1} = [1 - \frac{(\delta^{(n)}, H(\gamma, \omega)\,\delta^{(n)})}{(\delta^{(n)}, \delta^{(n)})}]^{-1} \qquad (52),$$

and

$$\rho_{n+1} = [1 - \frac{\nu_{n+1}}{\nu_n} \cdot \frac{(\delta^{(n)}, \delta^{(n)})}{(\delta^{(n-1)}, \delta^{(n-1)})} \cdot \frac{1}{\rho_n}]^{-1} \qquad (53).$$

From (51) and (52), we have

$$\delta^{(n+1)} = \rho_{n+1}[\nu_{n+1}H(\gamma, \omega)\delta^{(n)} + (1-\nu_{n+1})\delta^{(n)}] + (1-\rho_{n+1})\delta^{(n-1)} \qquad (54).$$

In addition, for this algorithm the error vector $\varepsilon^{(n)}$ can be expressed as

$$\varepsilon^{(n)} = Q_n(H(\gamma, \omega))\varepsilon^{(0)} \qquad (55),$$

where $Q_n(H(\gamma, \omega))$ is the n-degree matrix polynomial and relates to the following algebraic polynomials:

$$Q_0(x) = 1,$$

$$Q_1(x) = \nu_1 x + 1 - \nu_1$$

and

$$Q_{n+1}(x) = \rho_{n+1}[\nu_{n+1}x + 1 - \nu_{n+1}]Q_n(x) + (1-\rho_{n+1})Q_{n-1}(x) \tag{56}$$

7. Numerical Experiments

In numerical experiments, we treat the simple boundary value problem for elliptic partial differential equation or the so-called model problem[1] expressed as

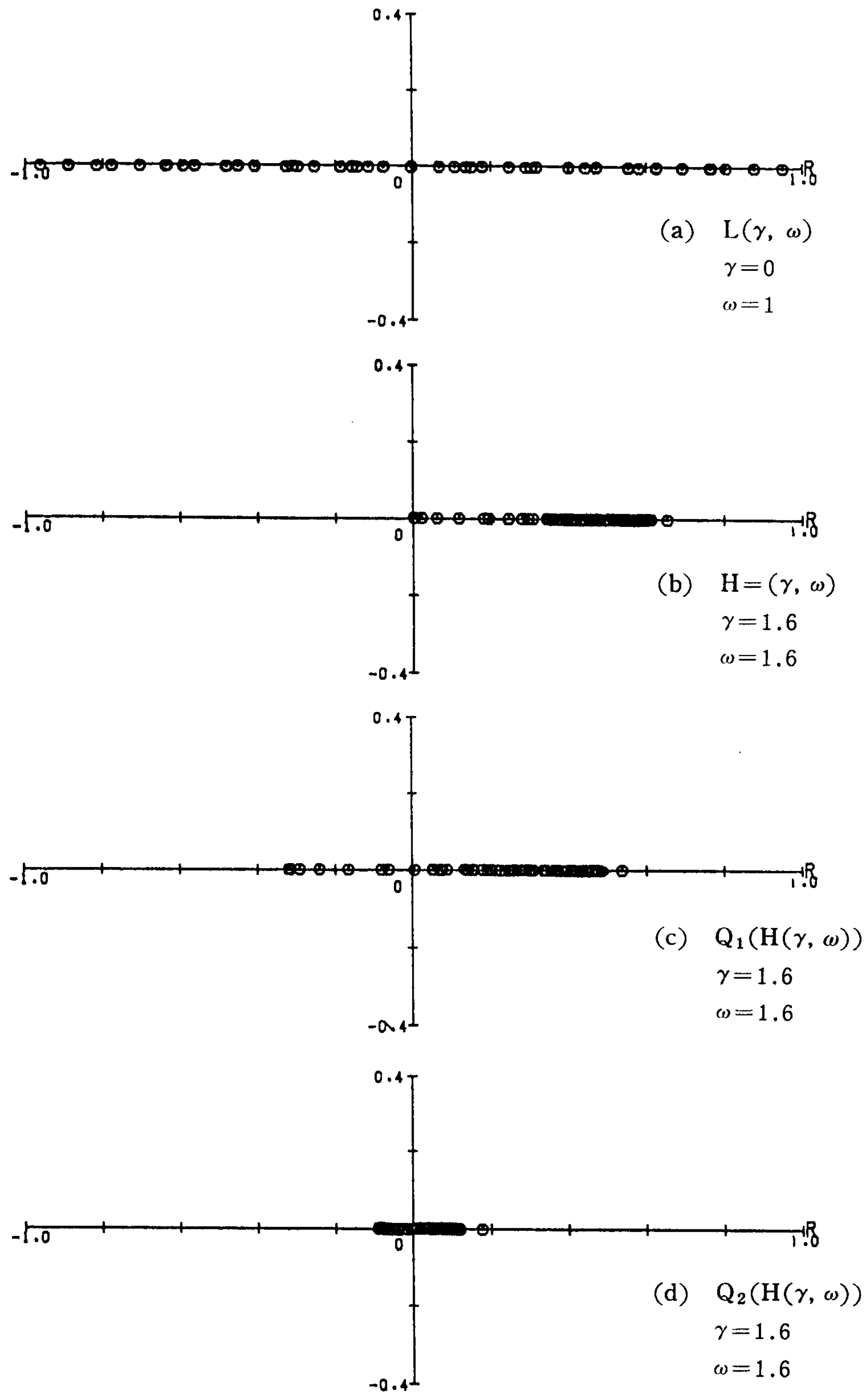

Fig. 1 Eigenvalues of iteration matrices $L(0, 1)$, $H(\gamma, \omega)$ and $Q_n(H(\gamma, \omega))$.

$$\nabla^2\phi(x, y) = 0 \ ; \quad (x, y) \in \Omega \tag{57}$$

and

$$\phi(x, y) = g(x, y) \ ; \quad (x, y) \in \partial\Omega \tag{58}$$,

where ∇^2 is the two-dimensional Laplacian, g(x, y) is the continuous function defined on the boundary $\partial\Omega$ and $\bar{\Omega}$ is the unit square domain. We use the well-known five-points difference formula. Fig. 1 shows the distribution of the eigenvalues of the Jacobi and SAOR iteration matrices and the matrix polynomial $Q_n(H(\gamma, \omega))$ generated during the SAOR-CG process. Fig. 2 shows the norm $\|\varepsilon^{(n)}\|$ of the error vectors, which is assumed to be continuous function of the iteration number n. Here, the mesh size $h=1/10$ is taken, and the acceleration and relaxation parameters are chosen to be $\gamma=1.6$ and $\omega=1.6$, respectively.

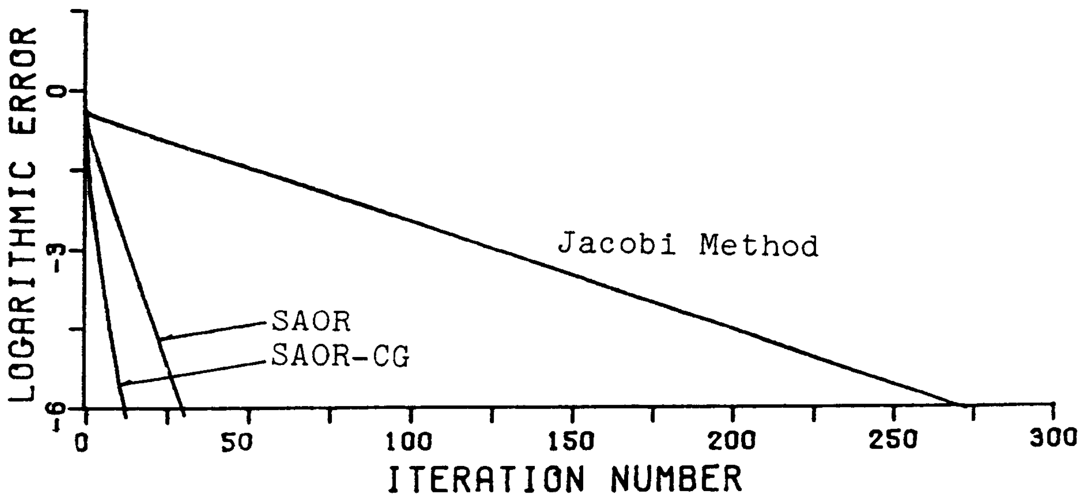

Fig. 2 Graph of $log \|\varepsilon^{(n)}\|$ versus iteration number n.

8. Conclusion

We proposed the SAOR method and presented the convergence theorem. As an extension of the SAOR method, we also presented the SAOR-CG algorithm. We confirmed the effectiveness of the SAOR method and the SAOR-CG algorithm by examining the distribution of the eigenvalues and the error vector. In the future, we would study on the adaptive procedure for the SAOR-CG algorithm.

References :

1) D.M. Young, Iterative Solution of Large Linear System, Academic Press, Inc., New York and London, 1971.

2) A. Hadjidimos, Accelerated overrelaxation method, Math. Comp., 32 (1978), pp. 149-157.

3) S. Yamada, M. Ikeuchi, H. Sawami and H. Niki, Convergence rate of accelerated over-relaxation method, Proceedings of The 23th National congress of The Information Pro-

cessing Society of Japan (IPSJ) held at The University of Tokyo, Tokyo, Oct. 14–16, 1981, pp. 893–894.

4) S. Yamada, M. Ikeuchi, H. Sawami and H. Niki, Convergence rate of symmetric accelerated overrelaxation method, Proceedings of The 24th National congress of IPSJ held at The Electric University of Tokyo, Tokyo, Mar. 22–24, 1982, pp. 897–898.

5) S. Yamada, M. Ikeuchi, H. Sawami and H. Niki, A symmetric AOR–CG algorithm, Proceedings of The 25th National congress of IPSJ held at Kyushu University, Fukuoka, Oct. 19–21, 1982, pp. 1177–1178.

6) L.A. Hageman and D.M. Young, Applied Iterative Methods, Academic Press, Inc., New York, 1981.