WAVE PROPAGATION ALONG GENERALIZED EXPONENTIAL LINE

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Abstract - For a generalized exponential transmission line the effect of the line-constants and the taper on the wave propagation is discussed. Especially the condition for distortionless propagation can be clarified.

The wave propagation along uniform transmission lines has been investigated in detail, whereas for nonuniform transmission lines a clear explanation has not been made. It thus is interesting to consider the effect of the line-constants and taper of a nonuniform transmission line on the wave-propagation. In the paper we deal with the generalized exponential-tapered transmission lines.

We consider a class of lossy nonuniform transmission lines, in which the time-invariants,

$$\sigma_{se} = R(z)/L(z) \quad \text{and} \quad \sigma_{sh} = G(z)/C(z), \tag{1}$$

are independent of z, where z denotes the physical length variable, and $L(z)$, $R(z)$, $C(z)$, and $G(z)$ are, respectively, the distributed inductance, resistance, capacitance, and conductance per unit length at the length, z, from the exciting end. In the generalized exponential-tapered transmission lines, the time-invariant,

$$\Delta_1 = \sqrt[4]{L/C} \; \frac{d^2}{dx^2} \sqrt[4]{C/L}, \tag{2}$$

and/or the time-invariant,

$$\Delta_2 = \sqrt[4]{C/L} \; \frac{d^2}{dx^2} \sqrt[4]{L/C} \tag{3}$$

are independent of the electrical length variable,

$$x = \int_0^z \sqrt{L(\zeta)\, C(\zeta)}\, d\zeta \tag{4}$$

The telegraph equation for a generalized exponential-tapered line can be written as

$$\frac{\partial^2}{\partial x^2}\{W(x)f(x,t)\}-W(x)\{\frac{\partial^2}{\partial t^2}f(x,t)+2p\frac{\partial}{\partial t}f(x,t)+q^2f(x,t)\}=0 \quad (5)$$

where t denotes the time variable, and

$2p=\sigma_{se}+\sigma_{sh}$,

$W=\sqrt{C/L}$ and $q^2=\sigma_{se}\sigma_{sh}+\Delta_1$ provided f denotes the voltage,

$W=\sqrt{L/C}$ and $q^2=\sigma_{se}\sigma_{sh}+\Delta_2$ provided f denotes the current.

If two initial conditions, $f(x, 0)$ and $f_t'(x, 0)$, and one boundary condition, $f(0, t)$, are given for the equation in (5), then the solution can be obtained by applying the technique of the two-dimensional Laplace transformation.[1] The result is

$$\begin{aligned}
f(x,t)=&\tfrac{1}{2}e^{-pt}W^{-1}(x)\{W(x+t)f(x+t,0)+W(x-t)f(x-t,0)\}\\
&+\tfrac{1}{2}pe^{-pt}W^{-1}(x)\int_{x-t}^{x+t}W(\xi)f(\xi,0)d\xi\\
&-\tfrac{1}{2}pke^{-pt}W^{-1}(x)\int_0^t J_1(k\tau)\int_{x-T(\tau)}^{x+T(\tau)}W(\xi)f(\xi,0)d\xi d\tau\\
&-\tfrac{1}{2}kte^{-pt}W^{-1}(x)\int_0^t J_1(k\tau)T^{-1}(\tau)[W\{x+T(\tau)\}f\{x+T(\tau),0\}\\
&\qquad\qquad+W\{x-T(\tau)\}f\{x-T(\tau),0\}]d\tau\\
&+\tfrac{1}{2}e^{-pt}W^{-1}(x)\int_{x-t}^{x+t}W(\xi)f_t'(\xi,0)d\xi\\
&-\tfrac{1}{2}ke^{-pt}W^{-1}(x)\int_0^t J_1(k\tau)\int_{x-T(\tau)}^{x+T(\tau)}W(x)f_t'(\xi,0)d\xi d\tau\\
&+W(0)e^{-px}W^{-1}(x)f(0,t-x)\\
&-kW(0)xW^{-1}(x)\int_x^t e^{-p\tau}J_1\{kX(\tau)\}X^{-1}(\tau)f(0,t-\tau)d\tau
\end{aligned} \quad (6)$$

where

$$W(-x)=W(x),\ f(-x,0)=-f(x,0),$$

$$f_t'(-x,0)=-f_t'(x,0),\ f(0,t)=0\ (t<0),$$

$$k=\sqrt{q^2-p^2},\ T(\tau)=\sqrt{t^2-\tau^2},\ X(\tau)=\sqrt{\tau^2-x^2}$$

The first six terms of the solution in (6) are caused by the initial stimulus, whereas the final two terms are caused by the boundary stimulus. The first of the six terms caused by the initial stimulus implies two waves moving in opposite directions along the transmission line. The first of the two terms caused by boundary stimulus signifies that the boundary stimulus, or the signal, propagates away from the excitor end. Their propagating velocities, which are equal each other, are independent of the electric length variable, x, but they depend on the physical length variable, z, that is,

$$\text{velocity} = 1/\sqrt{L(z)C(z)} \tag{7}$$

If the product of $L(z)$ and $C(z)$ is not a function of z, then the propagating velocity is also independent of z.

For a lossy transmission line, all the six terms caused by the initial stimulus decay with time elapsed, and the first of the two terms by the boundary stimulus decays spatially. Especially, for a lossless transmission line the second and third terms in (6) vanish, and all the remainders decay with neither time elapsed nor space passed.

Next, we examine the effect by the value of k. In the general case of $k\neq 0$, the integrals of Bessel function, J_1, appear in some terms, and these terms thus are vibratory. As stated above, in the lossy line of $k\neq 0$ all the terms caused by the initial stimulus decay with time elapsed and the first of the two terms by the boundary stimulus decays spatially, whereas the final term by the boundary stimulus does not decay but remains and it arises the cause of distortion. In other words, if $k\neq 0$, the boundary stimulus, or the signal, does not propagate simply as wave but leaves a residue serving as distortion after all the stimuli have passed through a point. In the special case of $k=0$, all the stimuli propagate in distortionless, generally, with decay with both time elapsed and space passed, for the vibratory function, J_1, vanishes.

A summary of above statements is as follows: All the terms caused by the initial stimulus decay with time elapsed. The first of the two terms caused by the boundary stimulus decays spatially, while the final term by the boundary stimulus does not decay but remains as a cause of distortion when $k\neq 0$. If $k=0$, then the final term is absent originally. Thus the expression, $k=0$, indicates the condition for distortionless propagation.

The condition, $k=0$, also can be written as

$$4\Delta = (\sigma_{se} - \sigma_{sh})^2 \tag{8}$$

where Δ denotes either Δ_1 or Δ_2 as stated above. Since both of the time-invariants, σ_{se} and σ_{sh}, are real, Δ must be real and non-negative in order to be distortionless. If the taper of the line is hyperbolic cosine-squared or exponential or hyperbolic sine-squared, then $\Delta > 0$. If the taper is uniform or square, then $\Delta = 0$. If trigonometric cosine-squared, then $\Delta < 0$. Therefore, in the case of the five tapers except the trigonometric taper the transmission lines can be distortionless provided the condition in (8) holds.

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Reference

1) D. Voelker and G. Doetsch, "Die Zweidimensionale Laplace-Transformation," Verlag Birkhauser, 1950