On the Solution of the Integral Inequality

$$xu(x) \leqq \int_0^x (\nu + \varepsilon(t))u(t)\,dt$$

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In the paper, "Sur l'equation intégrale

$$xu(x) = f(x) + \int_0^x K(x, t, u(t))\,dt."$$

Dr. Sato gave the following lemma.
"If the integal inequality

$$xu(x) \leqq \nu \int_0^x u(x)\,dt \qquad \nu > 0,\ x > 0,$$

has the continuous solution

$$0 \leqq u(x) = o(x^{\nu-1})$$

in the interval I : $0 \leqq x \leqq r$

it must necessarily be $u(x) \equiv 0$."

We can regard as a special case Dr. Nagumo's condition in his paper on the theory of ordinary differential equations, as is evident from the remark at the end of this paper.

Now I want to prove the similar proposition, extending Dr. Shimizu's condition.

Theorem. "Let $\varepsilon(x)$ be continuous in the interval I : $0 \leqq x \leqq r$, and $\varepsilon(x) \geqq 0$, and $\int_0^x \frac{\varepsilon(t)}{t}\,dt < +\infty$. If the integral inequality,

$$(1) \qquad xu(x) \leqq \int_0^x (\nu + \varepsilon(t))u(t)\,dt \qquad \nu > 0,\ x > 0,$$

has the continuous solution $u(x)$, $0 \leqq u(x) = o(x^{\nu-1})$ in the interval I, it must necessarily be $u(x) \equiv 0$."

Proof. It is plain that $u(x) \equiv 0$ is a solution of (1).
Now, let

$$(2) \qquad u(x) = x^{\nu-1} e^{\int_0^x \frac{\varepsilon(t)}{t}dt}\, w(x),$$

$u(x) \equiv 0$ is satisfied only when $w(x)$ is identically equal to zero.

From (2), for the continuous function $u(x)$ $0 \leqq u(x) = o(x^{\nu-1})$ in the interval we may define the continuous function $w(x)$ $0 \leqq w(x) = o(1)$ in the $0 < x \leqq$ r. Therefore, if we define $w(0) = 0$, $w(x)$ is continuous in I, and $w(x) \geqq 0$. Consequently, if for the continuous function $w(x)$, $0 \leqq w(x) = o(1)$ in the interval, $u(x)$, given by (2), satisfies the inequality (1). It is sufficient for the proof of the theorem that $w(x)\equiv 0$ should be established.

Then if we assume $w(x)\not\equiv o$, $w(x)$ must have a maximum value at $x=x_0$, so that, if we put $W=w(x_0)$, we know that W is positive for $0<x_0\leqq r$.

From (1) and (2), we get the next formulas.

$$x_0^{\nu}\, e^{\int_0^{x_0}\frac{\varepsilon(s)}{s}ds}\, W = x_0^{\nu}\, e^{\int_0^{x_0}\frac{\varepsilon(s)}{s}ds}\, w(x_0)$$

$$\leqq \int_0^{x_0} (\nu+\varepsilon(t))\, t^{\nu-1}\, e^{\int_0^{t}\frac{\varepsilon(s)}{s}ds}\, w(t)\,dt$$

$$< W\int_0^{x_0} (\nu+\varepsilon(t))\, t^{\nu-1}\, e^{\int_0^{t}\frac{\varepsilon(s)}{s}ds}\, dt = W x_0^{\nu}\, e^{\int_0^{x_0}\frac{\varepsilon(s)}{s}ds}$$

But, since $x_0>0$ and $e^{\int_0^{x_0}\frac{\varepsilon(s)}{s}ds}>0$, the above assumption is absurd.
So that this theorem has been proved.

Remark. If we put $xu(x)=w(x)$,

we get $$w(x)\leqq \int_0^x \frac{(\nu+\varepsilon(t))\, w(t)}{t}\, dt.$$

So that, we get the next corollary that is equivalent to this theorem.

Corollary. "Let $\varepsilon(x)$ be a function which satisfies the same condition as in the above theorem, then the continuous solution of the integral inequality

$$u(x)\leqq \int_0^x \frac{(\nu+\varepsilon(t))\,u(t)}{t}\,dt$$

in the interval I and $0\leqq u(x)=o(x^{\nu})$,
must necessarily be $u(x)\equiv 0$."

Dr. Shimizu's sufficient condition for the uniqueness of the solution of the differential equation may be shown as follows.

"If the function $f(x, y)$ is continuous in some neighbourhood of the point (a, b), and satisfies the inequality

$$|f(x,\bar{y})-f(x,y)|\leqq \frac{(1+\varepsilon(x-a))\,|\bar{y}-y|}{|x-a|}$$

then the solution of the differential equation $\frac{dy}{dx}=f(x,y)$ which has $y(a)=b$ as the initial condition, must be one and only one." By the way, $\varepsilon(x)$ is the function which satisfies the same condition as in this theorem.

Such a solution can be shown as

$$y(x)=b+\int_0^x f(t, y(t))\,dt$$

$a = b = 0$ and $x > 0$ without loss of generality.

Now, if there were two different solutions $y_1(x)$ and $y_2(x)$, it follows that

$$|y_1(x) - y_2(x)| \leqq \int_0^x \frac{(1+\varepsilon(t))\,|y_1(t) - y_2(t)|}{t} dt.$$

Since, from the hypothesis, $|y_1(x) - y_2(x)|$ is continuous, and not negative, and

$\lim_{x \to 0} \frac{y_1(x) - y_2(x)}{x} = y_1'(0) - y_2'(0) = f(0, 0) - f(0, 0) = 0,$

it follows, from the corollary, that $|y_1(x) - y_2(x)| \equiv 0$.

This is absured.

Thus Dr. Shimizu's condition has been proved.

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On the solution of the integral inequality

$$x.\, u(x) \leqq \int_0^x (\nu + \varepsilon(t))\, u(t)\, dt$$

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It is demonstrated that if $\varepsilon(x) \geqq 0$ on I: $0 \leqq x \leqq r$ with $\int_0^x \frac{\varepsilon(t)}{t} dt < \infty$, then the only solution of inequality in the title which is continuous on I and satisfies the condition $0 \leqq u(x) = o(x^{\nu-1})$ is $u(x) \equiv 0$

Applied to differential equations, this yields the uniqueness of the solution of a differential eqnation $y' = f(x, y)$, through (a, b) if $f(x, y)$ is continuous in a vicinity of (a, b) and satisfies there the inequlity

$$|f(x, \bar{y}) - f(x, y)| < \frac{(1 + \varepsilon(x-a))\,|\bar{y} - y|}{|x-a|}$$

a result due shimizu 〔Proc. Imp. Acad, Tokyo 4. 326-329 (1928)〕 generalizing a condition of Nagumo.

〔T. H. Hilderbraudt (Ann. Arbor. Mich.)〕

参考. Lipschitz の条件より本質的に弛い十分条件を与えるものとして，つぎの南雲の定理 (1926-27) がある.

定理 $f(x, y)$ が R に於いて一価連続で，有界にして且

$$|x - x_0| \cdot |f(x, y) - f(x, z)| \leqq |y - z|$$

を満足するならば，$x = x_0$ のとき，$y = y_0$ となる微分方程式A の解は高々 1 つ存在する.