THE BULLETIN OF OKAYAMA UNIVERSITY OF SCIENCE No. 53 A pp. 63-67 (2017)

Some Secant type Iterative methods by Approximation of Logarithmic Secant Method with Numerical Integrations

Michio Sakakihara

Department of Information Science
Okayama University of Science,
1-1 Ridai-cho, Kita-ku, Okayama 700-0005, Japan

(Received October 31, 2017; accepted December 4, 2017)

Numerical iterative methods for nonlinear equations are proposed which derived from the logarithmic secant method by applying some numerical integration schemes: the midpoint method, the trapezoidal method and the Simpson method. For the problems of a nonlinear equation with a double root, we present error analysis same as the secant method. By numerical experiments, it is shown that the proposed method is effective for the problem. Moreover, the proposed method is also effective for the problems of which the multiplicity of roots is greater than two.

Keywords: Secant method; multiple root; Simpson method.

1. Introduction

The secant iterative method for solving the equation such as

(1) $$f(x) = 0$$

is expressed as

(2) $$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} f(x_k)$$

with two initial values x_0, x_1 1). By Sakakihara 2), the logarithmic secant method :

(3) $$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{\ln(f(x_k)/f(x_{k-1}))}$$

is proposed to solve (1) approximately. (3) is rewritten into

(4) $$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} L\big(f(x_k), f(x_{k-1})\big)$$

where $L\big(f(x_k), f(x_{k-1})\big)$ is the logarithmic mean of $f(x_k), f(x_{k-1})$ such as

$$L\big(f(x_k), f(x_{k-1})\big) = \frac{f(x_k) - f(x_{k-1})}{\ln(f(x_k)/f(x_{k-1}))} .$$

It was shown that the logarithmic secant method is effective for the equation with multiple roots in 2). Even though the advantage, the necessary to evaluate logarithmic function in each step of iteration become expensive for the computing time. In order to avoid evaluating logarithmic function, we apply some approximate method for the logarithmic function. Applying the approximation of the integral form for the logarithmic mean, some new secant type iterative methods are proposed. Theorems of error analysis same as the case of the secant method are proposed for the problem with the double root. By the numerical experiments, the proposed methods are also effective for the problems which has the roots of which the multiplicity is greater than two.

2. Logarithmic secant method

The Newton-Raphson iterative method for solving (1)

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

is rewritten into

(5) $$x_{k+1} = x_k - [\ln[f(x_k)]]'$$

for $f(x_k) > 0$. By applying the first order finite difference to (5), we obtain (3).For the practical computation, the scheme :

$$x_{k+1} = x_k - \frac{2(x_k - x_{k-1})}{\ln\left(\left(f(x_k)/f(x_{k-1})\right)^2\right)}$$

is rather useful since the value $f(x_k)$ can takes a negative value.

3. Approximation of logarithmic mean by numerical integrations

The logarithmic mean of positive real values a, b is defined by

(6) $$L(a,b) = \frac{a-b}{\ln(a/b)}$$

The integral representation of the logarithmic mean is obtained as

(7) $$L(a,b) = (a-b)/\int_a^b \frac{1}{x} dx$$

Applying the midpoint, the trapezoidal, and the Simpson numerical scheme 3) to (6), we obtain

(8) $$L(a,b) \approx \frac{a+b}{2} \ ,$$

(9) $$L(a,b) \approx \frac{2ab}{a+b}$$

and

(10) $$L(a,b) \approx \frac{6ab(a+b)}{10ab + a^2 + b^2}$$

respectively.

4. New secant type iterations

The secant method (2) is derived by applied the rectangle approximation to (7). In order to *formulate new derivative free iterative method, the approximations of the logarithmic mean* described in the previous session are applicable. The new iterations are obtained as

(11) $$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} \frac{f(x_k) + f(x_{k-1})}{2} \ ,$$

(12) $$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} \frac{2f(x_k)f(x_{k-1})}{f(x_k) + f(x_{k-1})}$$

and

(13) $$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} \frac{6f(x_k)f(x_{k-1})\left(f(x_k) + f(x_{k-1})\right)}{10f(x_k)f(x_{k-1}) + f(x_k)^2 + f(x_{k-1})^2} \ ,$$

respectively.

5. Error Analysis

If α is the solution for (1), we have the Taylor series expansion at the point of α :

$$f(x) = f'(\alpha)(x-\alpha) + \frac{f''(\alpha)}{2}(x-\alpha)^2 + \cdots \ .$$

Set $e_k = x_k - \alpha$. The error of secant method is as follows :

Theorem 1. If x_k is the sequence generated by the secant iteration (2) which converges to the simple root of the equation (1), then we obtain the error estimate such as $e_{k+1} \approx K e_k e_{k-1}$. Moreover, if the sequence approaches to the double root of the equation (1), then we obtain that $e_{k+1} \approx K' e_k$. Here, K and K' are generic constants.

From the theorem, we obtain the following corollary by the usual definition of the convergence order.

Corollary 1. The order of convergence of the secant method for the simple root is of

$$\frac{1+\sqrt{5}}{2}$$

which is the golden number. Moreover, the order of convergence of the secant method is of first for the double root.

For the problem to seek the double root of the equation (1), we obtain the following error estimate theorems for the iteration (11) :

Theorem 2. If x_k is the sequence generated by the secant iteration (11) which converges to the double root of the equation (1), then we obtain the error estimate such as

$$e_{k+1} \approx e_k - \left(\frac{e_k + e_{k-1}}{2}\right) + \frac{e_k e_{k-1}}{e_k + e_{k-1}} .$$

Proof : The Taylor expansion of $f(x)$ at the double root α becomes

$$f(x_k) = f'(\alpha) e_k + \frac{f''(\alpha)}{2} {e_k}^2 + \cdots$$

where $e_k = x_k - \alpha$.

$$e_{k+1} \approx e_k - \frac{e_k - e_{k-1}}{(A_2 {e_k}^2) - (A_2 {e_{k-1}}^2)} \frac{(A_2 {e_k}^2) + (A_2 {e_{k-1}}^2)}{2} = e_k - \frac{1}{e_k + e_{k-1}} \frac{{e_k}^2 + {e_{k-1}}^2}{2} = e_k - \frac{1}{e_k + e_{k-1}} \frac{(e_k + e_{k-1})^2 - 2 e_k e_{k-1}}{2}$$

$$= e_k - \left(\frac{e_k + e_{k-1}}{2}\right) + \frac{e_k e_{k-1}}{e_k + e_{k-1}}$$

where $A_2 = \frac{f''(\alpha)}{2}$.

Theorem 3. If x_k is the sequence generated by the secant iteration (12) which converges to the double root of the equation (1), then we obtain the error estimate such as

$$e_{k+1} \approx e_k \left[1 - \frac{e_k {e_{k-1}}^2}{{e_k}^3 + {e_k}^2 e_{k-1} + e_k {e_{k-1}}^2 + {e_{k-1}}^3}\right] .$$

Proof : The proof of the theorem is same as theorem 2 such that

$$e_{k+1} \approx e_k - \frac{e_k - e_{k-1}}{(A_2 {e_k}^2) - (A_2 {e_{k-1}}^2)} \frac{(A_2 {e_k}^2)(A_2 {e_{k-1}}^2)}{(A_2 {e_k}^2) + (A_2 {e_{k-1}}^2)} = e_k - \frac{e_k - e_{k-1}}{({e_k}^2) - ({e_{k-1}}^2)} \frac{({e_k}^2)({e_{k-1}}^2)}{({e_k}^2) + ({e_{k-1}}^2)}$$

$$= e_k \left[1 - \frac{e_k {e_{k-1}}^2}{{e_k}^3 + {e_k}^2 e_{k-1} + e_k {e_{k-1}}^2 + {e_{k-1}}^3}\right] .$$

Theorem 4. If x_k is the sequence generated by the secant iteration (13) which converges to the double root of the equation (1), then we obtain the error estimate such as

$$e_{k+1} \approx e_k \left[1 - \frac{6(e_k{}^3 e_{k-1}{}^2 + e_k e_{k-1}{}^4)}{e_k{}^5 + e_k{}^4 e_{k-1} + 10 e_k{}^3 e_{k-1}{}^2 + 10 e_k{}^2 e_{k-1}{}^3 + e_k e_{k-1}{}^4 + e_{k-1}{}^5}\right]$$

Proof: The proof of the theorem is same as the previous theorems.

6. Numerical Experiments

In order to show that the proposed iterative methods are effective, we examine those methods to the problem :

$$f(x) = (x^m - x)^n \exp(ax) = 0 \tag{14}$$

where m and n are positive integers, a is a real number. The problem has the solution $x = 1$. The initial values for the iterations are $x_0 = 1.9$ and $x_1 = 1.85$. In Table 1, we illustrate the dependency on m and n of the number of iterations with the Newton-Raphson method with $x_0 = 1.9$.

Table 1 Number of iterations with Newton-Raphson method for (14) with $x_0 = 1.9$ and $a = 1$.

	m =2	m =10	m =50	m =100	m =500	m =1000
n =1	8	12	38	70	326	647
n =2	53	61	110	173	684	1325
n =3	89	100	173	268	1034	1995

The computations are carried out by the Maple18 with 30 digits. We set that the convergence criteria is $|x_k - 1| < 10^{-15}$. In Table 2, we compare the number of iterations for the iterative methods.

Table 2 Number of iterations with Eqs.(2), (11), (12) and (13) for (14) with $m = 2$ and $a = 1$.

Iterative method	n =2	n =3	n =4	n =5	n =10	n =50
(2)	75	127	178	229	482	2502
(11)	117	65	68	102	274	1550
(12)	50	91	131	171	371	1960
(13)	28	67	105	141	319	1732

Table 3 Number of iterations with Eqs.(2), (11), (12) and (13) for (14) with $m = 10$ and $a = 1$.

Iterative method	n =2	n =3	n =4	n =5	n =10	n =50
(2)	87	143	199	255	533	2755
(11)	127	71	85	120	307	1704
(12)	59	104	148	192	412	2159
(13)	36	79	120	160	356	1909

Table 4 Number of iterations with Eqs.(2), (11), (12) and (13) for (14) with $m = 100$ and $a = 100$.

Iterative method	n =2	n =3	n =4	n =5	n =10	n=50
(2)	87	143	199	255	533	–
(11)	127	71	85	120	307	–
(12)	59	104	148	192	412	–
(13)	36	79	120	160	356	–

7. Conclusion

By applying some numerical schemes to the logarithmic secant iteration, we proposed some secant type iterative methods in this paper. We obtain the error analysis for the present methods in the case that the equation (1) has a double root. Numerical experiments are shown for those methods in order to show the efficiency compared with the secant method and the Newton-Raphson method. From the numerical results, it is shown that Simpson approximation is most effective. For the case that $m = 10, n = 2$, the number of iterations for the proposed method with the Simpson approximation is about a half of the Newton-Raphson case. On the other hand, as the multiplicity of the root n is increasing, as the iterative method (11) is giving better performance.

References

1) J. F. Traub, Iterative Methods for the Solution of Equations, Printice-Hall, Englewood Cliffs. N.J. (1964).

2) M. Sakakihara, Secant type approach via approximation of logarithmic derivative, The Bulletin of Okayama University of Science, 52,69-71 (2016).

3) R. Plato, Concise Numerical Mathematics, Graduate Studies in Mathematics vol.57, American Mathematical Society (2003).