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A new metric to measure the braid index of a link

Kiyokazu Suto, Kanako Suto*, and Naoki Mizobuchi*

Department of Applied Mathmatics, Faculty of Science,
**Graduate School of Science,*
Okayama University of Science,
1-1 Ridai-cho, Kita-ku, Okayama 700-0005, Japan

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We will show that labeling of crossings of a link diagram with integers gives a new metric to measure the braid index of the link. We will also discuss about a relation between our new metric and the number of Seifert circles of the diagram.

Keywords: link; braid index; Seifert circle; labeling of crossings.

Introduction

In [3], we introduced a representation of link diagrams by attaching integers to crossings with certain manner. In this article, we will show that such labeling can be used to measure braid indices of links as follows.

First, we label crossings of a oriented link diagram with integers. Next, we choose starting points of the diagram, and walk through all the strands of the diagram after the given orientation.

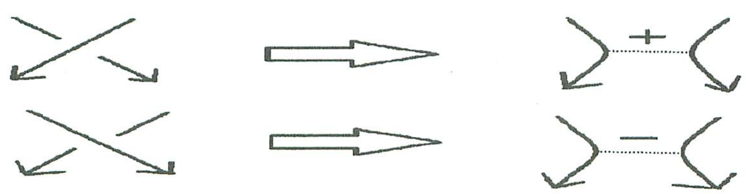
By ordering the integers attached to crossings according to the walk, we get a sequence of integers. This sequence is split into some number of increasing sequences.

For each link, we consider the minimum number of the increasing sequences among the all labeling, all choices of starting points, and all diagrams of the link.

By making use of Yamada's result [4] which mentions that any link diagram can be deformed to a closed braid with the same number of strands as the number of Seifert circles of the diagram, it is proved that the minimum number equals the braid index of the link.

1. Seifert System

Let L be an oriented link, and D be a diagram of L. By applying the following deformation



to all the crossings of D, we obtain a set of mutually disjoint oriented circles and dashed lines with signs connecting those circles. This system is called Seifert system of D, and each circle in the system is called Seifert circle. We denote by $s(D)$ the number of Seifert circles of the diagram D.

In [4], it is proved that D can be deformed to a closed braid with $s(D)$ strands, and that, as an immediate consequence of the result, the braid index of the link L equals the minimum of $s(D)$'s with D varying all the diagrams of L.

2. Increasing sequences of labels of crossings

We keep the same notation L and D as the previous section.

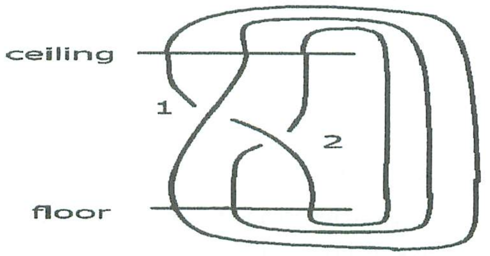
Let $L_1, \dots, L_m$ be the connected components of L, $D_1, \dots, D_m$ the respective images in D, $c(D)$ be the set of crossings of D, n be the number of elements of $c(D)$, and $f : c(D) \ni c \longrightarrow f(c) \in \{1, \dots, n\}$ be a one-to-one mapping.

We choose a point P_i on a strand of D_i different from any crossing for each $1 \leq i \leq n$. We start from P_1 and walk through D_1 after the given orientation until we arrive at P_1 again, and repeat such walks on all $D_2, \dots, D_m$.

By picking up integers $f(c)$ when we pass crossings c, and arranging the integers in that order, we get a sequence of integers. This sequence is split into maximal strictly increasing subsequences. Let $i(D;\ f\ ; P_1, \dots, P_m)$ be the number of the subsequences, and $i(L)$ be the minimum of $i(D;\ f\ ; P_1, \dots, P_m)$ for all diagrams D, all mappings f, and all choices of starting points $P_1, \dots, P_m$.

Let σ be a braid and $\hat{\sigma}$ be its diagram. As a typical example, we consider the case where D is the closure of $\hat{\sigma}$.

We can draw $\hat{\sigma}$ as strands between two parallel lines called a ceiling and a floor, in such a way that any line joining any pair of crossings is not parallel to the ceiling or the floor. Then there is an only one mapping f of $c(D)$ to $\{1, \dots, n\}$ such that $f(c) < f(c')$ if and only if c is closer to the ceiling than c'. Further we can choose all $P_1, \dots, P_m$ at the ceiling. Obviously in this case, $i(D;\ f; P_1, \dots, P_m)$ equals the number of strands of σ.



Let $b(L)$ be the braid index of L, that is, the minimal number of strands of braids of which closures are equivalent to L. Taking as σ a braid giving $b(L)$, we get the following evaluation.

Lemma 1. $i(L) \leq b(L)$.

On the other hand, we see that $s(D)$'s give lower bounds of $i(D;\ f; P_1, \dots, P_m)$'s as follows.

D can be regarded as an oriented 4-regular graph, considering crossings as vertices, and segments of strands joining crossings without any crossing on them other than the initial crossings and final crossings, as edges. It is obvious that $i(D;\ f; P_1, \dots, P_m)$ is the same as the number of edges e satisfying $f(c) \geq f(c')$ with c and c' the initial and final vertex of e respectively.

Take a Seifert circle C of D. Note that each dashed line of Seifert system of D represents a vertex of D viewed as an oriented graph, and each arc of C between dashed lines attaching to C represents an edge of D. If we goes along C after the give orientation, there must be an arc such that the corresponding edge satisfying the above condition. Thus we have

Lemma 2. $i(D;\ f; P_1, \dots, P_m) \geq s(D)$.

Combining the above two lemmas, and Yamada's result described in the previous section, we have obtained our main result.

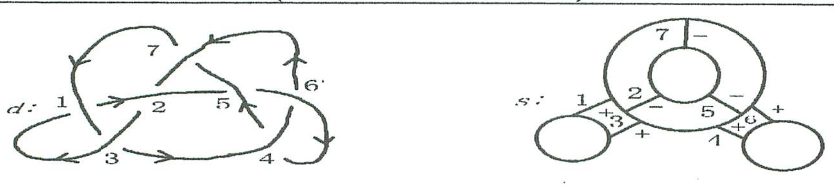
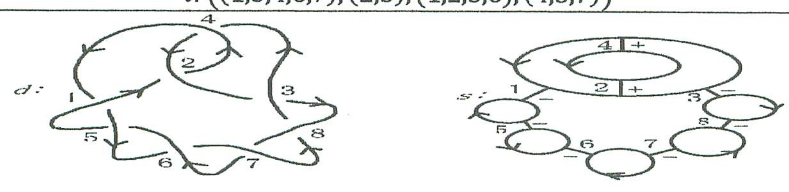
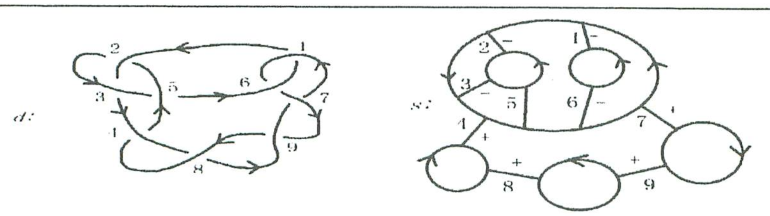
Theorem 3. $i(L)$ equals $b(L)$.

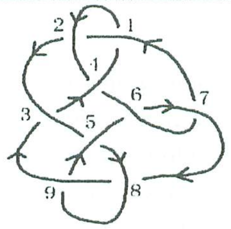
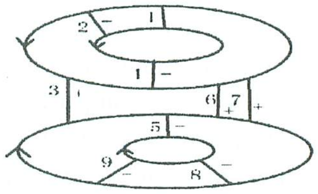
3. **Comparison of minimal numbers of maximal increasing sequences and of Seifert circles**

A natural question arises from our main result: whether or not $i(D;\ f; P_1, \ldots, P_m)$ equals $s(D)$ for a proper choices of f and $P_1, \ldots, P_m$?

We computed $s(D)$'s and the minimum $i(D;\ f; P_1, \ldots, P_m)$'s for knot diagrams $3_1, \ldots, 9_{28}$ in the well-known table of [2]. For these diagrams, $i(D;\ f; P_1)$ equals $s(D)$ if we chose f and P_1 properly.

We pick up some samples of our computations in the table below. In the table, the symbols in the left columns are names given to knots in [2], d's are diagrams with minimal crossings, s's are Seifert systems of the diagrams, and i's are sequences of labels with splittings to minimal numbers of maximal increasing subsequences.

3_1	 i: ((1,2,3), (1,2,3))
6_3	 i: ((1,2,3,5,7), (3,4), (1,2,4,5))
7_7	 i: ((1,3,4,6,7), (2,3), (1,2,5,6), (4,5,7))
8_1	 i: ((1,2,4), (1,5,6,7,8), (3,4), (2,3,8), (7), (6), (5))
9_8	 i: ((1,5,6,9), (8), (4,5), (2,3,5,6), (1,2,3,4,8,9), (6))

9_{28}	d:  s:  i: ((1,2,4,6,7), (1,23,5,8,9), (5,6,7,8,9), (3,4))

It is very likely that the answer to the above question is yes. We shall continue further investigation.

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