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Flatness and some other properties of a finitely generated extension of anti-integral elements over a Noetherian domain

KEN-ICHI YOSHIDA AND KIYOSHI BABA*

Department of Applied Mathematics
Okayama University of Science
1-1 Ridai-cho
Okayama 700-0005
JAPAN
and
**Department of Mathematics*
Faculty of Education and Welfare Science
Oita University
Oita 870-1192
JAPAN

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Abstract

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be anti-integral elements over a Noetherian domain R. Set $A = R[\alpha_1, \alpha_2, \ldots, \alpha_n]$. We investigate flatness, faithful flatness, exclusiveness, existence of a blowing-up point and unramification of the extension A/R under the condition $I_{\hat{\alpha}_1} + I_{\hat{\alpha}_2} + \cdots + I_{\hat{\alpha}_n} = R$ where

$$I_{\hat{\alpha}_i} = \{a \in R; a\alpha_j \in R[\alpha_i]\ (j = 1, 2, \ldots, n)\}.$$

Our results are generalizations of those of [2]

Let R be a Noetherian domain with quotient field K and $R[X]$ a polynomial ring over R in an indeterminate X. Let α be an element of an algebraic field extension of K and $\pi : R[X] \longrightarrow R[\alpha]$ the R-algebra homomorphism defined by $\pi(X) = \alpha$. Let $\varphi_\alpha(X)$ be the monic minimal polynomial of α over K with deg $\varphi_\alpha(X) = d$. Write

$$\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d,\ (\eta_1, \ldots, \eta_d \in K).$$

We define $I_{[\alpha]} := \bigcap_{i=1}^d (R :_R \eta)$ and $J_{[\alpha]} := I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$ where $(R :_R \eta) = \{c \in R; c\eta_i \in R\}$ and $(1, \eta_1, \ldots, \eta_d)$ is the R-module generated by $1, \eta_1, \ldots, \eta_d$. We also define $\widetilde{J}_{[\alpha]} := I_{[\alpha]}(\eta_0, \eta_1, \ldots, \eta_{d-1})$ where $\eta_0 = 1$. An element α is called an anti-integral element of degree d over R if $\operatorname{Ker}\pi = I_{[\alpha]}\varphi_\alpha(X)R[X]$. Let $\mathfrak{p}$ be an element of Spec$R$. It is easily verified that, if α is an anti-integral element over R, then α is also an anti-integral element over $R_\mathfrak{p}$. An element α is said to be a super-primitive element of degree d over R if $J_{[\alpha]} \not\subset \mathfrak{p}$ for every $\mathfrak{p} \in \mathrm{Dp}_1(R)$ where $\mathrm{Dp}_1(R) = \{\mathfrak{p} \in \mathrm{Spec}R; \mathrm{depth}R_\mathfrak{p} = 1\}$.

Our general reference for unexplained terms is [5].

Let R be a Noetherian domain and $\alpha_1, \alpha_2, \ldots, \alpha_n$ anti-integral elements over R. Set

$$A = R[\alpha_1, \alpha_2, \ldots, \alpha_n]$$

and define

$$I_{\hat{\alpha}_i} = \{a \in R; a\alpha_j \in R[\alpha_i]\ (j = 1, 2, \ldots, n)\}.$$

Then $I_{\hat{\alpha}_i}$ is an ideal of R. By the definition of $I_{\hat{\alpha}_i}$, we have the following:

Lemma 1. *Let $\mathfrak{p}$ be an element of* Spec$R$ *If $\mathfrak{p} \not\supset I_{\hat{\alpha}_i}$, then $A_{\mathfrak{p}} = R_{\mathfrak{p}}[\alpha_i]$.*

Lemma 2([7, Theorem 1.8]). *Let R be a Noetherian domain and α an anti-integral element over R. Let $\mathfrak{p}$ be an element of* Spec$R$. *Then the following two conditions are equivalent:*
(i) *$R[\alpha]_{\mathfrak{p}}/R_{\mathfrak{p}}$ is a flat extension.*
(ii) $\mathfrak{p} \not\supset J_{[\alpha]}$.

Proposition 3. *Let R be a Noetherian domain and $\alpha_1, \alpha_2, \dots, \alpha_n$ anti-integral elements over R. Set $A = R[\alpha_1, \alpha_2, \dots, \alpha_n]$ and assume that $I_{\hat{\alpha}_1} + I_{\hat{\alpha}_2} + \cdots + I_{\hat{\alpha}_n} = R$. Let $\mathfrak{p}$ be an element of* Spec$R$. *Then the following conditions are equivalent:*
(i) *$A_{\mathfrak{p}}/R_{\mathfrak{p}}$ is a flat extension.*
(ii) $\mathfrak{p} \not\supset I_{\hat{\alpha}_1} \cap J_{[\alpha_1]} + \cdots + I_{\hat{\alpha}_n} \cap J_{[\alpha_n]}$.

Proof. (i) $\Rightarrow$ (ii). Since $I_{\hat{\alpha}_1} + I_{\hat{\alpha}_2} + \cdots + I_{\hat{\alpha}_n} = R$, there exists an index i such that $\mathfrak{p} \not\supset I_{\hat{\alpha}_i}$. Then $A_{\mathfrak{p}} = R_{\mathfrak{p}}[\alpha_i]$ by Lemma 1. Condition (i) asserts that $\mathfrak{p}R_{\mathfrak{p}} \not\supset J_{[\alpha_i]}R_{\mathfrak{p}}$ by Lemma 2. Hence $\mathfrak{p} \not\supset J_{[\alpha_i]}$, and

$$\mathfrak{p} \not\supset I_{\hat{\alpha}_1} \cap J_{[\alpha_1]} + \cdots + I_{\hat{\alpha}_n} \cap J_{[\alpha_n]}.$$

(ii) $\Rightarrow$ (i). By the condition (ii), there exists an index i such that $\mathfrak{p} \not\supset I_{\hat{\alpha}_i} \cap J_{[\alpha_i]}$. Then $\mathfrak{p} \not\supset I_{\hat{\alpha}_i}$ and $\mathfrak{p} \not\supset J_{[\alpha_i]}$. Therefore $A_{\mathfrak{p}} = R_{\mathfrak{p}}[\alpha_i]/R_{\mathfrak{p}}$ is a flat extension by Lemmas 1 and 2. Q.E.D.

Theorem 4. *Let R be a Noetherian domain and $\alpha_1, \alpha_2, \dots, \alpha_n$ anti-integral elements over R. Set $A = R[\alpha_1, \alpha_2, \dots, \alpha_n]$ and assume that $I_{\hat{\alpha}_1} + I_{\hat{\alpha}_2} + \cdots + I_{\hat{\alpha}_n} = R$. Then the following conditions are equivalent:*
(i) *A/R is a flat extension.*
(ii) $I_{\hat{\alpha}_1} \cap J_{[\alpha_1]} + \cdots + I_{\hat{\alpha}_n} \cap J_{[\alpha_n]} = R$.

Proof. Since flatness is a local-global property, it is immediate from Proposition 3. Q.E.D.

Lemma 5([7, Proposition 3.7] and [5, (4.D) Theprem 3]). *Let R be a Noetherian domain and α an anti-integral element over R. Let $\mathfrak{p}$ be an element of* Spec$R$. *Then the following two conditions are equivalent:*
(i) *The extension $R_{\mathfrak{p}}[\alpha]/R_{\mathfrak{p}}$ is a faithfully flat extension.*
(ii) $\mathfrak{p} \not\supset \widetilde{J}_{[\alpha]}$.

Proposition 6. *Let R be a Noetherian domain and $\alpha_1, \alpha_2, \dots, \alpha_n$ anti-integral elements over R. Set $A = R[\alpha_1, \alpha_2, \dots, \alpha_n]$ and assume that $I_{\hat{\alpha}_1} + I_{\hat{\alpha}_2} + \cdots + I_{\hat{\alpha}_n} = R$. Let $\mathfrak{p}$ be an element of* Spec$R$. *Then the following two conditions are equivalent:*
(i) *$A_{\mathfrak{p}}/R_{\mathfrak{p}}$ is a faithfully flat extension.*
(ii) $\mathfrak{p} \not\supset I_{\hat{\alpha}_1} \cap \widetilde{J}_{[\alpha_1]} + \cdots + I_{\hat{\alpha}_n} \cap \widetilde{J}_{[\alpha_n]}$.

Proof. (i) $\Rightarrow$ (ii). By the assumption $I_{\hat{\alpha}_1} + I_{\hat{\alpha}_2} + \cdots + I_{\hat{\alpha}_n} = R$, there exists an index i such that $\mathfrak{p} \not\supset I_{\hat{\alpha}_i}$. Then $A_{\mathfrak{p}} = R_{\mathfrak{p}}[\alpha_i]$ by Lemma 1. Since $R_{\mathfrak{p}}[\alpha_i]/R_{\mathfrak{p}}$ is a faithfully flat extension, we have $\mathfrak{p}R_{\mathfrak{p}} \not\supset \widetilde{J}_{[\alpha_i]}R_{\mathfrak{p}}$ by Lemma 5. Hence $\mathfrak{p} \not\supset I_{\hat{\alpha}_i} \cap \widetilde{J}_{[\alpha_i]}$. Therefore $\mathfrak{p} \not\supset I_{\hat{\alpha}_1} \cap \widetilde{J}_{[\alpha_1]} + \cdots + I_{\hat{\alpha}_n} \cap \widetilde{J}_{[\alpha_n]}$.

(ii) $\Rightarrow$ (i). By the condition (ii), there exists an index i such that $\mathfrak{p} \not\supset I_{\hat{\alpha}_i} \cap \widetilde{J}_{[\alpha_i]}$. Therefore $A_{\mathfrak{p}} = R_{\mathfrak{p}}[\alpha_i]/R_{\mathfrak{p}}$ is a faithfully flat extension by Lemmas 1 and 5. Q.E.D.

Theorem 7. *Let R be a Noetherian domain and $\alpha_1, \alpha_2, \ldots, \alpha_n$ anti-integral elements over R. Set $A = R[\alpha_1, \alpha_2, \ldots, \alpha_n]$ and assume that $I_{\hat{\alpha}_1} + I_{\hat{\alpha}_2} + \cdots + I_{\hat{\alpha}_n} = R$. Then the following two conditions are equivalent:*
(i) *A/R is a faithfully flat extension.*
(ii) $I_{\hat{\alpha}_1} \cap \widetilde{J}_{[\alpha_1]} + \cdots + I_{\hat{\alpha}_n} \cap \widetilde{J}_{[\alpha_n]} = R$.
Furthermore, the implication (iii) $\Rightarrow$ (i) *holds:*
(iii) *$R[\alpha_1]/R, \ldots, R[\alpha_n]/R$ are all faithfully flat extensions.*

Proof. Since faithful flatness is a local-global property, the equivalence of the conditions (i) and (ii) is immediate from Proposition 6. We will prove the implication (iii) $\Rightarrow$ (i). Lemma 5 shows that $\widetilde{J}_{[\alpha_1]} = R, \ldots, \widetilde{J}_{[\alpha_n]} = R$. Moreover, $I_{\hat{\alpha}_1} + I_{\hat{\alpha}_2} + \cdots + I_{\hat{\alpha}_n} = R$. Hence $I_{\hat{\alpha}_1} \cap \widetilde{J}_{[\alpha_1]} + \cdots + I_{\hat{\alpha}_n} \cap \widetilde{J}_{[\alpha_n]} = R$. Therefore A/R is a faithfully flat extension. Q.E.D.

Let K be the quotient field of R. We say that A/R is an exclusive extension if $A \cap K = R$.

Lemma 8([8, Theorem 5]). *Let R be a Noetherian domain with quotient field K and α a super-primitive element over R. Assume that R contains an infinite field. Then the following two conditions are equivalent:*
(i) $\mathrm{grade}\widetilde{J}_{[\alpha]} > 1$ *where we define* $\widetilde{J}_{[\alpha]} = \infty$ *if* $\widetilde{J}_{[\alpha]} = R$.
(ii) *$R[\alpha]/R$ is an exclusive extension.*

Proposition 9. *Let R be a Noetherian domain with quotient field K. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be super-primitive elements over R. Set $A = R[\alpha_1, \alpha_2, \ldots, \alpha_n]$. Assume that $I_{\hat{\alpha}_1} + I_{\hat{\alpha}_2} + \cdots + I_{\hat{\alpha}_n} = R$ and R contains an infinite field. Then the following two conditions are equivalent:*
(i) $A \cap K = R$.
(ii) $\mathrm{grade}(I_{\hat{\alpha}_1} \cap \widetilde{J}_{[\alpha_1]} + \cdots + I_{\hat{\alpha}_n} \cap \widetilde{J}_{[\alpha_n]}) > 1$.

Proof. (i) $\Rightarrow$ (ii). Assume that there exists an element $\mathfrak{p}$ of $\mathrm{Dp}_1(R)$ such that $\mathfrak{p} \supset I_{\hat{\alpha}_1} \cap \widetilde{J}_{[\alpha_1]} + \cdots + I_{\hat{\alpha}_n} \cap \widetilde{J}_{[\alpha_n]}$. Since $I_{\hat{\alpha}_1} + I_{\hat{\alpha}_2} + \cdots + I_{\hat{\alpha}_n} = R$, there exists an index i such that $\mathfrak{p} \not\supset I_{\hat{\alpha}_i}$. Then $A_\mathfrak{p} = R_\mathfrak{p}[\alpha_i]$ by Lemma 1. The condition $A \cap K = R$ implies that $A_p \cap K = R_p$. Then Lemma 8 asserts that $\mathfrak{p}R_\mathfrak{p} \not\supset \widetilde{J}_{[\alpha_i]}R_\mathfrak{p}$. Hence $\mathfrak{p} \not\supset \widetilde{J}_{[\alpha_i]}$, and $\mathfrak{p} \not\supset I_{\hat{\alpha}_i} \cap \widetilde{J}_{[\alpha_i]}$. This is a contradiction.
(ii) $\Rightarrow$ (i). Assume that $A \cap K \supsetneq R$. Then there exists an element ζ of $A \cap K$ such that $R \not\ni \zeta$. Let $\mathfrak{p}$ be a prime divisor of I_ζ where $I_\zeta = \{a \in R; a\zeta \in R\}$. Then we know that $\mathrm{depth}R_p = 1$ (cf. [9, Proposition 1.10]). Hence $\mathfrak{p} \not\supset I_{\hat{\alpha}_1} \cap \widetilde{J}_{[\alpha_1]} + \cdots + I_{\hat{\alpha}_n} \cap \widetilde{J}_{[\alpha_n]}$ by the condition (ii). Then there exists an index i such that $\mathfrak{p} \not\supset I_{\hat{\alpha}_i} \cap \widetilde{J}_{[\alpha_i]}$. Then $A_\mathfrak{p} = R_\mathfrak{p}[\alpha_i]$ by Lemma 1. Lemma 8 shows that $A_\mathfrak{p} \cap K = R_\mathfrak{p}[\alpha] \cap K = R_\mathfrak{p}$. Hence $\zeta \in R_\mathfrak{p}$ and $\mathfrak{p} \not\supset I_\zeta$. This is absurd. Q.E.D.

Let $\varphi : \mathrm{Spec}R[\alpha_1, \alpha_2, \ldots, \alpha_n] \longrightarrow \mathrm{Spec}R$, and $\varphi_k : \mathrm{Spec}R[\alpha_k] \longrightarrow \mathrm{Spec}R$ $(k = 1, 2, \ldots, n)$ be contraction mappings, that is, $\varphi(\mathfrak{P}) = \mathfrak{P} \cap R$, $\varphi_k(\mathfrak{Q}) = \mathfrak{Q} \cap R$ for $\mathfrak{P} \in \mathrm{Spec}R[\alpha_1, \alpha_2, \ldots, \alpha_n]$, $\mathfrak{Q} \in \mathrm{Spec}R[\alpha_k]$ $(k = 1, 2, \ldots, n)$.

Proposition 10. *Let R be a Noetherian domain and $\alpha_1, \alpha_2, \ldots, \alpha_n$ anti-integral elements over R. Set $A = R[\alpha_1, \alpha_2, \ldots, \alpha_n]$. Let $\varphi : \mathrm{Spec}R[\alpha_1, \alpha_2, \ldots, \alpha_n] \longrightarrow \mathrm{Spec}R$, and $\varphi_k : \mathrm{Spec}R[\alpha_k] \longrightarrow \mathrm{Spec}R$ $(k = 1, 2, \ldots, n)$ be contraction mappings respectively. Assume that $I_{\hat{\alpha}_1} + I_{\hat{\alpha}_2} + \cdots + I_{\hat{\alpha}_n} = R$. Then*

$$\mathrm{Im}\,\varphi = \bigcap_{k=1}^{n} \mathrm{Im}\,\varphi_k,$$

Proof. It is clear that $\mathrm{Im}\,\varphi \subset \bigcap_{k=1}^{n} \mathrm{Im}\,\varphi_k$. We will prove that $\mathrm{Im}\,\varphi \supset \bigcap_{k=1}^{n} \mathrm{Im}\,\varphi_k$. Let $\mathfrak{p}$ be an element of $\bigcap_{k=1}^{n} \mathrm{Im}\,\varphi_k$. Assume that $\mathfrak{p} \notin \mathrm{Im}\,\varphi$. Then we can easily verify that $\mathfrak{p}A_{\mathfrak{p}} = A_{\mathfrak{p}}$. Since $I_{\hat{\alpha}_1} + I_{\hat{\alpha}_2} + \cdots + I_{\hat{\alpha}_n} = R$, there exists an index i such that $\mathfrak{p} \not\supset I_{\hat{\alpha}_i}$. Then $A_{\mathfrak{p}} = R_{\mathfrak{p}}[\alpha_i]$ by Lemma 1. Since $\mathfrak{p}A_{\mathfrak{p}} = A_{\mathfrak{p}}$, we get $\mathfrak{p}R_{\mathfrak{p}}[\alpha_i] = R_{\mathfrak{p}}[\alpha_i]$. This is a contradiction. Q.E.D.

Let A be an extension of R and $\mathfrak{p}$ an element of Spec$R$. We say that A is a blowing-up at $\mathfrak{p}$ or $\mathfrak{p}$ is a blowing-up point of A/R if the following two conditions are satisfied:

(1) $\mathfrak{p}A_{\mathfrak{p}} \cap R_{\mathfrak{p}} = \mathfrak{p}R_{\mathfrak{p}}$.

(2) $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ is isomorphic to a polynomial ring $(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}})[T]$.

Lemma 11([7, Corollary 1.10]). *Let R be a Noetherian domain and α an anti-integral element over R. Let $\mathfrak{p}$ be an element of* Spec$R$. *Then the following two conditions are equivalent:*

(i) *$\mathfrak{p}$ is a blowing-up point of $R[\alpha]/R$.*

(ii) $\mathfrak{p} \supset J_{[\alpha]}$.

Proposition 12. *Let R be a Noetherian domain and $\alpha_1, \alpha_2, \ldots, \alpha_n$ anti-integral elements over R. Set $A = R[\alpha_1, \alpha_2, \ldots, \alpha_n]$ and assume that $I_{\hat{\alpha}_1} + I_{\hat{\alpha}_2} + \cdots + I_{\hat{\alpha}_n} = R$. Let $\mathfrak{p}$ be an element of* Spec$R$. *Then the following two conditions are equivalent:*

(i) $\mathfrak{p} \supset I_{\hat{\alpha}_1} \cap J_{[\alpha_1]} + \cdots + I_{\hat{\alpha}_n} \cap J_{[\alpha_n]}$.

(ii) *$\mathfrak{p}$ is a blowing-up point of A/R.*

Proof. By Lemmas 2 and 11, we see that $\mathfrak{p}$ is not a blowing-up point of $R[\alpha]/R$ if and only if $R_{\mathfrak{p}}[\alpha]/R_{\mathfrak{p}}$ is a flat extension. Hence the proof is clear from Proposition 3. Q.E.D.

Theorem 13. *Let R be a Noetherian domain and $\alpha_1, \alpha_2, \ldots, \alpha_n$ anti-integral elements over R. Set $A = R[\alpha_1, \alpha_2, \ldots, \alpha_n]$ and assume that $I_{\hat{\alpha}_1} + I_{\hat{\alpha}_2} + \cdots + I_{\hat{\alpha}_n} = R$. Then the following three conditions are equivalent:*

(i) *A/R is a flat extension.*

(ii) *A/R has no blowing-up point.*

(iii) $I_{\hat{\alpha}_1} \cap J_{[\alpha_1]} + \cdots + I_{\hat{\alpha}_n} \cap J_{[\alpha_n]} = R$.

Proof. It is immediate from Propositions 3 and 12. Q.E.D.

Let A be a finite R-algebra. Let $\mathfrak{P}$ be an element of Spec(A) and $\mathfrak{p} = \mathfrak{P} \cap R$. The residue fields of $\mathfrak{P}$ and $\mathfrak{p}$ are denoted by $k(\mathfrak{P})$ and $k(\mathfrak{p})$ respectively. We say that $\mathfrak{P}$ is unramified over $\mathfrak{p}$ if the following two conditions hold:

(1) $\mathfrak{P}A_{\mathfrak{P}} = \mathfrak{p}A_{\mathfrak{P}}$.

(2) $k(\mathfrak{P})$ is a finite separable algebraic extension of $k(\mathfrak{p})$.

The extension A/R is called unramified if the following two conditions are satisfied:

(1) For every element $\mathfrak{p}$ of Spec(R), there are only finitely many elements of Spec(A) lying over $\mathfrak{p}$.

(2) If $\mathfrak{P}$ is an element of Spec(A) lying over $\mathfrak{p}$, then $\mathfrak{P}$ is unramified over $\mathfrak{p}$.

It is known that A/R is an unramified extension if and only if $\Omega_R(A) = (0)$ where $\Omega_R(A)$ stands for the differential module of A over R. ([6, Chapter 3, Theorem 14])

The extension A/R is said to be étale if A/R is an unramified and flat extension. (cf. [1, Chapter VI, Definition (4.1)] and [4, p. 100])

Let $\varphi'_\alpha(X)$ be the derivative of $\varphi_\alpha(X)$.

Lemma 14.(cf. [3, Theorem 8]) *Let R be a Noetherian domain and α an anti-integral element over R. Then the following two conditions are equivalent:*
(1) *$R[\alpha]/R$ is an unramified extension.*
(2) *$I_{[\alpha]}\varphi'_\alpha(\alpha)R[\alpha] = R[\alpha]$.*

Proposition 15. *Let R be a Noetherian domain and $\alpha_1, \alpha_2, \dots, \alpha_n$ anti-integral elements over R. Set $A = R[\alpha_1, \alpha_2, \dots, \alpha_n]$ and assume that $I_{\hat{\alpha}_1} + I_{\hat{\alpha}_2} + \cdots + I_{\hat{\alpha}_n} = R$. Set*

$$U = I_{\hat{\alpha}_1} \cap I_{[\alpha_1]}\varphi'_{\alpha_1}(\alpha_1)R[\alpha_1] + \cdots + I_{\hat{\alpha}_n} \cap I_{[\alpha_n]}\varphi'_{\alpha_n}(\alpha_n)R[\alpha_n].$$

Let $\mathfrak{p}$ be an element of Spec(R). *Then the following two conditions are equivalent:*
(i) $\mathfrak{p} \not\supset U$.
(ii) *$A_\mathfrak{p}/R_\mathfrak{p}$ is an unramified extension.*

Proof. (1) $\Rightarrow$ (ii). Since $\mathfrak{p} \not\supset U$, there exists an index i such that $\mathfrak{p} \not\supset I_{\hat{\alpha}_i} \cap I_{[\alpha_i]}\varphi'_{\alpha_i}(\alpha_i)R[\alpha_i]$. Then $A_\mathfrak{p} = R_\mathfrak{p}[\alpha_i]$ by Lemma 1. Besides, we have $I_{[\alpha_i]\mathfrak{p}}\varphi'_{\alpha_i}(\alpha_i)R_\mathfrak{p}[\alpha_i] = R_\mathfrak{p}[\alpha_i]$. Lemma 14 implies that $A_\mathfrak{p}/R_\mathfrak{p}$ is an unramified extension.

(ii) $\Rightarrow$ (i). By the assumption $I_{\hat{\alpha}_1} + I_{\hat{\alpha}_2} + \cdots + I_{\hat{\alpha}_n} = R$, there exists an index i such that $\mathfrak{p} \not\supset I_{\hat{\alpha}_i}$. Then $A_\mathfrak{p} = R_\mathfrak{p}[\alpha_i]$ by Lemma 1. Because $A_\mathfrak{p}/R_\mathfrak{p} = R_\mathfrak{p}[\alpha_i]/R_\mathfrak{p}$ is an unramified extension, we see that $I_{[\alpha_i]\mathfrak{p}}\varphi'_{\alpha_i}(\alpha_i)R_\mathfrak{p}[\alpha_i] = R_\mathfrak{p}[\alpha_i]$. This shows that $\mathfrak{p} \not\supset I_{[\alpha_i]}\varphi'_{\alpha_i}(\alpha_i)R[\alpha_i]$. Hence $\mathfrak{p} \not\supset I_{\hat{\alpha}_i} \cap I_{[\alpha_i]}\varphi'_{\alpha_i}(\alpha_i)R[\alpha_i]$. Therefore $\mathfrak{p} \not\supset U$. Q.E.D.

Proposition 16. *Let R be a Noetherian domain and $\alpha_1, \alpha_2, \dots, \alpha_n$ anti-integral elements over R. Set $A = R[\alpha_1, \alpha_2, \dots, \alpha_n]$ and assume that $I_{\hat{\alpha}_1} + I_{\hat{\alpha}_2} + \cdots + I_{\hat{\alpha}_n} = R$. If A/R is an unramified extension, then A/R is an étale extension.*

Proof. Let $\mathfrak{p}$ be an element of Spec(R). Since $I_{\hat{\alpha}_1} + I_{\hat{\alpha}_2} + \cdots + I_{\hat{\alpha}_n} = R$, there exists an index i such that $\mathfrak{p} \not\supset I_{\hat{\alpha}_i}$. Then $A_\mathfrak{p} = R_\mathfrak{p}[\alpha_i]$ by Lemma 1. By [3, Theorem 9], $A_\mathfrak{p}/R_\mathfrak{p}$ is a flat extension because $A_\mathfrak{p}/R_\mathfrak{p}$ is an unramified extension. Hence $A_\mathfrak{p}/R_\mathfrak{p}$ is an étale extension. Q.E.D.

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