Spectra of Vertex-Transitive Graphs and Hecke Algebras of Finite Groups

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1 Introduction

The determination of the spectrum of a graph, i.e. the eigenvalues of the adjacency matrix of a graph, is an important task for studying the combinatorial structure of the graph.[1),2)] In this paper we consider vertex-transitive graphs including Cayley graphs as special cases, and propose an algorithm of finding the spectra of those graphs. We derive an expression of the adjacency matrix of a vertex-transitive graph in terms of the irreducible representations of the Hecke algebra of its automorphism group.

This enables us to reducing our problem to the determination of the multiplicative structure and the character table of the Hecke algebra. We show the utility of our algorithm by means of several examples.

2 Basic structures of vertex-transitive graphs

Let $M = (V, E)$ be a finite simple graph, namely, a finite graph without loops or multiple edges. Here V is the set of vertices of M and E is the set of edges of M. Since M is a simple graph, each edge can be identified with the set of its extremal vertices. A bijection g of V is called a graph automophism of M if and only if it satisfies $\{u, v\} \in E$ if and only if $\{g \cdot u, g \cdot v\} \in E$. M is called a vertex-transitive graph if and only if the group G of graph automorphisms of M acts transitively on V.

Fix $v_0 \in V$ and put $H = \{h \in G\,;\, h \cdot v_0 = v_0\}$. Since G is transitive on V, there exists a canonical bijection between the set $G/H = \{xH\,;\, x \in G\}$ of all left H cosets of G and $G \cdot v_0 = V$. For $v \in V$, let $N(v)$ be the set of all vertices adjacent to v, that is, $N(v) = \{w \in V\,;\, \{v, w\} \in E\}$. Write $v = x \cdot v_0$ where $x \in G$. Then $N(v) = xN(v_0) = \{x \cdot u\,;\, u \in N(v_0)\}$, so that M is a regular graph with valency $|N(v_0)|$. Here and from now on, we denote the cardinality of a finite set X by $|X|$.

Put $S = \{g \in G\,;\, g \cdot v_0 \in N(v_0)\}$. Since $v_0 \notin N(v_0)$, it follows that $S \cap H = \emptyset$. Let $u \in N(v_0)$. Since $\{v_0, h \cdot u\} = \{h \cdot v_0, h \cdot u\}$ for $h \in H$ and $\{v_0, u\} \in E$, it follows that $h \cdot u \in N(v_0)$. Hence $N(v_0)$ is stable under H and $HSH = S$. Let $g \in S$. Since $\{v_0, g^{-1} \cdot v_0\} = \{g^{-1} \cdot v_0, g^{-1}(g \cdot v_0)\}$ and $\{v_0, g \cdot v_0\} \in E$, it follows that $g^{-1} \cdot v_0 \in N(v_0)$ and hence $g^{-1} \in S$. This means $S = S^{-1}$.

Let $H\backslash G/H = \{HgH\,;\ g \in G\}$ be the set of all H double cosets of G. Since $HSH = S$, it follows that S is a union of H double cosets, namely, there exists a subset Ω of $H\backslash G/H$ such that $S = \bigcup_{D\in\Omega} D$. From $S \cap H = \emptyset$ and $S = S^{-1}$, we conclude that $H \notin \Omega$ and $D \in \Omega$ implies $D^{-1} \in \Omega$. Conversely, let G be a finite group and H a subgroup of G. Let Ω be a subset of $H\backslash G/H$ such that $H \notin \Omega$ and $D \in \Omega$ implies $D^{-1} \in \Omega$. Put $S = \bigcup_{D\in\Omega} D$. Then one can construct a graph $M(G/H, \Omega)$ in the following manner.

Take G/H as the vertex set and $E(\Omega) = \{\{xH, yH\}\,;\ x^{-1}y \in S\}$ as the edge set. Then from $H \cap S = \emptyset$ and $S = S^{-1}$ we conclude that $M(G/H, \Omega)$ is a finite simple graph. Obviously G acts transitively on G/H, which leads to a natural action on $E(\Omega)$. Therefore $M(G/H, \Omega)$ is a vertex-transitive graph. Note that any vertex-transitive graph is isomophic to $M(G/H, \Omega)$ for suitable G, H and Ω. We remark that if $H = (1)$ then $\Omega = S$ and $M(G, \Omega)$ is nothing but the Cayley graph of G with respect to Ω. For a double coset $D = HgH \in H\backslash G/H$, let D/H be the set of all representatives of left H cosets contained in D. Clearly

$$D = \bigcup_{s\in D/H} sH. \tag{1}$$

Set $\mathrm{ind}(D) = |D/H|$. Since $N(xH) = \{yH\,;\ x^{-1}y \in S\}$ where $S = \bigcup_{D\in\Omega} D$, it follows that

$$N(xH) = \bigcup_{D\in\Omega} \{xsH\,;\ s \in D/H\} \quad \text{for } xH \in G/H. \tag{2}$$

Hence the valency of $M(G/H, \Omega)$ is given by $\sum_{D\in\Omega} \mathrm{ind}(D)$. Note that $M(G/H, \Omega)$ is connected if H and S generates G. If $\Omega_1, \Omega_2 \subset H\backslash G/H$ with $\Omega_1 \cap \Omega_2 = \emptyset$, both satisfying $H \notin \Omega_i$ and $D \in \Omega_i$ implies $D^{-1} \in \Omega_i$ $(i = 1, 2)$, then the edge set of $M(G/H, \Omega_1 \cup \Omega_2)$ is the disjoint union of the edge set $E(\Omega_1)$ of $M(G/H, \Omega_1)$ and the edge set $E(\Omega_2)$ of $M(G/H, \Omega_2)$.

Now we consider the adjacency matrix $A(G/H, \Omega)$ of $M(G/H, \Omega)$. It is a linear endomorphism on the vector space $\mathbf{C}^{G/H}$ of all $\mathbf{C}$-valued functions on G/H, whose definition is

$$A(G/H, \Omega)f(xH) = \sum_{yH\in N(xH)} f(yH) \quad \text{for } f \in \mathbf{C}^{G/H}.$$

From (2), we get

$$A(G/H, \Omega)f(xH) = \sum_{D\in\Omega}\sum_{s\in D/H} f(xsH). \tag{3}$$

3 The Hecke algebra $\mathcal{H}(G, H)$ and its characters

Let $\mathbf{C}G$ be the group algebra of G over $\mathbf{C}$. For a double coset $D \in H\backslash G/H$, define $\varepsilon(D) \in \mathbf{C}G$ by

$$\varepsilon(D) = |H|^{-1} \sum_{g\in D} g. \tag{4}$$

In particular for the trivial double coset H, we write $e = \varepsilon(H) = |H|^{-1}\sum_{h\in H} h$. Then e is an idempotent of $\mathbf{C}G$ such that $he = e = eh'$ for $h, h' \in H$. Put $\mathcal{H}(G, H) = e\mathbf{C}Ge$.

Then $\mathcal{H}(G, H)$ is a subalgebra of $\mathbf{C}G$, which we call the Hecke algebra of G with respect to H. We can check easily that $\varepsilon(D) = \mathrm{ind}(D)ege$ for $g \in D$, so that $\varepsilon(D) \in \mathcal{H}(G, H)$ for $D \in H\backslash G/H$. It is well known[4)] that $\{\varepsilon(D)\,;\, D \in H\backslash G/H\}$ forms a $\mathbf{C}$-basis of $\mathcal{H}(G, H)$.

Recall that there is a natural one to one correspondence between a representation of G and a $\mathbf{C}G$-module. Let (ρ, W) be a representation of G on a vector space W. Set $W^H = \{\xi \in W\,;\, \rho(h)\xi = \xi \text{ for } h \in H\}$ and $\rho_{\mathcal{H}}(e) = |H|^{-1}\sum_{h\in H} \rho(h)$. Then $\rho_{\mathcal{H}}(e)$ yields a projection of W onto W^H. Consequently W^H is a $\mathcal{H}(G, H)$-module via

$$\rho_{\mathcal{H}}(ege)\xi = \rho_{\mathcal{H}}(e)(\rho(g)\xi) \quad \text{where } g \in G \text{ and } \xi \in W^H. \tag{5}$$

Let $\widehat{G}$ be the set of all equivalence classes of irreducible representations of G, and $\widehat{G}^H$ the subset of $\widehat{G}$ consisting of irreducible representations (π, V_π) with $V_\pi^H \neq (0)$. It can be easily seen that $(\pi_{\mathcal{H}}, V_\pi^H)$ is an irreducible $\mathcal{H}(G, H)$-module. It is known[4)] that all the irreducible $\mathcal{H}(G, H)$-modules are of the form $(\pi_{\mathcal{H}}, V_\pi^H)$ for $(\pi, V_\pi) \in \widehat{G}^H$. The character $\chi_{\pi_{\mathcal{H}}}$ of $(\pi_{\mathcal{H}}, V_\pi^H)$ is defined by $\chi_{\pi_{\mathcal{H}}}(a) = \mathrm{Tr}(\pi_{\mathcal{H}}(a))$ for $a \in \mathcal{H}(G, H)$.

Since $\pi_{\mathcal{H}}(\varepsilon(D)) = |H|^{-1}\sum_{g\in D} \pi_{\mathcal{H}}(g)$, it follows that

$$\chi_{\pi_{\mathcal{H}}}(\varepsilon(D)) = |H|^{-1}\sum_{g\in D}\chi_\pi(g) \tag{6}$$

where χ_π is the character of π. The character table of $\mathcal{H}(G, H)$ is given by

$$(\chi_{\pi_{\mathcal{H}}}(\varepsilon(D)))_{\pi\in\widehat{G}^H, D\in H\backslash G/H}\,. \tag{7}$$

Note that the trivial representation $\mathbf{1}$ of G is contained in $\widehat{G}^H$ and it holds that $\chi_{1_{\mathcal{H}}}(\varepsilon(D)) = \mathrm{ind}(D)$. Furthermore it is obvious that $\chi_{\pi_{\mathcal{H}}}(e) = \dim V_\pi^H$.

4 The adjacency matrix $A(G/H, \Omega)$ and the representations of $\mathcal{H}(G, H)$

Let $\mathbf{C}^G$ be the vector space of all $\mathbf{C}$-valued functions on G. Then $\mathbf{C}^G$ is a left and a right $\mathbf{C}G$-module via $L(g)\varphi(x) = \varphi(g^{-1}x)$ and $R(g)\varphi(x) = \varphi(xg)$ where $g, x \in G$ and $\varphi \in \mathbf{C}^G$. Consider the right $\mathbf{C}G$-module $(R, \mathbf{C}^G)$. The subspace $(\mathbf{C}^G)^H$ of all right H-fixed vectors of $\mathbf{C}^G$ is clearly isomorphic to $\mathbf{C}^{G/H}$, so that $\mathbf{C}^{G/H}$ can be viewed as an $\mathcal{H}(G, H)$-module under $R_{\mathcal{H}}$. In particular we have by (4) and (1)

$$R_{\mathcal{H}}(\varepsilon(D))f(xH) = |H|^{-1}\sum_{g\in D} f(xgH) = \sum_{s\in D/H} f(xsH) \quad \text{for } f \in \mathbf{C}^{G/H}.$$

Combining this with (3), we obtain

Lemma 1. *The adjacency matrix $A(G/H, \Omega)$ of the vertex-transitive graph $M(G/H, \Omega)$ can be written as*

$$A(G/H, \Omega) = \sum_{D\in\Omega} R_{\mathcal{H}}(\varepsilon(D)). \tag{8}$$

Applying the well known theorem of Peter-Weyl, we have the direct sum decomposition

$$\mathbf{C}^{G/H} = \bigoplus_{\pi\in\widehat{G}^H} V_\pi^* \otimes V_\pi^H$$

where V_π^* is the dual space of V_π, and moreover

$$R_{\mathcal{H}}(a) = \bigoplus_{\pi \in \hat{G}^H} I \otimes \pi_{\mathcal{H}}(a) \quad \text{for } a \in \mathcal{H}(G, H) \tag{9}$$

where I is the identity endomorphism of V_π^*. From (9) and Lemma 1, we obtain the following results.

Theorem 1. (i) *For* $\pi \in \hat{G}^H$, *define a linear endomorphism* $A_\pi(G/H, \Omega)$ *on* V_π^H *by*

$$A_\pi(G/H, \Omega) = \sum_{D \in \Omega} \pi_{\mathcal{H}}(\varepsilon(D)). \tag{10}$$

Then the adjacency matrix $A(G/H, \Omega)$ *of the vertex-transitive graph* $M(G/H, \Omega)$ *can be written as*

$$A(G/H, \Omega) = \bigoplus_{\pi \in \hat{G}^H} I \otimes A_\pi(G/H, \Omega). \tag{11}$$

(ii) *Let* $\sigma(M(G/H, \Omega))$ *be the spectrum of* $M(G/H, \Omega)$, *namely, the set of all eigenvalues of* $A(G/H, \Omega)$. *Let* $\sigma(A_\pi(G/H, \Omega))$ *be the set of all eigenvalues of* $A_\pi(G/H, \Omega)$. *We denote by* $d(\pi)\sigma(A_\pi(G/H, \Omega))$ *the set of all eigenvalues of* $A_\pi(G/H, \Omega)$ *counted with multiplicity* $d(\pi)$ *where* $d(\pi) = \dim V_\pi$. *Then we have*

$$\sigma(M(G/H, \Omega)) = \bigcup_{\pi \in \hat{G}^H} d(\pi)\sigma(A_\pi(G/H, \Omega)).$$

Therefore our problem of determining $\sigma(M(G/H, \Omega))$ is reduced to that of finding $\sigma(A_\pi(G/H, \Omega))$ for $\pi \in \hat{G}^H$. In the sequal we propose an algorithm of determining $\sigma(A_\pi(G/H, \Omega))$. Put $d = \dim V_\pi^H$ and write $\sigma(A_\pi(G/H, \Omega)) = \{\lambda_1, \cdots, \lambda_d\}$. Consider the k-th power sum of the eigenvalues $\lambda_1, \cdots, \lambda_d$; $p_k = \sum_{i=1}^d \lambda_i^k$ $(k \geq 1)$. Then $p_k = \mathrm{Tr}(A_\pi(G/H, \Omega)^k)$. Write $\Omega = \{D_1, \cdots, D_r\}$ and $\varepsilon_i = \varepsilon(D_i)$ $(1 \leq i \leq r)$ for simplicity.

From the definition of $A_\pi(G/H, \Omega)$, we conclude that

$$A_\pi(G/H, \Omega)^k = \sum_{i_1, \cdots, i_k = 1}^{r} \pi_{\mathcal{H}}(\varepsilon_{i_1} \cdots \varepsilon_{i_k})$$

and hence

$$p_k = \sum_{i_1, \cdots, i_k = 1}^{r} \chi_{\pi_{\mathcal{H}}}(\varepsilon_{i_1} \cdots \varepsilon_{i_k}).$$

This means that the power sums of eigenvalues can be read from the multiplication table and the character table of $\mathcal{H}(G, H)$.

Recall that by Newton's famous formula the characteristic polynomial $\Phi(A_\pi(G/H, \Omega); \lambda)$ of $A_\pi(G/H, \Omega)$ is given by

$$\Phi(A_\pi(G/H, \Omega); \lambda) = (d!)^{-1} \begin{vmatrix} \lambda^d & \lambda^{d-1} & \lambda^{d-2} & \cdots & \lambda & 1 \\ p_1 & 1 & & & & \\ p_2 & p_1 & 2 & & 0 & \\ \vdots & \vdots & \ddots & \ddots & & \\ p_{d-1} & p_{d-2} & \cdots & p_1 & d-1 & \\ p_d & p_{d-1} & \cdots & p_2 & p_1 & d \end{vmatrix}.$$

Solving the characteristic equation $\Phi(A_\pi(G/H, \Omega); \lambda) = 0$, we get the eigenvalues of

$A_\pi(G/H, \Omega)$. The most simplest case is that (G, H) is a Gelfand pair. That is the case $\dim V_\pi^H = 1$ for all $(\pi, V_\pi) \in \widehat{G}^H$. In this case $\chi_{\pi\mathcal{H}} = \pi_{\mathcal{H}}$ and hence

$$\sigma(A_\pi(G/H, \Omega) = \left\{\sum_{D\in\Omega}\chi_{\pi\mathcal{H}}(\varepsilon(D))\right\}$$

and

$$\sigma(M(G/H, \Omega)) = \bigcup_{\pi\in\widehat{G}^H} d(\pi)\left\{\sum_{D\in\Omega}\chi_{\pi\mathcal{H}}(\varepsilon(D))\right\}.$$

5 Examples

In this section we take $G = S_5$ (the symmetric group on $\{1, 2, 3, 4, 5\}$). Every element of G can be written as a product of pairwise disjoint cycles. We denote by $(i_1 i_2 \cdots i_r)$ the r-cycle on $\{i_1, i_2, \cdots, i_r\} \subset \{1, 2, 3, 4, 5\}$. It is well known[3] that the irreducible representations of G and the conjugacy classes of G are parametrized by the partitions of 5, so that

$$\widehat{G} = \{(5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1)\}.$$

We recall that

$$\dim V_\pi = 1, 4, 5, 6, 5, 4, 1$$
$$\text{for } \pi = (5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1) \text{ respectively.}$$

In the following, we write the corresponding irreducible character by the same symbol; for example $\chi_{(4,1)} = (4,1)$. For a graph M, we write the spectrum $\sigma(M)$ of M as follows;

$$\sigma(M) = \begin{pmatrix} \lambda_1, & \lambda_2, & \cdots, & \lambda_s \\ m_1, & m_2, & \cdots, & m_s \end{pmatrix}$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_s$ are all the distinct eigenvalues of the adjacency matrix of M and $m_1, m_2, \cdots, m_s$ are the corresponding multiplicities. In the sequel, we simply write $M(\Omega)$ for the vertex-transitive graph $M(G/H, \Omega)$ defined by $\Omega \subset H\backslash G/H$.

Example 1. Let $G = S_5$ and $H = \{\sigma^i\,;\ 0 \leq i \leq 4\}$ where $\sigma = (12345) \in G$. Then $|G/H| = 24$ and $H\backslash G/H = \{D_i\,;\ 0 \leq i \leq 7\}$ where

$$\begin{array}{llll} D_0 = H, & D_1 = H(12)(35)H, & D_2 = H(1243)H, & D_3 = H(1254)H, \\ D_4 = H(12)H, & D_5 = H(13)H, & D_6 = H(123)H, & D_7 = H(124)H. \end{array}$$

Since $D_1^{-1} = D_1, D_2^{-1} = D_3, D_3^{-1} = D_2, D_4^{-1} = D_4, D_5^{-1} = D_5, D_6^{-1} = D_6, D_7^{-1} = D_7$ it follows that there are 64 possible choices of Ω in $H\backslash G/H$.

Put $\varepsilon_i = \varepsilon(D_i)$ $(0 \leq i \leq 7)$ and $e = \varepsilon(D_0)$. Then the multiplication table of $\mathcal{H}(G, H)$ is given as follows;

	e	ε_1	ε_2	ε_3	ε_4	ε_5	ε_6	ε_7
e	e	ε_1	ε_2	ε_3	ε_4	ε_5	ε_6	ε_7
ε_1	ε_1	e	ε_3	ε_2	ε_5	ε_4	ε_7	ε_6
ε_2	ε_2	ε_3	ε_1	e	ε_7	ε_6	ε_4	ε_5
ε_3	ε_3	ε_2	e	ε_1	ε_6	ε_7	ε_5	ε_4
ε_4	ε_4	ε_5	ε_6	ε_7	$5e+\alpha$	$5\varepsilon_1+\alpha$	$5\varepsilon_2+\beta$	$5\varepsilon_3+\beta$
ε_5	ε_5	ε_4	ε_7	ε_6	$5\varepsilon_1+\alpha$	$5e+\alpha$	$5\varepsilon_3+\beta$	$5\varepsilon_2+\beta$
ε_6	ε_6	ε_7	ε_5	ε_4	$5\varepsilon_3+\beta$	$5\varepsilon_2+\beta$	$5e+\alpha$	$5\varepsilon_1+\alpha$
ε_7	ε_7	ε_6	ε_4	ε_5	$5\varepsilon_2+\beta$	$5\varepsilon_3+\beta$	$5\varepsilon_1+\alpha$	$5e+\alpha$

where $\alpha = 2\varepsilon_6+2\varepsilon_7$ and $\beta = 2\varepsilon_4+2\varepsilon_5$.

Decomposing the induced $\mathbf{C}G$-module $\mathbf{C}^{G/H}$ into irreducibles, we have

$$\hat{G}^H = \{(5), (3,2), (3,1,1), (2,2,1), (1,1,1,1,1)\}.$$

Using the character table of S_5[5], we can compute the right hand side of (6) and obtain the character table of $\mathcal{H}(G, H)$. The result is

	e	ε_1	ε_2	ε_3	ε_4	ε_5	ε_6	ε_7
(5)	1	1	1	1	5	5	5	5
(3,2)	1	1	−1	−1	1	1	−1	−1
(3,1,1)	2	−2	0	0	0	0	0	0
(2,2,1)	1	1	1	1	−1	−1	−1	−1
(1,1,1,1,1)	1	1	−1	−1	−5	−5	5	5 .

We illustrate our method of determining $\sigma(M(\Omega))$ with some examples.

Let $\Omega = \{D_4\}$. From the character table we find that $\dim V_\pi^H = \chi_{\pi\mathcal{H}}(e) = 1$ for $\pi =$ (5), (3,2), (2,2,1) and (1,1,1,1,1). For those π, we have $\sigma(A_\pi(M(\{D_4\}))) = \{\chi_{\pi\mathcal{H}}(\varepsilon_4)\}$. From the character table we read $\chi_{\pi\mathcal{H}}(\varepsilon_4) = 5, 1, -1, -5$ for $\pi =$ (5), (3,2), (2,2,1) and (1,1,1, 1,1) respectively. Since $\dim V_{(3,1,1)}^H = 2$, we have to compute the power sums p_1 and p_2 in this case. But using the multiplication table and the character table, we have $p_1 = (3,1,1)(\varepsilon_4) = 0$ and $p_2 = (3,1,1)(\varepsilon_4^2) = (3,1,1)(5e+2\varepsilon_6+2\varepsilon_7) = 10$.

Consequently the characteristic polynomial $\Phi(A_{(3,1,1)}(\{D_4\})\,;\,\lambda)$ is equal to λ^2-5 and hence $\sigma(A_{(3,1,1)}(\{D_4\})) = \{\sqrt{5}, -\sqrt{5}\}$. Considering the dimensions of the members of $\hat{G}^H$, we get

$$\sigma(M(\{D_4\})) = \begin{pmatrix} 5, & \sqrt{5}, & 1, & -1, & -\sqrt{5}, & -5 \\ 1, & 6\,, & 5, & 5, & 6\,, & 1 \end{pmatrix}.$$

Since the multiplicity of the maximal eigenvalue 5 is 1, it follows that $M(\{D_4\})$ is a connected 5-regular graph, and since the distribution of the eigenvalues is symmetric with respect to the origin, we conclude that $M(\{D_4\})$ is a bipartite graph.

Take $\Omega = \{D_6\}$. Then $\chi_{\pi\mathcal{H}}(\varepsilon_6) = 5, -1, -1, 5$ for $\pi =$ (5), (3,2), (2,2,1) and (1,1,1,1,1) respectively. Since $(3,1,1)(\varepsilon_6) = 0$ and $(3,1,1)(\varepsilon_6^2) = 10$, we have $\sigma(A_{(3,1,1)}(\{D_6\})) = \{\sqrt{5},$

$-\sqrt{5}\}$. Therefore we obtain

$$\sigma(M(\{D_6\}))=\begin{pmatrix}5, & \sqrt{5}, & -1, & -\sqrt{5}\\ 2, & 6\ , & 10, & 6\end{pmatrix}.$$

Note that $M(\{D_6\})$ is the disjoint union of two skelton graphs of the icosahedron.

Take $\Omega=\{D_1, D_2, D_6\}$. Then $\chi_{\pi\mathcal{H}}(\varepsilon_1+\varepsilon_2+\varepsilon_6)=7, -1, 1, 5$ for $\pi=(5), (3,2), (2,2,1)$ and $(1,1,1,1,1)$ respectively. From the multiplication table, it follows that $(\varepsilon_1+\varepsilon_2+\varepsilon_6)^2 = 6e+\varepsilon_1+2\varepsilon_3+\varepsilon_4+\varepsilon_5+2\varepsilon_6+4\varepsilon_7$. Therefore $p_1=(3,1,1)(\varepsilon_1+\varepsilon_2+\varepsilon_6)=-2$ and $p_2=(3,1,1)((\varepsilon_1+\varepsilon_2+\varepsilon_6)^2)=10$. The characteristic polynomial of $A_{(3,1,1)}(\{D_1, D_2, D_6\})$ is equal to $\lambda^2+2\lambda-3$ and hence the eigenvalues are $1, -3$. Consequently we have

$$\sigma(M(\{D_1, D_2, D_6\}))=\begin{pmatrix}7, & 5, & 1, & -1, & -3\\ 1, & 1, & 11, & 5, & 6\end{pmatrix}.$$

In this way we can determine the spectra of all the vertex-transitive graphs on G/H arising from Ω in $H\backslash G/H$.

Example 2. Let $G=S_5$ and $H=S_2\times S_3=\{1, (12)\}\times\{1, (34), (35), (45), (345), (354)\}$. Then $|G/H|=10$ and $H\backslash G/H=\{D_0=H, D_1=H(13)H, D_2=H(13)(24)H\}$. Since $D_1^{-1}=D_1$ and $D_2^{-1}=D_2$, it follows that all the possible choices of Ω are $\emptyset$, $\{D_1\}$, $\{D_2\}$ and $\{D_1, D_2\}$. We find that the corresponding vertex-transitive graphs are $M(\emptyset)=\bar{K}_{10}$, $M(\{D_1\})=J(5,2)$, $M(\{D_2\})=O_3$ and $M(\{D_1, D_2\})=K_{10}$ respectively.

Here K_{10} is the complete graph on 10 vertices, $\bar{K}_{10}$ is the complement of K_{10}, $J(5,2)$ is the Johnson graph and O_3 is Peterson graph. Put $e=\varepsilon(D_0)$, $\varepsilon_1=\varepsilon(D_1)$ and $\varepsilon_2=\varepsilon(D_2)$. Then the multiplication table of $\mathcal{H}(G, H)$ is given by

	e	ε_1	ε_2
e	e	ε_1	ε_2
ε_1	ε_1	$6e+3\varepsilon_1+4\varepsilon_2$	$2(\varepsilon_1+\varepsilon_2)$
ε_2	ε_2	$2(\varepsilon_1+\varepsilon_2)$	$3e+\varepsilon_1$.

This yields that $\mathcal{H}(G, H)$ is commutative, namely, (G,H) is a Gelfand pair.

Decomposing $\mathbf{C}^{G/H}$ into irreducibles, we have $\hat{G}^H=\{(5), (4,1), (3,2)\}$. Using the character table of S_5, we obtain the character table of $\mathcal{H}(G, H)$;

	e	ε_1	ε_2
(5)	1	6	3
(4,1)	1	1	−2
(3,2)	1	−2	1 .

As in Example 1, we can deduce $\sigma(M(\Omega))$ from the character table of $\mathcal{H}(G, H)$. The results are

$$\sigma(M(\{\emptyset\})) = \begin{pmatrix} 0 \\ 10 \end{pmatrix}, \qquad \sigma(M(\{D_1\})) = \begin{pmatrix} 6, & 1, & -2 \\ 1, & 4, & 5 \end{pmatrix},$$

$$\sigma(M(\{D_2\})) = \begin{pmatrix} 3, & 1, & -2 \\ 1, & 5, & 4 \end{pmatrix}, \quad \sigma(M(\{D_1, D_2\})) = \begin{pmatrix} 9, & -1 \\ 1, & 9 \end{pmatrix}.$$

Example 3. Let $G = S_5$ and $H = S_3 = \{1, (12), (13), (23), (123), (132)\}$. Then $|G/H| = 20$ and $H\backslash G/H = \{D_i\,;\ 0 \le i \le 6\}$, where

$$D_0 = H,\quad D_1 = H(45)H,\quad D_2 = H(34)H,\quad D_3 = H(35)H,\quad D_4 = H(345)H,$$
$$D_5 = H(354)H \quad \text{and} \quad D_6 = H(24)(35)H.$$

Since $D_i^{-1} = D_i$ $(i = 1, 2, 3, 6)$, $D_4^{-1} = D_5$ and $D_5^{-1} = D_4$, there are 32 possible choices of $\Omega \subset H\backslash G/H$. Put $\varepsilon_i = \varepsilon(D_i)$ $(1 \le i \le 6)$ and $e = \varepsilon(D_0)$. Then we can show that $\mathcal{H}(G, H)$ is an algebra generated by ε_1 and ε_2 with the relations

$$\varepsilon_1^2 = e, \quad \varepsilon_2^2 = 3e + 2\varepsilon_2,$$
$$(\varepsilon_1\varepsilon_2)^2 = \varepsilon_2\varepsilon_1 + \varepsilon_2\varepsilon_1\varepsilon_2 - \varepsilon_1\varepsilon_2\varepsilon_1,$$
$$(\varepsilon_2\varepsilon_1)^2 = \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_1\varepsilon_2 - \varepsilon_1\varepsilon_2\varepsilon_1.$$

Note that

$$\varepsilon_3 = \varepsilon_1\varepsilon_2\varepsilon_1, \quad \varepsilon_4 = \varepsilon_2\varepsilon_1, \quad \varepsilon_5 = \varepsilon_1\varepsilon_2 \quad \text{and} \quad \varepsilon_6 = \varepsilon_2\varepsilon_1\varepsilon_2 - \varepsilon_1\varepsilon_2\varepsilon_1.$$

As in the previous examples, we have

$$\hat{G}^H = \{(5), (4,1), (3,2), (3,1,1)\}.$$

The character table of $\mathcal{H}(G, H)$ is given by

	e	ε_1	ε_2	ε_3	ε_4	ε_5	ε_6
(5)	1	1	3	3	3	3	6
(4,1)	2	0	2	2	−1	−1	−4
(3,2)	1	1	−1	−1	−1	−1	2
(3,1,1)	1	−1	−1	−1	1	1	0

We list up the spectra of several vertex-transitive graphs on G/H;

$$\sigma(M(\{D_1\})) = \begin{pmatrix} 1, & -1 \\ 10, & 10 \end{pmatrix}, \quad \sigma(M(\{D_2\})) = \sigma(M(\{D_3\})) = \begin{pmatrix} 3, & -1 \\ 5, & 15 \end{pmatrix},$$

$$\sigma(M(\{D_6\})) = \begin{pmatrix} 6, & 2, & 0, & -4 \\ 1, & 5, & 10, & 4 \end{pmatrix},$$

$$\sigma(M(\{D_1, D_2\})) = \sigma(M(\{D_1, D_3\})) = \begin{pmatrix} 4, & 3, & 0, & -1, & -2 \\ 1, & 4, & 5, & 4, & 6 \end{pmatrix},$$

$$\sigma(M(\{D_1, D_6\})) = \begin{pmatrix} 7, & 3, & -1, & -3 \\ 1, & 5, & 10, & 4 \end{pmatrix}, \quad \sigma(M(\{D_2, D_3\})) = \begin{pmatrix} 6, & 3, & 1, & -2 \\ 1, & 4, & 4, & 11 \end{pmatrix},$$

$$\sigma(M(\{D_4, D_5\})) = \begin{pmatrix} 6, & 2, & 1, & -2, & -3 \\ 1, & 6, & 4, & 5, & 4 \end{pmatrix},$$

$$\sigma(M(\{D_2, D_4, D_5\})) = \begin{pmatrix} 9, & \sqrt{6}, & 1, & -\sqrt{6}, & -3 \\ 1, & 4, & 6, & 4, & 5 \end{pmatrix},$$

$$\sigma(M(\{D_2, D_3, D_6\})) = \begin{pmatrix} 12, & 3, & 0, & -2, & -3 \\ 1, & 4, & 5, & 6, & 4 \end{pmatrix}.$$

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